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FIXED-POINT ITERATION PROCESSES FOR NON-LIPSCHITZIAN MAPPINGS OF ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE

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We study convergences of Mann and Ishikawa iteration processes for mappings of asymptotically quasi-nonexpansive type in Banach spaces.

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1. Introduction and preliminaries. Let D be a nonempty subset of a real Banach space X and $T : D \to D$ a nonlinear mapping. The mapping T is said to be asymptotically quasi-nonexpansive (see [5]) if $F(T) \neq \varnothing$ and there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\left\|T^n x - p\right\| \le (1+k_n)\|x-p\| \tag{1.1}$$

for all $x \in D$, $p \in F(T)$, and $n \in \mathbb{N}$. The mapping T is said to be asymptotically nonexpansive (see [3]) if there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\left\|T^n x - T^n y\right\| \le (1+k_n)\|x-y\| \tag{1.2}$$

for all $x, y \in D$ and $n \in \mathbb{N}$. The mapping T is said to be a mapping of asymptotically nonexpansive type [4] if

$$\limsup_{n\to\infty} \sup_{x\in D} \left(\left\|T^n x - T^n y\right\| - \|x-y\|\right) \le 0 \tag{1.3}$$

for any $y \in D$.

In 1973, Petryshyn and Williamson [7] gave necessary and sufficient conditions for Mann iterative sequence to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [2] extended the results of [7] and gave the necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Recently, Liu [5] extended results of [2, 7] and gave the necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points of asymptotically quasi-nonexpansive mappings.

First, we introduce the concept of class of mappings of asymptotically quasi-nonexpansive type: the mapping T is said to be a mapping of asymptotically quasi-nonexpansive type if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{x\in D} \left(\|T^n x - p\| - \|x - p\| \right) \le 0 \quad \text{for any } p \in F(T). \tag{1.4}$$

Remark 1.1. If T is a mapping of asymptotically nonexpansive type with $F(T) \neq \emptyset$, then T is a mapping of asymptotically quasi-nonexpansive type.

Remark 1.2. If D is bounded and T is an asymptotically quasi-nonexpansive mapping, then T is a mapping of asymptotically quasi-nonexpansive type. In fact, if T is an asymptotically quasi-nonexpansive mapping, then there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - p\| \le (1 + k_n) \|x - p\| \tag{1.5}$$

for all $x \in D$, $p \in F(T)$, and $n \in \mathbb{N}$, which implies

$$\sup_{x\in D} \left\{ \|T^n x - T^n y\| - \|x - y\| \right\} \le k_n \cdot \operatorname{diam} D \tag{1.6}$$

for any $y \in F(T)$ and $n \in \mathbb{N}$. Hence

$$\limsup_{n\to\infty} \sup_{x\in D} \left(\|T^n x - T^n y\| - \|x - y\| \right) \le 0 \quad \text{for any } y \in F(T). \tag{1.7}$$

We observe from Remarks 1.1 and 1.2 that the class of mappings of asymptotically nonexpansive type is an intermediate class between the class of mappings of asymptotically quasi-nonexpansive type and that of mappings of asymptotically nonexpansive type with nonempty fixed-point sets. Let

$$\begin{aligned}
C_1 &= \{T : T : D \longrightarrow D \text{ is a nonexpansive mapping}\},\\
C_2 &= \{T : T : D \longrightarrow D \text{ is a quasi-nonexpansive mapping}\},\\
C_3 &= \{T : T : D \longrightarrow D \text{ is an asymptotically nonexpansive mapping}\},\\
C_4 &= \{T : T : D \longrightarrow D \text{ is an asymptotically quasi-nonexpansive mapping}\},\\
C_5 &= \{T : T : D \longrightarrow D \text{ is a mapping of asymptotically nonexpansive type}\},\\
C_6 &= \{T : T : D \longrightarrow D \text{ is a mapping of asymptotically quasi-nonexpansive type}\}.
\end{aligned} \tag{1.8}$$

Then we have the following implications:

$$\begin{array}{ccc}
C_1 & \Longrightarrow & C_2 \\
\Downarrow & & \Downarrow \\
C_3 & \Longrightarrow & C_4 \\
\Downarrow & & \Downarrow \\
C_5 & \Longrightarrow & C_6.
\end{array} \tag{1.9}$$

In this paper, we are mainly interested in the problem of approximation of fixed points of the more general class of mappings of asymptotically quasi-nonexpansive type than that of asymptotically quasi-nonexpansive mappings. The purpose of this paper is to continue discussion concerning convergence of Mann and Ishikawa iteration processes for mappings of asymptotically quasi-nonexpansive type in Banach spaces. We give necessary and sufficient conditions for the Mann and Ishikawa iteration processes to converge to fixed points of mappings of asymptotically quasi-nonexpansive type. Further, we obtain extensions of various results obtained quite recently by Deng [1], Ghosh and Denath [2], Liu [5], and Tan and Xu [9, 10] to more general types of space as well as families of operators.

We say that a Banach space X satisfies *Opial's condition* [6] if, for each sequence $\{x_n\}$ in X weakly convergent to a point x and for all $y \neq x$,

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||. \tag{1.10}$$

The examples of Banach spaces which satisfy Opial's condition are Hilbert spaces, and all $L^p[0,2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [6].

Let D be a nonempty closed convex subset of a Banach space X. Then $I - T$ is *demiclosed at zero* if, for any sequence $\{x_n\}$ in D, condition $x_n \to x$ weakly and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ implies $(I - T)x = 0$.

2. Main results. In this section, we establish some weak and strong convergences for mappings of asymptotically quasi-nonexpansive type in Banach spaces.

Lemma 2.1. *Let D be a nonempty subset of a normed space X and let $T : D \to E$ be a mapping of asymptotically quasi-nonexpansive type. For two given real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$, let a sequence $\{x_n\}$ in D be defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \qquad n = 1,2,\ldots. \end{aligned} \tag{2.1}$$

If p is a fixed point of T, then

(a) $\|x_{n+1} - p\| \leq \|x_n - p\| + (1+\beta_n)\sup_{x\in D}(\|T^n x - p\| - \|x - p\|)$, $n = 1,2,\ldots,$

(b) $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. Let p be a fixed point of T.

(a) From (2.1), we have

$$\begin{aligned} ||x_{n+1} - p|| &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||T^n y_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n (||T^n y_n - p|| \\ &\quad - \|y - p\|) + \alpha_n ||y_n - p|| \end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_n)||x_n-p|| + (||T^n y_n - p|| - ||y_n - p||) \\
&\quad + \alpha_n ((1-\beta_n)||x_n - p|| + \beta_n ||T^n x_n - p||) \\
&\le ||x_n - p|| + (||T^n y_n - p|| - ||y_n - p||) \\
&\quad + \beta_n (||T^n x_n - p|| - ||x_n - p||) \\
&\le ||x_n - p|| + (1+\beta_n) \sup_{x\in D} (||T^n x - p|| - \|x - p\|).
\end{aligned} \tag{2.2}$$

(b) For $m, n \in \mathbb{N}$, we have

$$\begin{aligned}
||x_{n+m} - p|| &\le ||x_{n+m-1} - p|| + 2\sup_{x\in D} (||T^{n+m-1} x - p|| - \|x - p\|) \\
&\le ||x_{n+m-1} - p|| + 2\sup_{x\in D} (||T^m x - p|| - \|x - p\|) \\
&\le ||x_{n+m-2} - p|| + 4\sup_{x\in D} (||T^m x - p|| - \|x - p\|) \\
&\le \cdots \le ||x_n - p|| + 2n\sup_{x\in D} (||T^m x - p|| - \|x - p\|).
\end{aligned} \tag{2.3}$$

Hence, for $n \in \mathbb{N}$,

$$\begin{aligned}
\limsup_{m\to\infty} ||x_m - p|| &\le ||x_n - p|| + 2n \limsup_{m\to\infty} (||T^m x - p|| - \|x - p\|) \\
&\le ||x_n - p||.
\end{aligned} \tag{2.4}$$

It follows that

$$\limsup_{m\to\infty} ||x_m - p|| \le \liminf_{n\to\infty} ||x_n - p||. \tag{2.5}$$

Thus $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Lemma 2.2. *Let D and T be as in Lemma 2.1. For a given real sequence $\{\alpha_n\}$ in $[0,1]$, let a sequence $\{x_n\}$ in D be defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n = 1,2,\ldots. \tag{2.6}$$

If p is a fixed point of T, then

(a) $\|x_{n+1} - p\| \le \|x_n - p\| + \sup_{x\in D}(\|T^n x - p\| - \|x - p\|)$, $n = 1,2,\ldots$,

(b) $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Theorem 2.3. *Let X be a Banach space which satisfies Opial's condition and let D be a weakly compact subset of X. Let T and $\{x_n\}$ be as in Lemma 2.1. Suppose that T has a fixed point, $I - T$ is demiclosed at zero, and $\{x_n\}$ is an approximating fixed-point sequence for T, that is, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. First, we show that $\omega_w(x_n) \subset F(T)$. Let $x_{n_k} \to x$ weakly. By assumption, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since $I - T$ is demiclosed at zero,

$x \in F(T)$. By Opial's condition, $\{x_n\}$ possesses only one weak limit point, that is, $\{x_n\}$ converges weakly to a fixed point of T. □

Theorem 2.4. *Let X be a Banach space which satisfies Opial's condition and let D be a weakly compact subset of X. Let T and $\{x_n\}$ be as in Lemma 2.2. Suppose that T has a fixed point, $I - T$ is demiclosed at zero, and $\{x_n\}$ is an approximating fixed-point sequence for T, that is, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Remark 2.5. Theorem 2.3 improves Theorem 2 of Deng [1] for mappings of asymptotically quasi-nonexpansive type. Theorem 2.4 generalizes Theorem 2.1 of Schu [8].

Theorem 2.6. *Let D be a closed subset of Banach space, let $T : D \to D$ be a mapping of asymptotically quasi-nonexpansive type, and $F(T)$ be nonempty closed set. For two given real sequences $\{\alpha_n\}$ and $\{\beta\}$ in $[0,1]$, let the Ishikawa iterative sequence $\{x_n\}$ in D be defined by (2.1). Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_n d(x_n, F(T)) = 0$.*

Proof. Let $\{x_n\}$ converge strongly to a point $z \in F(T)$. Then $\lim_n d(x_n, F(T)) = 0$. Conversely, suppose $\liminf_n d(x_n, F(T)) = 0$. From Lemma 2.1(a),

$$||x_{n+1} - p|| \le ||x_n - p|| + 2 \sup_{x\in D} (||T^n x - p|| - \|x - p\|) \tag{2.7}$$

for any $n \in \mathbb{N}$ and $p \in F(T)$. Since T is a mapping of asymptotically quasi-nonexpansive type, we have

$$\lim_n \sup_{k\ge n} \left\{ \sup_{x\in D} (||T^k x - p|| - \|x - p\|) \right\} \le 0. \tag{2.8}$$

Hence, there exists a positive integer n_0 and a sequence $\{a_n\}$ of positive real numbers with $\lim_n a_n = 0$ such that

$$\sup_{k\ge n} \left\{ \sup_{x\in D} (||T^k x - p|| - \|x - p\|) \right\} \le a_n \tag{2.9}$$

for any $n \ge n_0$. Without loss of generality, we can assume that $a_n = 1/2n^2$. Hence,

$$\sup_{k\ge n} \left\{ \sup_{x\in D} (||T^k x - p|| - \|x - p\|) \right\} \le \frac{1}{2n^2} \tag{2.10}$$

for any $n \ge n_0$. It follows from (2.7) that

$$||x_{n+1} - p|| \le ||x_n - p|| + \frac{1}{n^2} \tag{2.11}$$

for all $n \geq n_0$, that is,

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \frac{1}{n^2} \tag{2.12}$$

for all $n \geq n_0$. Hence for $n, m \geq n_0$, we have

$$d(x_{n+m}, F(T)) \leq d(x_n, F(T)) + \sum_{i=n}^{n+m-1} \frac{1}{i^2}. \tag{2.13}$$

Using [10, Lemma 1, page 303], we obtain that $\lim_n d(x_n, F(T))$ exists, and it follows from $\liminf_n d(x_n, F(T)) = 0$ that $\lim_n d(x_n, F(T)) = 0$. Thus, $\lim_n d(x_n, F(T)) = 0$. For each $\varepsilon > 0$, there exists a natural number m_0 such that

$$d(x_n, F(T)) < \frac{\varepsilon}{3} \tag{2.14}$$

for all $n \geq m_0$. Then there exists a $p' \in F(T)$ such that $d(x_n, p') < \varepsilon/2$ for all $n \geq m_0$. If $n, m \geq m_0$, then

$$d(x_n, x_m) \leq d(x_n, p') + d(p', x_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{2.15}$$

This shows that $\{x_n\}$ is a Cauchy sequence in D. Let $\lim_n x_n = v \in D$. Since $F(T) \subset D$ is closed and $\lim_n d(x_n, F(T)) = 0$, we conclude that $v \in F(T)$. This completes the proof. □

As a consequence of Theorem 2.6, we obtain the following result.

Theorem 2.7. *Let D be a closed subset of Banach space, let $T : D \to D$ be a mapping of asymptotically quasi-nonexpansive type, and let $F(T)$ be a nonempty closed set. For a given sequence $\{\alpha_n\}$ in $[0,1]$, let the Mann iterative sequence $\{x_n\}$ in D be defined by (2.6). Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_n d(x_n, F(T)) = 0$.*

Remark 2.8. Theorems 2.6 and 2.7 extend corresponding results of Ghosh and Debnath [2], Liu [5], and Petryshyn and Williamson [7] from quasi-nonexpansive or asymptotically quasi-nonexpansive mapping to large class of non-Lipschitzian mappings.

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