ON AN INEQUALITY OF DIANANDA

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We consider certain refinements of the arithmetic and geometric means. The results generalize an inequality of P. Diananda.

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1. Introduction. Let $P_{n,r}(\mathbf{x})$ be the generalized weighted means

$$P_{n,r}(\mathbf{x}) = \left(\sum_{i=1}^{n} q_i x_i^r \right)^{1/r}, \tag{1.1}$$

where $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \to 0^+$ and where $q_i > 0$, $1 \le i \le n$, are positive real numbers with $\sum_{i=1}^{n} q_i = 1$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. In this note, we let $q = \min q_i$ and always assume $n \ge 2$ and $0 \le x_1 < x_2 < \cdots < x_n$.

We define $A_n(\mathbf{x}) = P_{n,1}(\mathbf{x})$, $G_n(\mathbf{x}) = P_{n,0}(\mathbf{x})$, and $H_n(\mathbf{x}) = P_{n,-1}(\mathbf{x})$ and we will write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$, A_n for $A_n(\mathbf{x})$, and similarly for other means when there is no risk of confusion.

For mutually distinct numbers r, s, and t and any real number α and β, we define

$$\Delta_{r,s,t,\alpha,\beta} = \left| \frac{P_{n,r}^{\alpha} - P_{n,t}^{\alpha}}{P_{n,r}^{\beta} - P_{n,s}^{\beta}} \right|, \tag{1.2}$$

where we interpret $P_{n,r}^0 - P_{n,s}^0$ as $\ln P_{n,r} - \ln P_{n,s}$. When $\alpha = \beta$, we define $\Delta_{r,s,t,\alpha}$ to be $\Delta_{r,s,t,\alpha,\alpha}$. As a limit case

$$\Delta_{r,s,t,0} = \lim_{\alpha \to 0} \Delta_{r,s,t,\alpha} = \left| \frac{(\ln (P_{n,r}/P_{n,t}))}{(\ln (P_{n,r}/P_{n,s}))} \right|. \tag{1.3}$$

Bounds for $\Delta_{r,s,t,\alpha,\beta}$ have been studied by many mathematicians. For the case $\alpha \ne \beta$, we refer the reader to [2, 5, 7] for the detailed discussions. When $\alpha = \beta$, we can bound $\Delta_{r,s,t,\alpha}$ in terms of r, s, and t only, due to the following result of Hsu [6] (see also [1]).

THEOREM 1.1. *For $r > s > t > 0$,*

$$1 < \Delta_{r,s,t,1} < \frac{s(r-t)}{t(r-s)}. \tag{1.4}$$

It is also interesting to consider the following bounds:

$$f_{r,s,t,\alpha}(q) \geq \Delta_{r,s,t,\alpha} \geq g_{r,s,t,\alpha}(q), \tag{1.5}$$

where $f_{r,s,t,\alpha}(q)$ is a decreasing function of q and $g_{r,s,t,\alpha}(q)$ is an increasing function of q.

The cases $r = 1$, $s = 0$, $t = -1$, $\alpha = 0$, $q_i = 1/n$, $1 \leq i \leq n$, in (1.5) with $f_{1,0,-1,0}(q) = n$ and $g_{1,0,-1,0}(q) = n/(n-1)$, are the famous Sierpiński's inequality [9].

Another cases, $r = 1$, $s = 1/2$, $t = 0$, and $\alpha = 1$ with $f_{1,1/2,0,1}(q) = 1/q$ and $g_{1,1/2,0,1}(q) = 1/(1-q)$ were proved by Diananda [3, 4] (see also [1, 8]), originally stated as

$$\frac{1}{q}\Sigma_n \geq A_n - G_n \geq \frac{1}{1-q}\Sigma_n, \tag{1.6}$$

where $\Sigma_n = \sum_{1 \leq i < j \leq n} q_i q_j (x_i^{1/2} - x_j^{1/2})^2$.

The main purpose of this note is to generalize Diananda's result, which is given by Theorem 3.1.

2. Lemmas

LEMMA 2.1. *For* $0 \leq q \leq 1/2$,

$$\frac{r-1}{r} - (1 - q^{r-1}) \leq 0 \quad (r \geq 2), \tag{2.1}$$

$$\left|\frac{r-1}{r}\right| \geq \left|1 - (1-q)^{r-1}\right| \quad (0 < r \leq 2) \tag{2.2}$$

with equality holding if and only if $r = 2$ *and* $q = 1/2$.

PROOF. We prove (2.1) here and the proof for (2.2) is similar. It suffices to prove (2.1) for $q = 1/2$, which is equivalent to $2^r \geq 2r$. Notice that the two curves $y = 2^r$ and $y = 2r$ only intersect at $r = 1$ and $r = 2$ in which cases they are equal and the conclusion then follows. □

LEMMA 2.2. *For* $0 < q \leq 1$, *the function*

$$f(q) = \left|\frac{q}{1 - (1-q)^{r-1}}\right| \tag{2.3}$$

is decreasing for $0 < r \neq 1 < 2$ *and increasing for* $r > 2$.

PROOF. We prove the case $1 < r \neq 2$ here and the case $0 < r < 1$ is similar. We have

$$f'(q) = \frac{1 - (1-q)^{r-1} - q(r-1)(1-q)^{r-2}}{(1 - (1-q)^{r-1})^2} \tag{2.4}$$

and by the mean value theorem, $1 - (1-q)^{r-1} = q(r-1)\eta^{r-2}$, where $1 - q < \eta < 1$, which implies $f'(q) \leq 0$ for $1 < r < 2$ and $f'(q) \geq 0$ for $r > 2$. □

LEMMA 2.3. *For $0 < r \neq 1 < 2$ and $0 < q \leq 1/2$,*

$$\left| \frac{1/2}{1 - 1/r} \right| < \left| \frac{q}{1 - (1-q)^{r-1}} \right|. \tag{2.5}$$

If $r > 2$, (2.5) is valid with ">" instead of "<".

PROOF. We prove the case $1 < r < 2$ here and the other cases are similar. By Lemma 2.2, it suffices to show (2.5) for $q = 1/2$. In this case, (2.5) is equivalent to (2.2). □

3. The main theorems

THEOREM 3.1. *For any $t \neq 0$,*

$$\Delta_{t,t/r,0,t} \geq \frac{1}{1 - q^{r-1}} \quad (r \geq 2), \tag{3.1}$$

$$\Delta_{t,t/r,0,t} \leq \left| \frac{1}{1 - (1-q)^{r-1}} \right| \quad (0 < r \neq 1 \leq 2) \tag{3.2}$$

with equality holding if and only if $n = 2$, $x_1 = 0$, and $q_2 = q$ for (3.1) and $n = 2$, $x_1 = 0$, and $q_1 = q$ for (3.2), except in the trivial cases $r = n = 2$ and $q_1 = q_2 = 1/2$.

PROOF. Since the proofs of (3.1) and (3.2) are very similar, we only prove (3.1) here and we just point out that (2.2) is needed for the proof of (3.2). The case $r = 2$ was treated in [3], so we will assume that $r > 2$ from now on. First, consider the case $t = 1$ and define

$$D_n(\mathbf{x}) = (1 - q^{r-1})(A_n - G_n) - (A_n - P_{n,1/r}), \tag{3.3}$$

and then we have

$$\frac{1}{q_n} \frac{\partial D_n}{\partial x_n} = (1 - q^{r-1})\left(1 - \frac{G_n}{x_n}\right) - \left(1 - \left(\frac{P_{n,1/r}}{x_n}\right)^{1-1/r}\right). \tag{3.4}$$

By a change of variables: $x_i/x_n \to x_i$, $1 \leq i \leq n$, we may assume $0 \leq x_1 < x_2 < \cdots < x_n = 1$ in (3.4) and rewrite it as

$$g_n(x_1,\ldots,x_{n-1}) := (1 - q^{r-1})(1 - G_n) - \left(1 - (P_{n,1/r})^{1-1/r}\right). \tag{3.5}$$

We want to show that $g_n \geq 0$. Let $\mathbf{a} = (a_1,\ldots,a_{n-1}) \in [0,1]^{n-1}$ be the point in which the absolute minimum of g_n is reached.

We may assume $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. If $a_i = a_{i+1}$ for some $1 \leq i \leq n-2$ or $a_{n-1} = 1$, by combing a_i with a_{i+1} and q_i with q_{i+1} or a_{n-1} with 1 and q_{n-1} with q_n, we can reduce the determination of the absolute minimum of g_n to

that of g_{n-1} with different weights. Thus, without loss of generality, we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$. If **a** is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$ and (3.5) is reduced to

$$g_n = 1 - q^{r-1} - \left(1 - (P_{n,1/r})^{1-1/r}\right). \tag{3.6}$$

It follows that $g_n \geq 0$ is equivalent to $P_{n,1/r} \geq q^r$ while the last inequality is easily verified with equality holding if and only if $n = 2$, $a_1 = 0$, and $q_2 = q$. Thus $g_n \geq 0$ for this case.

Now we may assume $a_1 > 0$ and **a** is an interior point of $[0,1]^{n-1}$, then we obtain

$$\nabla g_n(a_1,\ldots,a_{n-1}) = 0 \tag{3.7}$$

such that $a_1,\ldots,a_{n-1}$ solve the equation

$$-(1 - q^{r-1})\frac{G_n}{x} + \left(1 - \frac{1}{r}\right)(P_{n,1/r})^{-1/r}\left(\frac{P_{n,1/r}}{x}\right)^{1-1/r} = 0. \tag{3.8}$$

The above equation has at most one root (regarding G_n and $P_{n,1/r}$ as constants), so we only need to show that $g_n \geq 0$ for the case $n = 2$. Now by letting $0 < x_1 = x < x_2 = 1$ in (3.5), we get

$$\frac{1}{q_1}g_2'(x) = h(x)x^{1/r-1}, \tag{3.9}$$

where

$$h(x) = \frac{r-1}{r}(q_1 x^{1/r} + q_2)^{r-2} - (1 - q^{r-1})x^{q_1 - 1/r}. \tag{3.10}$$

If $1/r \geq q_1$, then

$$\begin{aligned} h'(x) &= \frac{(r-1)(r-2)}{r^2}q_1 x^{1/r-1}(q_1 x^{1/r} + q_2)^{r-3} \\ &\quad - (1 - q^{r-1})\left(q_1 - \frac{1}{r}\right)x^{q_1 - 1/r - 1} \geq 0 \end{aligned} \tag{3.11}$$

which implies

$$h(x) \leq h(1) = \frac{r-1}{r} - (1 - q^{r-1}) < 0 \tag{3.12}$$

for $r > 2$ and $q \leq 1/2$ by Lemma 2.1, and thus $g_2(x) \geq g_2(1) = 0$.

If $q_1 > 1/r$, we have

$$\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+}\left(\frac{r-1}{r}(q_1 x^{1/r} + q_2)^{r-2} - (1 - q^{r-1})x^{q_1 - 1/r}\right) > 0, \tag{3.13}$$

$$\begin{aligned} \lim_{x \to 1^-} h(x) &= \lim_{x \to 1^-}\left(\frac{r-1}{r}(q_1 x^{1/r} + q_2)^{r-2} - (1 - q^{r-1})x^{q_1 - 1/r}\right) \\ &= \frac{r-1}{r} - (1 - q^{r-1}) < 0. \end{aligned} \tag{3.14}$$

Notice here that any positive root of $h(x)$ also satisfies the equation

$$P(x) = q_1 x^{1/r} + q_2 - (C x^{q_1 - 1/r})^{1/(r-2)} = 0, \tag{3.15}$$

where $C = r(1 - q^{r-1})/(r-1)$.

It is easy to see that $P'(x)$ can have at most one positive root. Thus by Rolle's theorem, $P(x)$ hence $h(x)$ can have at most two roots in $(0,1)$. Equations (3.13) and (3.14) further imply $h(x)$, hence $g_2'(x)$ has exactly one root x_0 in $(0,1)$. Since (3.14) shows that $g_2'(1) < 0$, $g_2(x)$ takes its maximum value at x_0. Thus $g_2(x) \geq \min\{g_2(0), g_2(1)\} = 0$.

Thus we have shown $g_n \geq 0$, hence $\partial D_n / \partial x_n \geq 0$ with equality holding if and only if $n = 1$ or $n = 2$, $x_1 = 0$, and $q_2 = q$. By letting x_n tend to x_{n-1}, we have $D_n \geq D_{n-1}$ (with weights $q_1, \ldots, q_{n-2}, q_{n-1} + q_n$). Since $1 - q^{r-1}$ is a decreasing function of q, it follows by induction that $D_n > D_{n-1} > \cdots > D_2 = 0$ when $x_1 = 0$ and $q_2 = q$ in D_2 or else $D_n > D_{n-1} > \cdots > D_1 = 0$. Since we assume that $n \geq 2$ in this note, this completes the proof for $t = 1$.

Now for an arbitrary t, a change of variables $x_i \to x_i^t$ in the above cases leads to the desired conclusion. □

We remark here that the constants in (3.1) and (3.2) are best possible by considering the cases $n = 2$, $x_1 = 0$, and $q_2 = q$ or $q_1 = q$. Also when $n = 2$, we conclude from the proof of Lemma 2.1 and $\lim_{x_1 \to x_2} \Delta_{t,t/r,0,t} = r/(r-1)$ that an upper bound in the form of (3.2) does not hold for $\Delta_{1,1/r,0,1}$ when $r > 2$. Similarly, a lower bound in the form of (3.1) does not hold for $1 < r < 2$.

For $t = 1$, rewrite (3.1) as

$$A_n - G_n \geq \frac{1}{1 - q^{r-1}}(A_n - P_{n,1/r}). \tag{3.16}$$

When $n = 2$, we have

$$\lim_{x_1 \to x_2} \frac{(A_2 - P_{2,1/2})/(1-q)}{(A_2 - P_{2,1/r'})/(1 - q^{r'-1})} = \frac{(1/2)/(1-q)}{(1 - 1/r')/(1 - q^{r'-1})}. \tag{3.17}$$

By considering $q = 0, 1/2$, we find that the right-hand side of (3.16) is not comparable for $r = 2$ and any $r' > 2$.

However, for the comparison of the left-hand side of (3.2), we have the following theorem.

THEOREM 3.2. *For any $t \neq 0$ and $0 < r \neq 1 < 2$,*

$$\left| \frac{q}{1 - (1-q)^{r-1}} \right| \geq \Delta_{t,t/r,t/2,t}. \tag{3.18}$$

If $r \geq 2$, (3.18) is valid with "$\leq$" instead of "$\geq$" with equality holding in all the cases if and only if $n = 2$, $x_1 = 0$, and $q_1 = q$.

PROOF.   Since the proofs are similar, we only prove the case $1 < r < 2$ here. Notice by Lemma 2.2 that $q/(1-(1-q)^{r-1})$ is decreasing with respect to q, so we can prove by induction as we did in the proof of Theorem 3.1. First consider the case $t = 1$ and define

$$E_n(\mathbf{x}) = q(A_n - P_{n,1/r}) - (1-(1-q)^{r-1})(A_n - P_{n,1/2}), \qquad (3.19)$$

so

$$\frac{1}{q_n}\frac{\partial E_n}{\partial x_n} = q\left(1 - \left(\frac{P_{n,1/r}}{x_n}\right)^{1-1/r}\right) - (1-(1-q)^{r-1})\left(1 - \left(\frac{P_{n,1/2}}{x_n}\right)^{1/2}\right). \quad (3.20)$$

By a change of variables: $x_i/x_n \to x_i$, $1 \le i \le n$, we may assume $0 < x_1 < x_2 < \cdots < x_n = 1$ in (3.20) and rewrite it as

$$h_n(x_1,\ldots,x_{n-1}) := q\left(1 - (P_{n,1/r})^{1-1/r}\right) - (1-(1-q)^{r-1})(1 - P_{n,1/2}^{1/2}). \quad (3.21)$$

We want to show that $h_n \ge 0$. Let $\mathbf{a} = (a_1,\ldots,a_{n-1}) \in [0,1]^{n-1}$ be the point in which the absolute minimum of h_n is reached. Similar to the proof of Theorem 3.1, we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$. If $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$, and we can regard h_n as a function of $a_2,\ldots,a_{n-1}$, then we obtain

$$\nabla h_n(a_2,\ldots,a_{n-1}) = 0. \qquad (3.22)$$

Otherwise $a_1 > 0$, $\mathbf{a}$ is an interior point of $[0,1]^{n-1}$ and

$$\nabla h_n(a_1,\ldots,a_{n-1}) = 0. \qquad (3.23)$$

In either cases, $a_2,\ldots,a_{n-1}$ solve the equation

$$-q\left(1 - \frac{1}{r}\right)(P_{n,1/r})^{-1/r}\left(\frac{P_{n,1/r}}{x}\right)^{1-1/r} + \frac{1}{2}(1-(1-q)^{r-1})x^{-1/2} = 0. \quad (3.24)$$

The above equation has at most one root (regarding $P_{n,1/r}$ as a constant), so we only need to show $h_n \ge 0$ for the case $n = 3$ with $0 = x_1 < x_2 = x < x_3 = 1$ in (3.21). In this case we regard h_3 as a function of x and we get

$$\frac{1}{q_2}h_3'(x) = -q\frac{r-1}{r}\left(q_2 x^{1/r} + q_3\right)^{r-2}x^{1/r-1} + \frac{1}{2}(1-(1-q)^{r-1})x^{-1/2}. \quad (3.25)$$

Let x be a critical point, then $h_3'(x) = 0$. Similar to the proof of Theorem 3.1, there can be at most two roots in $[0,1]$ for $h_3'(x) = 0$.

Further notice that

$$\lim_{x\to 1^-}\frac{1}{q_2}h_3'(x) = -q\frac{r-1}{r}(1-q_1)^{r-2} + \frac{1-(1-q)^{r-1}}{2} < 0 \qquad (3.26)$$

by Lemma 2.3 and

$$\lim_{x \to 0^+} \frac{1}{q_2} h_3'(x) = +\infty. \tag{3.27}$$

It then follows that $h_3'(x)$ has exactly one root x_0 in $(0,1)$ and $h_3'(1) < 0$ implies that $h_3(x)$ takes its maximum value at x_0. Thus

$$h_3(x) \geq \min\{h_3(0), h_3(1)\} \geq 0, \tag{3.28}$$

where the last inequality follows from Lemma 2.2. Thus $D_n \geq 0$ with equality holding if and only if $n = 2$, $x_1 = 0$, and $q_1 = q$, and a change of variables $x_i \to x_i^t$ completes the proof. □

Notice here for $1 < r < 2$, by setting $t = 1$ and letting $q \to 0$ in (3.18) while noticing that $q/(1 - (1 - q)^{r-1})$ is a decreasing function of q, we get

$$\Delta_{1,1/r,1/2,1} \leq \frac{1}{r-1}, \tag{3.29}$$

a special case of Theorem 1.1, which shows that, in this case, Theorem 3.2 refines Theorem 1.1.

We end the note by refining a result of the author [5].

THEOREM 3.3. *If $x_1 \neq x_n$, $n \geq 2$, then for $1 > s \geq 0$ and $\sigma_{n,1} = \sum_{i=1}^n q_i(x_i - A_n)^2$,*

$$\frac{P_{n,s}^{1-s} - x_1^{1-s}}{2x_1^{1-s}(A_n - x_1)} \sigma_{n,1} - q \frac{(A_n - P_{n,s})^2}{2(A_n - x_1)}$$
$$> A_n - P_{n,s} > \frac{x_n^{1-s} - P_{n,s}^{1-s}}{2x_n^{1-s}(x_n - A_n)} \sigma_{n,1} + q \frac{(A_n - P_{n,s})^2}{2(x_n - A_n)}. \tag{3.30}$$

PROOF. We will prove the right-hand side inequality and the proofs for the left-hand side inequality are similar. Let

$$F_n(\mathbf{x}) = (x_n - A_n)(A_n - P_{n,s}) - \frac{x_n^{1-s} - P_{n,s}^{1-s}}{2x_n^{1-s}} \sigma_{n,1} - q \frac{(A_n - P_{n,s})^2}{2}. \tag{3.31}$$

We want to show by induction that $F_n \geq 0$. We have

$$\frac{\partial F_n}{\partial x_n} = \left(1 - q_n - q q_n \left(1 - \left(\frac{P_{n,s}}{x_n}\right)^{1-s}\right)\right)(A_n - P_{n,s})$$
$$- \frac{1-s}{2x_n} \left(\frac{P_{n,s}}{x_n}\right)^{1-s} \left(1 - \left(\frac{x_n}{P_{n,s}}\right)^s q_n\right) \sigma_{n,1} \tag{3.32}$$
$$\geq (1 - q_n) \left(\frac{P_{n,s}}{x_n}\right)^{1-s} \left(A_n - P_{n,s} - \frac{1-s}{2x_n} \sigma_{n,1}\right) \geq 0,$$

where the last inequality holds by [5, Theorem 2.1]. Thus by a similar induction process as the one in the proof of Theorem 3.1, we have $F_n \geq 0$. Since not all the x_i's are equal, we get the desired result. □

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