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SYNGE-BEIL AND RIEMANN-JACOBI JET STRUCTURES WITH APPLICATIONS TO PHYSICS

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In the framework of geometrized first-order jet approach, we study the Synge-Beil generalized Lagrange jet structure, derive the canonic nonlinear and Cartan connections, and infer the Einstein-Maxwell equations with sources; the classical ansatz is emphasized as a particular case. The Lorentz-type equations are described and the attached Riemann-Jacobi structures for two certain uniparametric cases are presented.

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1. Preliminaries. Let $(T, h_{\alpha\beta})$ and (M, γ_{ij}) be two $\mathscr{C}^\infty$ pseudo-Riemannian manifolds of dimensions m and n, respectively. We denote by $\zeta = (E = j^1(T, M), \pi, T \times M)$ the first-order jet bundle of mappings $\varphi : T \to M$, with the local coordinates

$$(t^\alpha, x^i, y^A)_{(\alpha,i,A)\in I_*} \equiv (y^\mu)_{\mu\in I}. \tag{1.1}$$

Throughout the paper, we consider the sets

$$\begin{gathered} I = I_h \cup I_v, \qquad I_h = I_{h_1} \cup I_{h_2}, \qquad I_{h_1} = \overline{1, m}, \qquad I_{h_2} = \overline{m+1, m+n}, \\ I_v = \overline{m+n+1, N}, \qquad I_* = I_{h_1} \times I_{h_2} \times I_v, \qquad N = m+n+mn; \end{gathered} \tag{1.2}$$

the indices will implicitly take the values

$$\alpha, \beta, \ldots \in I_{h_1}, \qquad i, j, \ldots \in I_{h_2}, \qquad A, B, \ldots \in I_v, \qquad \lambda, \mu, \ldots \in I. \tag{1.3}$$

For $A = m+n+n(i-m-1)+\alpha$, we will denote $A \equiv \binom{i}{\alpha}$ and $y^A \equiv x^{\binom{i}{\alpha}} = \partial x^i / \partial t^\alpha$.

We endow E with the *sub-Riemannian Synge-Beil metric* (see [9])

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t) g_{ij}(t, x, y), \tag{1.4}$$

where

$$g_{ij}(t, x, y) = \gamma_{ij}(x) + \varepsilon U_i(t, x, y) U_j(t, x, y), \quad \forall i, j \in I_{h_2}, \ \varepsilon \in \{\pm 1\}, \tag{1.5}$$

and $U_i(t,x,y)$ is a distinguished 1-form on E (see [1]). We call $(E,\tilde{g})$ *the Synge-Beil (SB) jet model*. The inverse of g_{ij} is $g^{ij} = \gamma^{ij} - \varepsilon\Theta U^i U^j$, where $U^i = \gamma^{ij}U_j$, $\Theta = (1+U_*)^{-1}$, and the star index denotes transvection with U^i.

We remark the important particular Synge-Beil uniparametric (SBU) *autonomous normalized case*, where $m = 1$, $s = t^1 = t$, and $h_{11} = 1$, for which we can use *the Finsler-Lagrange tangent space notations* from [5]. Shifting the indices left by one unit (hence, $I_{h_2} = \overline{1,n}$, $I_v = \overline{n+1,2n}$), we have $y^A \equiv y^{\binom{i}{1}} \overset{\text{not}}{=} y^i$. In this case, considering

$$U_i = \left[k\left(1-n^{-2}(x,y)\right)\right]^{1/2} y_i, \quad k > 0, \tag{1.6}$$

we encounter three important extensively studied cases.

(I) The *Synge classical framework* (see [10]), obtained for $\varepsilon = 1$ and $k = 1$, where $y_i = \gamma_{ij}y^j$, $n(x,y)$ is the refraction index of relativistic optics (see [7, 9]), and the direction $y = X(x)$ is provided by a vector field $X \in \mathscr{X}(M)$.

(II) If the potentials U_i in (1.5) are 0-homogeneous relative to y, in the limit case $n \to \infty$ with $\varepsilon = 1$, $k \in \mathscr{F}(M)$, we have

$$g_{ij}(x,y) = \gamma_{ij}(x) + k \cdot U_i(x,y)U_j(x,y), \tag{1.7}$$

and we may consider the Finsler fundamental function $F = \sqrt{L}$, where

$$L = g_{ij}(x,y)y^i y^j. \tag{1.8}$$

This is the relativistic Beil-type metric (see [3, 4]) with the two intensively studied subcases

$$U_i \in \left\{\left(\gamma_{jk}v^j v^k\right)^{-1/2} v_i,\ \left(s_j(x)v^j\right)^{-1} v_i\right\}, \tag{1.9}$$

where $s \in \mathscr{X}^*(M)$, and $v \in X^*(TM)$ is 0-homogeneous in y.

(III) The *generalized Lagrange model* of relativistic optics studied by Miron and Kawaguchi (see [6, 7]) is obtained as limit case $n \to \infty$ with $\varepsilon = 1$, $k = 1/c^2$ (c = speed of light), where the metric is

$$g_{ij}(x,y) = \gamma_{ij}(x) + c^{-2} \cdot y_i y_j, \quad \forall i,j \in I_{h_2}. \tag{1.10}$$

Considering the general SB-jet case (1.5), we can fix a priori on E a *nonlinear connection* $N = \{N_\mu^A\}_{\mu\in I_h, A\in I_v}$ of coefficients

$$N_{\beta}^{\binom{i}{\alpha}} = -\left|\begin{matrix}\gamma \\ \alpha\beta\end{matrix}\right| y^{\binom{i}{\gamma}}, \qquad N_{j}^{\binom{i}{\alpha}} = \left|\begin{matrix}i \\ jk\end{matrix}\right| y^{\binom{k}{\alpha}}. \tag{1.11}$$

However, an open question (see [9]) addresses the physical significance of choosing an alternative target-nonlinear connection coefficients provided by the spray attached to the Lagrangian

$$L = \tilde{g}_{AB} y^A y^B, \tag{1.12}$$

given by $\tilde{N}_j^{\binom{i}{\alpha}} = N_j^{\binom{i}{\alpha}} + (\varepsilon/2) g^{ik} \partial_\alpha (U_k U_j)$.

The fixed nonlinear connection leads to a splitting $TE = HE \oplus VE$, where $VE = \operatorname{Ker} \pi_*$, and to the associated local adapted basis of vector fields [1, 9]

$$\mathcal{B} = \left\{ \delta_\alpha \equiv \partial_\alpha - N_\alpha^A \delta_A,\ \delta_i \equiv \partial_i - N_i^A \delta_A,\ \delta_A \equiv \dot{\partial}_A = \frac{\partial}{\partial y^A} \right\}_{(\alpha,i,A) \in I_*} \equiv \{\delta_\mu\}_{\mu \in I}, \tag{1.13}$$

where $\partial_\alpha = \partial/\partial t^\alpha$, $\partial_i = \partial/\partial x^i$, of dual basis

$$\begin{aligned} \mathcal{B}^* = \Big\{ & \delta^\alpha \equiv dt^\alpha,\ \delta^i \equiv dx^i, \\ & \delta^A \delta y^A = dy^A + N_\alpha^A dt^\alpha + N_i^A dx^i \Big\}_{(\alpha,i,A) \in I_*} \equiv \{\delta^\mu\}_{\mu \in I}. \end{aligned} \tag{1.14}$$

For N fixed, a linear connection $\nabla = \{L_{\mu\nu}^\lambda\}_{\lambda,\mu,\nu \in I}$ in E has the adapted coefficients provided by $\delta^\lambda(\nabla_{\delta_\nu} \delta_\mu) = L_{\mu\nu}^\lambda$ for all $\lambda, \mu, \nu \in I$; these split into $3^3 = 27$ distinct subsets according to the three index subsets I_{h_1}, I_{h_2}, and I_v.

We endow E with the metric

$$G = \underbrace{h_{\alpha\beta}(t) dt^\alpha \otimes dt^\beta}_{h} + \underbrace{g_{ij}(t,x,y) dx^i \otimes dx^j}_{g} + \underbrace{\tilde{g}_{AB}(t,x,y) \delta y^A \otimes \delta y^B}_{\tilde{g}} \tag{1.15}$$

with g_{ij} given in (1.5). The *Cartan linear connection* has the four essential sets of coefficients

$$\begin{aligned} & L_{\beta\gamma}^\alpha = \left| \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right|, \qquad L_{jk}^i = \left| \begin{matrix} i \\ jk \end{matrix} \right| + \tilde{U}_{jk}^i, \\ & L_{j\alpha}^i = \frac{\left[\varepsilon (U^i U_j)_{;\alpha} - \Theta U^i U^m (U_m U_j)_{;\alpha} \right]}{2}, \\ & L_{jA}^i \equiv L_{j\binom{k}{\alpha}}^i = y^{im} U_{\binom{j}{\alpha}km} - U_{\binom{j}{\alpha}km} \Theta U^i U^m \end{aligned} \tag{1.16}$$

with

$$\begin{aligned} \tilde{U}_{jk}^i &= y^{im} U_{jkm} - \varepsilon\Theta (y_{jk*} - U_{jk*}) U^i, \\ U_{ijm} &= \frac{\varepsilon [\delta_{\{j}(U_m U_{i\}}) - \delta_m (U_i U_j)]}{2}, \\ U_{\binom{i}{\alpha}jm} &= \frac{\varepsilon \left[\delta_{\binom{\{j}{\alpha}} (U_m U_{i\}}) - \delta_{\binom{m}{\alpha}} (U_i U_j) \right]}{2}, \\ y_{ijm} &= |ij;m| = \frac{(\partial_{\{j} y_{mi\}} - \partial_m y_{ij})}{2}, \end{aligned} \tag{1.17}$$

where we denote by α and k the natural covariant derivatives on $(T,h_{\alpha\beta})$ and (M,γ_{ij}), respectively, by $\left|{}^{\alpha}_{\beta\gamma}\right|$ and $\left|{}^{i}_{jk}\right|$ the Christoffel symbols of the metrics h and γ, respectively, and $\tau_{[i\cdots j]} = \tau_{i\cdots j} - \tau_{j\cdots i}$, $\tau_{\{i\cdots j\}} = \tau_{i\cdots j} + \tau_{j\cdots i}$.

The torsion and the curvature of ∇ adapted coefficients are given by

$$\delta^{\lambda}(\mathcal{T}(\delta_\nu,\delta_\mu)) = T^{\lambda}_{\mu\nu}, \quad \delta^{\lambda}(\mathcal{R}(\delta_\nu,\delta_\mu)\delta_\rho) = R^{\lambda}_{\rho\mu\nu}, \quad \forall\lambda,\mu,\nu,\rho \in I. \tag{1.18}$$

In the Cartan connection case, the essential associated *torsion coefficients* are (see [9])

$$\begin{gathered} T^{i}_{\alpha j} = -L^{i}_{j\alpha}, \qquad T^{\binom{i}{\alpha}}_{\beta\binom{k}{\gamma}} = -\delta^{\gamma}_{\alpha} L^{i}_{j\beta}, \qquad T^{\binom{i}{\alpha}}_{j\binom{k}{\gamma}} = -\delta^{\gamma}_{\alpha}\tilde{U}^{i}_{jk}, \qquad T^{i}_{jA} = L^{i}_{jA}, \\ T^{\binom{i}{\alpha}}_{\binom{j}{\beta}\binom{k}{\gamma}} = \delta_{\alpha}{}^{[\beta} L^{i}{}_{j\binom{k}{\gamma]}}, \qquad T^{\binom{i}{\alpha}}_{\beta\gamma} = -\rho^{\delta}_{\alpha\beta\gamma}\mathcal{Y}^{\binom{i}{\delta}}, \\ T^{\binom{i}{\alpha}}_{jk} = \rho^{i}_{jkl}\mathcal{Y}^{\binom{l}{\alpha}}, \qquad T^{\binom{i}{\alpha}}_{\beta j} = 0, \end{gathered} \tag{1.19}$$

where $\rho^{\delta}_{\alpha\beta\gamma}$ and ρ^{i}_{jkl} are the curvature components of the metrics h and γ respectively. The *nonholonomy coefficients* $\omega^{\lambda}_{\mu\nu}$ given by $[\delta_\mu,\delta_\nu] = \omega^{A}_{\mu\nu}\delta_A$, for all $\mu,\nu \in I$, are related to torsion via $T^{\lambda}_{\mu\nu} = L^{\lambda}_{[\mu\nu]} + \omega^{\lambda}_{\mu\nu}$, for all $\lambda,\mu,\nu \in I$, and the essential curvature N-tensor fields (for explicit expressions, see [9]) are

$$R^{\lambda}_{\mu\nu\pi} = \delta_{[\pi} L^{\lambda}_{\mu]\nu} + L^{\sigma}{}_{\mu[\nu} L^{\lambda}_{\sigma\pi]} + L^{\lambda}_{\mu\sigma}\omega^{\sigma}_{\nu\pi}. \tag{1.20}$$

Denoting by $|\alpha$, $|i$, $|A$, and $|\lambda$ the covariant derivations given by ∇_{δ_μ}, for $\mu \in I_{h_1}, I_{h_2}, I_v$, and I, respectively, *the Ricci identities* for $X \in \mathcal{X}(E)$ and $\theta \in \mathcal{X}^*(E)$ are

$$\begin{gathered} X^{\lambda}_{|[\mu|\nu]} = R^{\lambda}_{\sigma\mu\nu}X^{\sigma} - T^{\sigma}_{\mu\nu}X^{\lambda}_{|\sigma}, \\ \theta_{\lambda|[\mu|\nu]} = R^{\sigma}_{\lambda\mu\nu}\theta_\sigma + T^{\sigma}_{\mu\nu}\theta_{\lambda|\sigma}, \quad \forall\lambda,\mu,\nu \in I. \end{gathered} \tag{1.21}$$

The adapted components of the Ricci tensor field are given by $R_{\lambda\mu} = R^{\nu}_{\lambda\mu\nu}$ and the scalar of curvature is $R \equiv G^{\mu\nu}R^{\nu}_{\lambda\mu\nu} = R_h + R_g + R_v$, where

$$\begin{gathered} R_h = h^{\alpha\beta}\rho^{\gamma}_{\alpha\beta\gamma}, \qquad R_g = \left(\gamma^{ij} - \varepsilon\Theta U^i U^j\right)\left(\rho^{k}_{ijk} + U^{k}_{ijk}\right), \\ R_v = \tilde{g}^{AB}R_{AB}, \end{gathered} \tag{1.22}$$

and $U^{i}_{jkl} = \tilde{U}^{i}_{j[k|l]} + \tilde{U}^{m}{}_{j[k}\tilde{U}^{i}_{ml]} + L^{i}_{j\binom{m}{\alpha}}\rho^{m}_{pkl}\mathcal{Y}^{\binom{p}{\alpha}}$.

2. Einstein-Maxwell equations. Denoting by $E_{\mu\nu} = R_{\mu\nu} + (1/2)RG_{\mu\nu}$ the Einstein N-tensor field, *the Einstein equations with sources*

$$E_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}, \quad \mu,\nu \in I, \tag{2.1}$$

split

$$\begin{gathered} R_{\alpha\beta} - \frac{1}{2} R h_{\alpha\beta} = \kappa \mathcal{T}_{\alpha\beta}, \\ R_{ij} - \frac{1}{2} R g_{ij} = \kappa \mathcal{T}_{ij}, \\ R_{AB} - \frac{1}{2} R g_{AB} = \kappa \mathcal{T}_{AB}, \\ 0 = \mathcal{T}_{\alpha i}, \qquad 0 = \mathcal{T}_{\alpha A}, \\ R_{i\alpha} = \kappa \mathcal{T}_{i\alpha}, \qquad R_{A\alpha} = \kappa \mathcal{T}_{A\alpha}, \\ R_{iA} = \kappa \mathcal{T}_{iA}, \qquad R_{Ai} = \kappa \mathcal{T}_{Ai}, \end{gathered} \tag{2.2}$$

where $\mathcal{T} = \mathcal{T}_{\mu\nu} \delta^\mu \otimes \delta^\nu \in \mathcal{T}^0_2(E)$ is the *energy-momentum tensor field* and κ is the cosmological constant. They satisfy the *conservation laws*

$$E^\mu_{\nu|\mu} = \kappa \mathcal{T}^\mu_{\nu|\mu}, \quad \forall \mu \in I = I_{h_1} \cup I_{h_2} \cup I_v, \tag{2.3}$$

where the indices are raised by means of the metric G in (1.15).

We note that for $U_i \equiv 0$, the Einstein equations reduce to the classical ones on $(T \times M, h + g)$. Also, if one considers *the extended electromagnetic* 2*-form*

$$F = F_{A\mu} \delta y^A \wedge \delta y^\mu, \tag{2.4}$$

then the Lagrangian density $\mathcal{L} = (L + F^{\lambda\mu} F_{\lambda\mu}) \sqrt{\det h}$ with L given in (1.12) provides by the Hilbert-Palatini variation the Einstein equations of the form (2.1) with the energy-momentum tensor field

$$T_{\mu\nu} = F_{\mu\rho} F^\rho_\mu - \frac{1}{4} G_{\mu\nu} F^{\rho\pi} F_{\rho\pi}, \quad \mu,\nu \in I. \tag{2.5}$$

In the extended potential Miron-Tatoiu approach [8], the electromagnetic tensor field F has the essential components

$$\begin{aligned} F_{A\beta} &\equiv F_{\binom{i}{\alpha}\beta} = \frac{1}{2} \left(h^{\alpha\gamma} g_{ik} y^{\binom{k}{[\gamma}} \right)_{|\beta]}, \\ F_{Aj} &\equiv F_{\binom{i}{\alpha} j} = \frac{1}{2} d_{\binom{[i}{\alpha} j]} = \left(\gamma_{[ik} \tilde{U}^k_{mj]} + \varepsilon U_{[i} \tilde{U}^k_{mj]} U_k \right) h^{\alpha\beta} y^{\binom{m}{\beta}}, \\ F_{AB} &\equiv F_{\binom{i}{\alpha}\binom{j}{\beta}} = \frac{1}{2} \tilde{g}_{\binom{[i}{\alpha} C} y^C_{|\binom{j]}{\beta}} = \frac{1}{2} U_{\binom{m}{\beta}[ji]} h^{\alpha\gamma} y^{\binom{m}{\gamma}}, \end{aligned} \tag{2.6}$$

derived from the *deflection tensor fields* (detailed in [9])

$$d^A_\mu = \delta^A \nabla_{\delta_\mu} \mathcal{C}, \quad \mu \in I, \ A \in I_v, \tag{2.7}$$

where $\mathcal{C} = y^A \delta_A$ is the Liouville field, by lowering the indices and anti-symmetrization; where the raising/lowering of the indices are assumed to be performed via the metric G. The extended electromagnetic tensor field (2.6) satisfies the following theorem.

Theorem 2.1. *The 2-form F is subject to the two sets of the Maxwell extended equations with sources*

$$
\begin{gathered}
\underset{\alpha\beta\gamma}{S} F_{\binom{i}{\alpha}\beta|\gamma} = \frac{1}{2}\underset{\alpha\beta\gamma}{S} h^{\alpha\varepsilon} g_{ik} y^{\binom{k}{[\varepsilon}}_{|\beta]|\gamma}, \\
F_{\binom{i}{\alpha}j|\beta} = \frac{1}{2}\Big[d_{\binom{[i}{\alpha}\beta|j]} + \Big(y_{\binom{m}{\alpha}} L^m{}_{[i\beta}\Big)_{|j]} - d_{\binom{i}{\alpha}m} L^m_{j\beta}\Big], \\
F_{\binom{i}{\alpha}\binom{j}{\beta}|\gamma} = \frac{1}{2}\Big[d_{\binom{[i}{\alpha}\gamma|\binom{j]}{\beta}} + y_{\binom{m}{\alpha}} \partial_{\binom{[j}{\beta}} L^m_{i]\gamma} \\
-\Big(d_{\binom{[i}{\alpha}\binom{m}{\varepsilon}} + L^k{}_{[i\binom{m}{\varepsilon}} y_{\binom{k}{\alpha}}\Big) L^m_{j]\delta}\delta^\gamma_\beta\Big], \\
\underset{ijk}{S} F_{\binom{i}{\alpha}j|k} = -\frac{1}{2}\underset{ijk}{S}\Big[d_{\binom{i}{\alpha}\binom{m}{\beta}} + L^p_{i\binom{m}{\beta}} y_{\binom{p}{\alpha}}\Big]\rho^m_{jkl} y^{\binom{l}{\beta}}, \\
\underset{ijk}{S}\Big[F_{\binom{i}{\alpha}j|\binom{k}{\beta}} + F_{\binom{i}{\alpha}\binom{j}{\beta}|k}\Big] = 0, \qquad F_{\binom{i}{\alpha}\binom{j}{\beta}|\binom{k}{\gamma}} \equiv 0,
\end{gathered}
\tag{2.8}
$$

$$
\tilde{g}^{BC} F_{B\alpha|C} = -4\pi J_\alpha, \qquad \tilde{g}^{BC} F_{Bi|C} = -4\pi J_i, \qquad G^{\lambda\mu} F_{A\lambda|\mu} = 4\pi J_A, \tag{2.9}
$$

where we denoted by $J = J_\mu \delta^\mu \in \mathcal{X}^*(E)$ the adapted dual electric current and by S the cyclic summation of the corresponding indices below.

The last Maxwell equations in (2.8) were first derived in [9]. Note that in the SBU case with $U_i \equiv 0$, the equations above provide in particular the classical Maxwell equations with sources

$$
\underset{ijk}{S} F_{ij;k} = 0, \qquad \gamma^{ij} F_{ik;j} = -4\pi J_k. \tag{2.10}
$$

3. Extended Lorentz equations. The *extended Lorentz equations* associated to the SB model are $G_{\nu\rho}(\nabla \mathcal{V}^\rho/ds) = F_{A\nu}\mathcal{V}^A$ [2]; denoting $F^\mu_A = G^{\mu\nu}F_{A\nu}$, for all $\mu \in I$; they can be rewritten as

$$
\frac{\nabla \mathcal{V}^\mu}{ds} = F^\mu_A \mathcal{V}^A, \tag{3.1}
$$

where

$$
\begin{gathered}
\mathcal{V} = \mathcal{V}^\mu \delta_\mu, \\
\{\mathcal{V}^\mu\}_{\mu\in I} \equiv \left(\frac{dt^\alpha}{ds}, \frac{dx^i}{ds}, \frac{\delta y^A}{ds} = \frac{dy^A}{ds} + N^A_\beta \frac{dt^\beta}{ds} + N^A_j \frac{dx^j}{ds}\right)_{(\alpha,i,A)\in I_*}
\end{gathered}
\tag{3.2}
$$

is *the covariant velocity* along the trajectory of the moving test-particle

$$
c : J \subset \mathbb{R} \longrightarrow E, \qquad c(s) = \big(t(s), x(s), y(s)\big), \quad \forall s \in J. \tag{3.3}
$$

We have denoted $\nabla \mathcal{V}^\mu/ds = \delta \mathcal{V}^\mu/ds + L^\mu_{\nu\rho}\mathcal{V}^\nu \mathcal{V}^\rho$, for all $\mu \in I$, and will further use the dot notation for expressing the s-derivation. The explicit Lorentz equations were described in [2]. In the SBU case with U_i dependent on x only, the electromagnetic tensors (2.6) have the components

$$F_A^\alpha = 0, \qquad F_A^i \equiv g^{ij} F_{\binom{k}{1} j} = g^{ij} g_{[kl} \tilde{U}^l_{mj]} y^{\binom{m}{1}}, \qquad F_A^B = 0, \tag{3.4}$$

and the Lorentz equations (3.1) reduce to [2]

$$\begin{gathered} \ddot{t}^\alpha + \left|\begin{matrix} \alpha \\ \beta\gamma \end{matrix}\right| \dot{t}^\beta \dot{t}^\gamma = 0, \\ \ddot{x}^i + \left|\begin{matrix} i \\ jk \end{matrix}\right| \dot{x}^j \dot{x}^k = g^{ij} g_{[kl} \tilde{U}^l_{mj]} y^{\binom{m}{1}} \mathcal{V}^{\binom{k}{1}}, \qquad \dot{\mathcal{V}}^A = 0, \end{gathered} \tag{3.5}$$

and hence are characterized by constant vertical adapted velocity vector field.

4. Riemann-Jacobi structure and energy-dependent Lagrangians. We further consider in the SBU framework two particular cases.

(I) *The Miron-Kawaguchi generalized Lagrange case*, for g given by

$$g_{ij}(x,y) = \gamma_{ij}(x) + k \cdot y_i y_j \tag{4.1}$$

with $k \in \mathcal{F}(M)$, where the Lagrangian (1.12) becomes $\tilde{L} = y_0(1 + k y_0)$ and the null index denotes transvection with y. In this case, the Legendre transformation is given locally by $(x,y) \in TM \to (x,p) \in T^*M$, $p_i = \partial L/\partial y^i = 2y_i(1+2ky_0)$, $i = \overline{1,n}$, and is a local diffeomorphism on the set

$$D = \{(x,y) \mid y \neq 0,\ 1+2ky_0 \neq 0\} \subset TM. \tag{4.2}$$

The associated Hamiltonian is

$$H \equiv y^i \frac{\partial L}{\partial y^i} - L = y_0(1+3ky_0) \tag{4.3}$$

and the local *Riemann-Jacobi structure* $(TU, \hat{g})$ provided by the directional variables $U^i = y^i$ is defined by the scaled metric

$$\hat{g}_{ij} \equiv \left(H + \frac{1}{2} U_*\right) \delta_{ij} = \frac{3}{2}(y_0 + 2ky_0^2)\delta_{ij}, \tag{4.4}$$

where $U_* = \delta_{ij} U^i U^j$.

(II) *The flat local Lagrange space with potential energy-dependence* endowed with Riemann-Jacobi generalized Lagrange metric, with the square-type Lagrangian (see [11])

$$\hat{L} = \frac{1}{2}\delta^{ij}(y_i - U_i)(y_j - U_j) = \frac{1}{2} y_0 + U_0 + f, \tag{4.5}$$

where $f = (1/2)U_*$ and the indices are raised/lowered by means of the Kronecker flat Euclidean metric. Here, the Hamiltonian is $H = (1/2)y_0 - f$ and the Legendre transformation provides momenta as potential shifts of direction $p_i = y_i - U_i$. For obtaining the h-paths associated to the Kern nonlinear connection [5, Theorem 7.4.1, page 113], we apply for $g_{ij} = \delta_{ij}$, $a = c = 1/2$, and $b = 1$ the following lemma.

Lemma 4.1. *Let (M, γ_{ij}) be a (pseudo-)Riemannian space. Then*

(a) *the spray and the Kern nonlinear connection of the Lagrangian*

$$L'' = ay_0 + bU_0 + cU_*, \quad a, b, c \in \mathbb{R}, \tag{4.6}$$

with $U_i \in \mathcal{X}^*(M)$, for raising/lowering performed using the metric γ_{ij} and $U_* = U_i U^i$, are, respectively, given by

$$\begin{aligned} G^i(x,y) &\equiv \frac{\gamma^{ij}(\dot{\partial}_j \partial_k L'' \cdot y^k - \partial_j L'')}{4} \\ &= \frac{a}{2}\gamma^i_{00} + \frac{1}{4}\{b[\gamma^{ia}(y^k U^j \partial_{[k}\gamma_{a]j} - y_k \partial_a U^k) + y^k \partial_k U^i] \\ &\qquad - c\gamma^{ia}(U^j U^k \partial_a \gamma_{jk} + 2U_j \partial_a U^j)\}, \end{aligned} \tag{4.7}$$

where γ^i_{jk} are the Christoffel symbols of γ_{ij} and

$$N^i_j(x,y) \equiv \frac{\partial G^i}{\partial y^j} = a\gamma^i_{j0} + \frac{b}{4}[\gamma^{ia}(\partial_{[j}\gamma_{a]k} \cdot y_{jk} \cdot \partial_a U^k) + \partial_j U^i]; \tag{4.8}$$

(b) *the Euler-Lagrange equations associated to L'' are the spray equations*

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0, \qquad y^i = \dot{x}^i, \quad i = \overline{1,n}, \tag{4.9}$$

and the h-paths are given by

$$\ddot{x}^i + N^i_j(x,\dot{x})\dot{x}^j = 0, \qquad y^i = \dot{x}^i, \quad i = \overline{1,n}, \tag{4.10}$$

with the spray and nonlinear connection determined above.

Then the Kern spray for $g_{ij} = (1/2) \cdot (\mathrm{Hess}_y L)_{ij} = \delta_{ij}$ is

$$G^i = \frac{\delta^{ij}(\partial_{[j}U^k{}_{k]} + U_k \partial_j U^k)}{4}, \tag{4.11}$$

and denoting $\Omega_{jk} = \partial_{[j}U_{k]}$, this is rewritten as $G^i = \delta^{ij}(\Omega_{j0} + \partial_j f)$. The associated nonlinear connection is then $N^i_j = (1/4)\Omega^i_j$, and its autoparallel curves (the h-paths) satisfy

$$\ddot{x}^i = \frac{1}{4}\Omega^i_j \dot{x}^j. \tag{4.12}$$

Then we have the following theorem.

Theorem 4.2. *The h-paths described by (4.12) are as well:*

(a) *the extended Lorentz curves (see [2]) particularized to the almost Riemann Lagrange special (ARLS) jet case associated to the flat metric δ_{ij} and to the potential $U_i/2$;*

(b) *the solutions of the Lorentz-Udriste force law (see [11]) of the Riemann-Jacobi-Lagrange structure $(M = \mathbb{R}^n, \hat{g}_{ij}, 4N^i_j)$, where $\hat{g}$ is the Riemann-Jacobi metric*

$$\hat{g}_{ij} = (H + f)\delta_{ij} = \frac{y_0}{2}\delta_{ij}; \tag{4.13}$$

(c) *the stationary curves (see [5]) of the reduced Lagrangian $\hat{L}' = (1/2)y_0 + U_0$.*

Proof. For (a) and (b), the Riemann-Jacobi and the flat metrics have null Christoffel symbols; for (c), we apply the lemma for $g_{ij} = \delta_{ij}$, $a = 1/2$, $b = 1$, and $c = 0$. □

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