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AVERAGING OF MULTIVALUED DIFFERENTIAL EQUATIONS

G. GRAMMEL

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Nonlinear multivalued differential equations with slow and fast subsystems are considered. Under transitivity conditions on the fast subsystem, the slow subsystem can be approximated by an averaged multivalued differential equation. The approximation in the Hausdorff sense is of order $O(\epsilon^{1/3})$ as $\epsilon \to 0$.

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1. Introduction. We consider the nonlinear perturbed multivalued differential equation

$$\dot{z}(t) \in \epsilon F(z(t), y(t)), \qquad \dot{y}(t) \in G(y(t)), \tag{1.1}$$

where $\epsilon > 0$ denotes the small perturbation parameter, $t \in [0, T/\epsilon]$ the time variable, $z(\cdot)$ the slow motion, and $y(\cdot)$ the fast motion.

The fundamental task in perturbation theory is the construction of a limit system which represents the situation of a vanishing perturbation parameter.

In the single-valued case, $F(z,y) = \{f(z,y)\}$ and $G(y) = \{g(y)\}$, that is, in the case of perturbed ordinary differential equations, this construction requires ergodicity properties of the fast subsystem. For instance, if the fast subsystem has a unique invariant measure μ on the compact invariant set N, then the trajectories of the averaged system

$$\dot{z}(t) = \epsilon \int_N f(z(t), y)\, d\mu(y) \tag{1.2}$$

uniformly approximate the slow trajectories of (1.1) on the time intervals $[0, T/\epsilon]$. Once an approximation by an averaged system is achieved, the next task is the determination of approximation orders. Here, one mainly has to know how fast the unique invariant measure can be realized by single trajectories, that is, how fast the unique invariant measure can be approximated by occupation measures.

In the multivalued case, an appropriate notion of invariant measure has been introduced in [1] in order to construct an averaged system. Averaging approaches for the order reduction of differential inclusions with two time scales have been used in [4, 6, 7, 9]. However, to the best knowledge of the author, aside from the periodic case (see [8]), the problem of determining approximation orders for perturbed multivalued differential equations has not been discussed yet.

The main purpose of the present paper is to show that, under transitivity conditions on the fast motion, approximation orders $O(\epsilon^{1/3})$, as $\epsilon \to 0$, can be achieved. This result complements an analogous result on singularly perturbed differential equations in [5], where the same approximation order is deduced from certain mixing properties of the fast flow.

2. Preliminaries and main result.

The setting is as follows.

Assumption 2.1. The state space of (1.1) is $\mathbb{R}^m \times \mathbb{R}^n$. There is a compact subset $N \subset \mathbb{R}^n$ such that $\mathbb{R}^m \times N$ is invariant with respect to (1.1). The set field (F,G) is Lipschitz continuous in the Hausdorff sense on $\mathbb{R}^m \times \mathbb{R}^n$ with Lipschitz constant $L \geq 0$, and has compact, convex, and nonvoid images. The function F is bounded on $\mathbb{R}^m \times N$; there is a constant $P \geq 0$ such that $F(z,y) \subset B_P(0) \subset \mathbb{R}^m$ for all $(z,y) \in \mathbb{R}^m \times N$.

The initial conditions are

$$z(0) = z^0 \in \mathbb{R}^m, \qquad y(0) = y^0 \in N. \tag{2.1}$$

In the sequel, we focus on the fast subsystem

$$\dot{y}(t) \in G(y(t)), \qquad y(0) = y^0 \in N. \tag{2.2}$$

We introduce the *solution map*

$$S_G : N \longrightarrow \mathcal{P}(C([0,\infty);N)) \tag{2.3}$$

which maps every $y^0 \in N$ to the set of solutions of the fast subsystem. Here, a solution is a Lipschitz continuous curve $t \mapsto y(t)$ with $y(0) = y^0$, which fulfils the inclusion of (2.2) for almost all $t \geq 0$.

Then the averaged inclusion is constructed in the following way. First, we define for all $(z,y^0) \in \mathbb{R}^m \times N$ and $S > 0$ the *finite time average*:

$$F_S(z,y^0) := \operatorname{cl}\left(\bigcup_{y(\cdot)\in S_G(y^0)} \frac{1}{S}\int_0^S F(z,y(t))\,dt\right), \tag{2.4}$$

where the integral is understood in the usual sense as the union of the integrals of all measurable selections $v(\cdot) \in F(z,y(\cdot))$. The following transitivity assumption is crucial.

Assumption 2.2. For any $y, y^0 \in N$, there is a time $t \geq 0$ and a solution $y(\cdot) \in S_G(y^0)$ of the fast subsystem with $y = y(t)$.

Lemma 2.3. *Suppose that Assumptions 2.1 and 2.2 are fulfilled. There is a limit set field F_0 on $\mathbb{R}^m$ with closed, uniformly bounded, convex, and nonvoid images such that uniformly in $z \in \mathbb{R}^m$ and $y^0 \in N$, the finite time averages satisfy the estimate*

$$d_H(F_S(z,y^0), F_0(z)) = O(S^{-1/2}), \quad \text{as } S \longrightarrow \infty. \tag{2.5}$$

This limit set field F_0 on $\mathbb{R}^m$ defines the averaged differential inclusion

$$\dot{z}(t) \in \epsilon F_0(z(t)), \quad z(0) = z^0. \tag{2.6}$$

In order to formulate the approximation theorem in a concise way, we define the solution maps for the original and the averaged system. Here, $\mathcal{P}(X)$ denotes the power set of a set X. The function

$$S_{(\epsilon F,G)} : \mathbb{R}^m \times N \longrightarrow \mathcal{P}(C([0,T/\epsilon];\mathbb{R}^m \times N)) \tag{2.7}$$

maps every $(z^0, y^0) \in \mathbb{R}^m \times N$ to the set of solutions of (1.1) and (2.1), and

$$S_{\epsilon F_0} : \mathbb{R}^m \longrightarrow \mathcal{P}(C([0,T/\epsilon];\mathbb{R}^m)) \tag{2.8}$$

maps every $z^0 \in \mathbb{R}^m$ to the set of solutions of (2.6). We remark that the set-valued maps $S_{(\epsilon F,G)}$, S_G, and $S_{\epsilon F_0}$ have compact images; compare, for example, [2, Theorem 3.5.2]. Therefore, we can use the Hausdorff metric, which we denote by $d_H(\cdot,\cdot)$, for the images of the solution map.

Theorem 2.4. *Suppose that Assumptions 2.1 and 2.2 are fulfilled. Then the following estimate is valid:*

$$d_H(\Pi S_{(\epsilon F,G)}(z^0,y^0), S_{\epsilon F_0}(z^0)) = O(\epsilon^{1/3}), \quad \text{as } \epsilon \longrightarrow 0, \tag{2.9}$$

uniformly in $(z^0,y^0) \in \mathbb{R}^m \times N$, where $\Pi : C([0,T/\epsilon],\mathbb{R}^m \times N) \to C([0,T/\epsilon],\mathbb{R}^m)$ is the projection.

What follows is a short discussion on the assumptions.

Assumption 2.1 is standard. First, we mention that the system (1.1) has a particular structure since the fast flow is decoupled. Without this structure, it may happen that the averaged system is not Lipschitz anymore, and no approximation orders can be expected. Concerning the fast flow, Lipschitz continuity is not really needed and could be replaced by upper semicontinuity. In this case, we would need to introduce a uniform bound on $t \geq 0$ in Assumption 2.2.

We have already mentioned that Theorem 2.4 complements a result in [5], where the same approximation order is achieved for ordinary differential equations with decoupled mixing fast flows. But Theorem 2.4 is by no means a multivalued generalization. This is due to the fact that the transitivity condition, formulated in Assumption 2.2, is a typical multivalued feature and reduces to a periodicity condition in the single-valued case. However, the regularity conditions used in the present paper are weaker than in [5], where the vector fields are of class C^1.

3. Proofs

Proof of Lemma 2.3. First, we show that the time $t \geq 0$, which is needed for the transition from one point $y^0 \in N$ to another point $y \in N$ (see Assumption 2.2), is uniformly bounded. For $t \geq 0$ and $y^0 \in N$, we set

$$A(t,y^0) := \bigcup_{y(\cdot)\in S_G(y^0)} \{y(t)\} \tag{3.1}$$

and note that $A(t,y^0) \subset N$ is compact in N according to Assumption 2.1. By Baire's theorem, there is an $n \in \mathbb{N}$ such that $A(n,y^0)$ has interior points in N. Then the claim follows by standard compactness arguments.

As a consequence, we obtain the estimate

$$d_H\left(F_S(z,y_1^0),F_S(z,y_2^0)\right) = O\left(S^{-1}\right), \quad \text{as } S \longrightarrow \infty, \tag{3.2}$$

uniformly in $z \in \mathbb{R}^m$ and $y_1^0, y_2^0 \in N$. We conclude that

$$d_H\left(\frac{1}{k}\sum_{i=1}^{k} F_S(z,y^0), F_{kS}(z,y^0)\right) = O\left(S^{-1}\right), \quad \text{as } S \longrightarrow \infty, \tag{3.3}$$

uniformly in $k \in \mathbb{N}$, holds. Moreover, it follows by a well-known fact from convex analysis that the estimate

$$d_H\left(\frac{1}{k}\sum_{i=1}^{k} F_S(z,y^0), \operatorname{conv} F_S(z,y^0)\right) = O\left(k^{-1}\right), \quad \text{as } k \longrightarrow \infty, \tag{3.4}$$

uniformly in $S > 0$, is true. Combining the last two estimates, we obtain

$$d_H\left(F_{S^2}(z,y^0), \operatorname{conv} F_S(z,y^0)\right) = O\left(S^{-1}\right), \quad \text{as } S \longrightarrow \infty. \tag{3.5}$$

From this we conclude that $F_S(z,y^0)$ is a Cauchy sequence, as $S \to \infty$, and the lemma's statement follows. □

Proof of Theorem 2.4. (I) Let $(T,\epsilon) \in \mathbb{R}^+ \times \mathbb{R}^+$. We divide the time interval in subintervals of the form $[t_l, t_{l+1}]$ which all have the same length $S_\epsilon > 0$, except for the last one which may be smaller. Accordingly, the index l is an element of the index set $I_\epsilon := \{0,\dots,[T/(\epsilon S_\epsilon)]\}$. Later, we will define a map $\epsilon \mapsto S_\epsilon$.

(II) We take some initial values $(z^0, y^0) \in \mathbb{R}^m \times N$ and a solution $(z_\epsilon(\cdot), y_\epsilon(\cdot)) \in S_{(\epsilon F,G)}(z^0, y^0)$. We have

$$z_\epsilon(t_{l+1}) = z_\epsilon(t_l) + \int_{t_l}^{t_{l+1}} \dot{z}_\epsilon(t)\,dt, \tag{3.6}$$

where $\dot{z}_\epsilon(\cdot) \in \epsilon F(z_\epsilon(\cdot), y_\epsilon(\cdot))$ is a measurable selection. For $l \in I_\epsilon$, we set $\xi_0 := z^0$ and

$$\xi_{l+1} := \xi_l + \int_{t_l}^{t_{l+1}} w_l(t)\,dt, \tag{3.7}$$

where $w_l(\cdot) \in \epsilon F(\xi_l, y_\epsilon(\cdot))$ is a particular measurable selection to be specified later.

We define for all $l \in I_\epsilon$,

$$\begin{aligned} \Delta_l &:= \|\xi_l - z_\epsilon(t_l)\|, \\ d_l &:= \max_{t_l \le t \le t_{l+1}} \|z_\epsilon(t) - \xi_l\|. \end{aligned} \tag{3.8}$$

We observe that

$$d_l \le \Delta_l + \epsilon S_\epsilon P. \tag{3.9}$$

According to the Filippov lemma (cf., e.g., [3, Proposition 3.4(b)]), there is a measurable selection $w_l(\cdot) \in \epsilon F(\xi_l, y_\epsilon(\cdot))$ with

$$\|\dot{z}_\epsilon(t) - w_l(t)\| \le \epsilon L d_l \tag{3.10}$$

for almost all $t \in [t_l, t_{l+1}]$. We conclude that

$$\begin{aligned} \Delta_{l+1} &\le \Delta_l + \int_{t_l}^{t_{l+1}} \|\dot{z}_\epsilon(t) - w_l(t)\|\,ds \\ &\le \Delta_l + \epsilon S_\epsilon L d_l \\ &\le \Delta_l(1 + \epsilon S_\epsilon L) + \epsilon^2 S_\epsilon^2 P. \end{aligned} \tag{3.11}$$

Considering that $\Delta_0 = 0$ and $l \le T/\epsilon S_\epsilon$, we obtain

$$\Delta_l \le \epsilon^2 S_\epsilon^2 P \sum_{i=0}^{l-1} (1 + \epsilon S_\epsilon L)^i \le \epsilon S_\epsilon P T e^{TL}. \tag{3.12}$$

According to Lemma 2.3, we can choose a $v_l \in \epsilon F_0(\xi_l)$ such that

$$\left\| v_l - \frac{1}{S_\epsilon} \int_{t_l}^{t_{l+1}} w_l(s)\,ds \right\| \le \epsilon O(S_\epsilon^{-1/2}). \tag{3.13}$$

For all $l \in I_\epsilon$, we define $\eta_0 := z^0$ and

$$\eta_{l+1} := \eta_l + S_\epsilon v_l. \tag{3.14}$$

We interpolate piecewise linearly and set, for $t \in [t_l, t_{l+1}]$,

$$\eta_l(t) := \eta_l + (t - t_l) v_l. \tag{3.15}$$

Obviously, we have for all $l \in I_\epsilon$,

$$\|\eta_l - \xi_l\| \le T O(S_\epsilon^{-1/2}). \tag{3.16}$$

For $t \in [t_l, t_{l+1}]$, we have

$$\begin{aligned} \operatorname{dist}(\dot\eta_l(t), \epsilon F_0(\eta_l(t))) &\le d_H(\epsilon F_0(\xi_l), \epsilon F_0(\eta_l)) + d_H(\epsilon F_0(\eta_l), \epsilon F_0(\eta_l(t))) \\ &\le \epsilon L T O(S_\epsilon^{-1/2}) + \epsilon^2 L S_\epsilon P. \end{aligned} \tag{3.17}$$

According to the Filippov theorem, there is a solution $z_0(\cdot) \in S_{\epsilon F_0}(z^0)$ of the averaged system (2.6) with

$$\|z_0(t_l) - \eta_l\| \le (L T^2 O(S_\epsilon^{-1/2}) + T L \epsilon S_\epsilon P) e^{LT}. \tag{3.18}$$

By (3.12), (3.16), and (3.18), we can estimate

$$\begin{aligned} \|z_\epsilon(t_l) - z_0(t_l)\| &\le \epsilon S_\epsilon P T e^{TL} + T O(S_\epsilon^{-1/2}) \\ &\quad + (L T^2 O(S_\epsilon^{-1/2}) + T L \epsilon S_\epsilon P) e^{LT}. \end{aligned} \tag{3.19}$$

(III) We take an initial value $z^0 \in \mathbb{R}^m$ and a solution $z_0(\cdot) \in S_{\epsilon F_0}(z^0)$ of the averaged differential inclusion (2.6). We furthermore choose an arbitrary initial value $y^0 \in N$. Then we have for all $l \in I_\epsilon$ and almost all $t \in [t_l, t_{l+1}]$,

$$\operatorname{dist}(\dot z_0(t), \epsilon F_0(z_0(t_l))) \le \epsilon^2 L S_\epsilon P. \tag{3.20}$$

By the convexity of $F_0(z_0(t_l))$ we even have for all $l \in I_\epsilon$,

$$\operatorname{dist}\left(\frac{1}{S_\epsilon}\int_{t_l}^{t_{l+1}} \dot{z}_0(t)\,dt, \epsilon F_0(z_0(t_l))\right) \le \epsilon^2 L S_\epsilon P. \tag{3.21}$$

For all $l \in I_\epsilon$, we choose $v_l \in \epsilon F_0(z_0(t_l))$ in a way that

$$\left\|\frac{1}{S_\epsilon}\int_{t_l}^{t_{l+1}} \dot{z}_0(t)\,dt - v_l\right\| \le \epsilon^2 L S_\epsilon P \tag{3.22}$$

and define $\delta_0 := z^0$ and

$$\delta_{l+1} := \delta_l + S_\epsilon v_l. \tag{3.23}$$

Then we can estimate for all $l \in I_\epsilon$,

$$\|z_0(t_l) - \delta_l\| \le T\epsilon L S_\epsilon P. \tag{3.24}$$

For $l \in I_\epsilon$, we define successively $\xi_0 := z^0$ and

$$\xi_{l+1} := \xi_l + S_\epsilon w_l, \tag{3.25}$$

where $w_l \in \epsilon F_{S_\epsilon}(\xi_l, y(t_l))$ is chosen such that

$$\begin{aligned}\|v_l - w_l\| &\le \operatorname{dist}(v_l, \epsilon F_0(\delta_l)) + d_H(\epsilon F_0(\delta_l), \epsilon F_0(\xi_l)) \\ &\quad + d_H(\epsilon F_0(\xi_l), \epsilon F_{S_\epsilon}(\xi_l, y(t_l))) \\ &\le \epsilon^2 L^2 T S_\epsilon P + \epsilon L\|\delta_l - \xi_l\| + \epsilon O(S_\epsilon^{-1/2}).\end{aligned} \tag{3.26}$$

Notice that, by the choice of the $w_l \in \epsilon F_{S_\epsilon}(\xi_l, y(t_l))$, we obtain successively a trajectory $y_\epsilon(\cdot)$ of the fast subsystem of (1.1). Then we have for all $l \in I_\epsilon$,

$$\begin{aligned}\|\delta_{l+1} - \xi_{l+1}\| &\le \|\delta_l - \xi_l\| + S_\epsilon\|v_l - w_l\| \\ &\le \|\delta_l - \xi_l\|(1 + \epsilon S_\epsilon L) + \epsilon^2 S_\epsilon^2 L^2 T P + \epsilon S_\epsilon O(S_\epsilon^{-1/2}).\end{aligned} \tag{3.27}$$

Since $l \le T/(\epsilon S_\epsilon)$ for all $l \in I_\epsilon$ and $\|\delta_0 - \xi_0\| = 0$, we conclude that

$$\begin{aligned}\|\delta_l - \xi_l\| &\le (\epsilon^2 S_\epsilon^2 L^2 T P + \epsilon S_\epsilon O(S_\epsilon^{-1/2}))\sum_{i=0}^{l-1}(1 + \epsilon S_\epsilon L)^i \\ &\le (L^2 T\epsilon S_\epsilon P + O(S_\epsilon^{-1/2}))e^{LT}.\end{aligned} \tag{3.28}$$

Similarly, as in part (II) of this proof, we can estimate for all $l \in I_\epsilon$,

$$||\xi_l - z_\epsilon(t_l)|| \le \epsilon S_\epsilon P T e^{LT} \tag{3.29}$$

for a solution $z_\epsilon(\cdot)$ of the slow subsystem of (1.1). By (3.24), (3.28), and (3.29), we can estimate

$$||z_0(t_l) - z_\epsilon(t_l)|| \le \epsilon S_\epsilon TLP + (\epsilon S_\epsilon L^2 TP + O(S_\epsilon^{-1/2}))e^{TL} + \epsilon S_\epsilon PTe^{LT}. \tag{3.30}$$

(IV) Considering that, for $t \in [t_l, t_{l+1}]$, we have

$$||z_0(t) - z_0(t_l)|| \le \epsilon S_\epsilon P, \qquad ||z_\epsilon(t) - z_\epsilon(t_l)|| \le \epsilon S_\epsilon P, \tag{3.31}$$

the claim follows by (3.19) and (3.30), setting

$$S_\epsilon := \epsilon^{-2/3}. \tag{3.32}$$

□

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G. Grammel: Center for Mathematics M6, Technical University München, Arcisstr. 21, 80290 München, Germany

E-mail address: grammel@appl-math.tu-muenchen.de