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INVARIANT MANIFOLD OF HYPERBOLIC-ELLIPTIC TYPE FOR NONLINEAR WAVE EQUATION

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It is shown that there are plenty of hyperbolic-elliptic invariant tori, thus quasiperiodic solutions for a class of nonlinear wave equations.

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1. Introduction and results. In this paper, we deal with the existence of the invariant tori of the nonlinear wave equation

$$u_{tt} = u_{xx} - V(x)u - f(u) \tag{1.1}$$

subject to Dirichlet boundary conditions

$$u(t,0) = 0 = u(t,\pi), \quad -\infty < t < +\infty, \tag{1.2}$$

where the potential V is in the square-integrable function space $L^2[0,\pi]$ and f is a real analytic, odd function of u of the form

$$f(u) = au^3 + \sum_{k\geq 5} f_k u^k, \quad a \neq 0. \tag{1.3}$$

This class of equations comprises the sine-Gordon, the sinh-Gordon, and the ϕ^4-equation, given by

$$V(x)u + f(u) = \begin{cases} \sin u, \\ \sinh u, \\ u + u^3, \end{cases} \tag{1.4}$$

respectively.

The existence of solutions, periodic in time, for nonlinear wave (NLW) equations has been studied by many authors. A wide variety of methods such as bifurcation theory and variational techniques have been brought on this problem. See [2, 3, 7, 10, 11, 12], for example. There are, however, relatively less methods to find a quasiperiodic solutions of NLW. The KAM (Kolmogorov-Arnold-Moser) theory is a very powerful tool in order to construct families of quasiperiodic solutions, which are on an invariant manifold, for some nearly integrable Hamiltonian systems of finitely or infinitely many degrees of freedom. Some partial differential equations such as (1.1) may be viewed as an infinitely dimensional Hamiltonian system. On this line, Wayne [13] obtained the time-quasiperiodic solutions of (1.1) when the potential V is lying on the outside of the set of some "bad" potentials. In [13], the set of all potentials is given some Gaussian measure and then the set of bad potentials is of small measure. However, this excludes the constant-value potential $V(x) \equiv m \in \mathbb{R}^+$. Bobenko and Kuksin [1], Kuksin [4], and Pöschel [9] (in alphabetical order) investigated this case. In order to get a family of n-dimensional invariant tori by an infinitely dimensional version of KAM theorem developed by Kuksin [4] and Pöschel [9], it is necessary to assume that there are n parameters in the Hamiltonian corresponding to (1.1). When $V(x) \equiv m > 0$, these parameters can be extracted from the nonlinear term $f(u)$ by Birkhoff normal form. Therefore, it was shown that for arbitrarily given positive integer n, there are a family of n-dimensional elliptic invariant tori when $V(x) \equiv m > 0$. See [9] for the details. By [9, Remark 7, page 274], the same result holds also true for the parameter values $-1 < m < 0$. A natural question is whether or not the same result holds true for the potential $V(x) \equiv m < -1$. The aim of this present paper is to give an answer to the question.

From now on, we assume that $V(x) \equiv m \in (-\infty, -1)$. To give the statement of our results, we need to introduce some notations. We study (1.1) as an infinitely dimensional Hamiltonian system. Following Pöschel [9], the phase space we may take, for example, is the product of the usual Sobolev spaces $\mathcal{W} = H_0^1([0,\pi]) \times L^2([0,\pi])$ with coordinates u and $v = u_t$. The Hamiltonian is then

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \int_0^\pi g(u)\,dx, \tag{1.5}$$

where $A = d^2/dx^2 + m$, $g = \int_0 f(s)\,ds$, and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in L^2. The Hamiltonian equations of motion are

$$u_t = \frac{\partial H}{\partial v} = v, \qquad v_t = -\frac{\partial H}{\partial u} = -Au - f(u). \tag{1.6}$$

Our aim is to construct time-quasiperiodic solutions of small amplitude. Such quasiperiodic solutions can be written in the form

$$u(t,x) = U(\omega_1 t, \dots, \omega_n t, x), \tag{1.7}$$

where $\omega_1, \dots, \omega_n$ are rationally independent real numbers which are called the basic frequency of u, and U is an analytic function of period 2π in the first n arguments. Thus, u admits a Fourier series expansion

$$u(t,x) = \sum_{k \in \mathbb{Z}^n} e^{\sqrt{-1}\langle k, \omega\rangle t} U_k(x), \tag{1.8}$$

where $\langle k,\omega\rangle = \sum_j k_j\omega_j$. Since the quasiperiodic solutions, to be constructed, are of small amplitude, (1.1) may be considered as the linear equation $u_{tt} = u_{xx} - mu$ with a small nonlinear perturbation f. For $j \in \mathbb{N}$, let

$$\phi_j = \sqrt{\frac{2}{\pi}}\sin jx, \qquad \lambda_j = \sqrt{j^2+m} \tag{1.9}$$

be the basic modes and frequencies of the linear system subject to Dirichlet boundary conditions, respectively. Then every solution of the linear system is the superposition of their harmonic oscillations and of the form

$$u(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x), \quad q_j(t) = y_j\cos\left(\lambda_j t + \phi_j^0\right) \tag{1.10}$$

with amplitude $y_j \geq 0$ and initial phase ϕ_j^0. The solution $u(t,x)$ is periodic, quasiperiodic, or almost periodic depending on whether one, finitely many, or infinitely many modes are excited, respectively. In particular, for the choice

$$J = \{j_0+1,\ j_0+2,\dots,j_0+n\} \subset \mathbb{N}, \quad \text{with } (j_0+1)^2+m > 0, \tag{1.11}$$

of finitely many modes, there is an invariant $2n$-dimensional linear subspace E_J that is completely foliated into rational tori with frequencies $\lambda_{j_0+1},\dots,\lambda_{j_0+n}$,

$$\begin{aligned} E_J &= \{(u,v) = (q_{j_0+1}\phi_{j_0+1}+\cdots+q_{j_0+n}\phi_{j_0+n}, \dot{q}_{j_0+1}\phi_{j_0+1}+\cdots+\dot{q}_{j_0+n}\phi_{j_0+n})\} \\ &= \bigcup_{y\in\bar{\mathbb{P}}^n} \mathcal{T}_j(y), \end{aligned} \tag{1.12}$$

where $\mathbb{P}^n = \{y \in \mathbb{R}^n : y_j > 0 \text{ for } 1 \leq j \leq n\}$ is the positive quadrant in $\mathbb{R}^n$ and

$$\mathcal{T}_J(y) = \{(u,v) : q_{j_0+j}^2 + \lambda_{j_0+j}^{-2}\dot{q}_{j_0+j}^2 = y_j, \text{ for } 1 \leq j \leq n\}. \tag{1.13}$$

Upon restoring the nonlinearity f, the invariant manifold E_J with their quasiperiodic solutions will not persist in their entirety due to resonance among the modes and the strong perturbing effect of f for large amplitudes. In a sufficiently small neighborhood of the origin, however, there does persist a large Cantor subfamily of rotational n-tori which are only slightly deformed. More exactly, we have the following theorem.

Theorem 1.1. *Suppose that the linear term $V(x) \equiv m$ and the nonlinearity f is of form (1.3). Then for all $m \in (-\infty,-1)\setminus\{-j^2 : j \in \mathbb{Z}\}$, all $n \in \mathbb{N}$ with $n \geq 5$ and $J = \{j_0+1,\dots,j_0+n\} \subset \mathbb{N}$ with $j_0^2+m < 0$ and $(j_0+1)^2+m > 0$, there is a Cantor set $\mathscr{C} \subset \mathbb{P}^n$, a family of n-tori*

$$\mathcal{T}_J(\mathscr{C}) = \bigcup_{y\in\mathscr{C}} \mathcal{T}_J(y) \subset E_J \tag{1.14}$$

over $\mathscr{C}$, and a Lipschitz continuous embedding

$$\Phi : \mathscr{T}_J[\mathscr{C}] \hookrightarrow H_0^1([0,\pi]) \times L^2([0,\pi]) = \mathscr{W} \tag{1.15}$$

which is a higher-order perturbation of the inclusion map $\Phi_0 : E_J \hookrightarrow \mathscr{W}$ restricted to $\mathscr{T}_J[\mathscr{C}]$, such that the restriction of Φ to each $\mathscr{T}_J(y)$ in the family is an embedding of a rotational invariant n-torus for the nonlinear equation (1.1).

Remark 1.2. The image $\Phi(\mathscr{T}_J[\mathscr{C}])$ of $\mathscr{T}_J[\mathscr{C}]$ we call a Cantor manifold of rotational n-tori. This manifold is hyperbolic-elliptic since there are a finite number of nonreal basic frequencies for the linear system $u_{tt} = u_{xx} - mu$ with $m < -1$. Note that the manifold obtained by Pöschel [9] is elliptic.

Remark 1.3. The Cantor set $\mathscr{C}$ has full density at the origin. That is,

$$\lim_{r \to 0} \frac{\text{meas}\,(\mathscr{C} \cap B_r)}{\text{meas}\,(\mathbb{P}^n \cap B_r)} = 1, \tag{1.16}$$

where $B_r = \{y : \|y\| < r\}$, and meas denotes the n-dimensional Lebesgue measure for sets.

Remark 1.4. We can also deal with the more general choice $J = \{j_1 < j_2 < \cdots < j_n\}$ and $n \ge 1$ at the cost of excluding some set of m values.

Remark 1.5. We do not know what happens to the potential $V(x) \equiv m \in \{-j^2 : j \in \mathbb{Z}\}$. In particular, very little is known about the case $m = 0$ in which (1.1) is "complete resonant" (cf. [5, 9]). When $m \in \{-j^2 : j \in \mathbb{Z}\}$ and $m \neq 0$, there is a zero-frequency for the linear system. According to our knowledge, it does not seem that the existing KAM theorem can handle this case.

2. An infinitely dimensional KAM theorem

2.1. Statement of the theorem. Consider small perturbations of an infinitely dimensional Hamiltonian in the parameter dependent normal form

$$N = \sum_{1 \le j \le n} \omega_j(\xi) y_j + \sum_{j \ge 1} \Omega_j(\xi) z_j \bar{z}_j \tag{2.1}$$

on a phase space

$$\mathscr{P}^{a,p} = \hat{\mathbb{T}}^n \times \mathbb{C}^n \times \ell^{a,p} \times \ell^{a,p} \ni (x, y, z, \bar{z}), \tag{2.2}$$

where $\hat{\mathbb{T}}^n$ is the complexification of the usual n-torus $\mathbb{T}^n$ with $1 \le n < \infty$, and $\ell^{a,p}$ is the Hilbert space of all complex sequence $w = (w_1, w_2, \ldots)$ with

$$\|w\|_{a,p}^2 = \sum_{j \ge 1} |w_j|^2 j^{2p} e^{2aj} < \infty, \quad a, p > 0. \tag{2.3}$$

Here the phase space $\mathcal{P}^{a,p}$ is endowed with the symplectic form $dx \wedge dy - \sqrt{-1}dz \wedge d\bar{z}$. The tangent frequencies $\omega = (\omega_1,\dots,\omega_n)$ and the normal frequencies $\Omega = (\Omega_1,\Omega_2,\dots) \in \mathbb{R}^{\mathbb{N}}$ depend on n-parameters $\xi \in \mathbb{O} \subset \mathbb{R}^n$, $\mathbb{O}$ a given compact set of positive Lebesgue measure. In [8], all Ω_j's are positive. In our case, there are a finite number of negative Ω_j's.

The Hamiltonian equation of motion of N are

$$\dot{x} = \omega(\xi), \qquad \dot{y} = 0, \qquad \dot{u} = \Omega(\xi)v, \qquad \dot{v} = -\Omega(\xi)u, \tag{2.4}$$

where $(\Omega u)_j = \Omega_j u_j$. Hence, for each $\xi \in \mathbb{O}$, there is an invariant n-dimensional torus $\mathcal{T}_0^n = \mathbb{T}^n \times \{0\} \times \{0\}$ with frequencies $\omega(\xi)$. The aim is to prove the persistence of the torus $\mathcal{T}_0^n$, for most values of parameter $\xi \in \mathbb{O}$ (in the sense of Lebesgue measure), under small perturbations P of the Hamiltonian H_0. To this end, the following assumptions are required.

Assumption 2.1 (nondegeneracy)**.** The real map $\xi \mapsto \omega(\xi)$ is a lipeomorphism between $\mathbb{O}$ and its image, that is, a homomorphism which is Lipschitz continuous in both directions. Moreover, for integral vectors $(k,l) \in \mathbb{Z}^n \times \hat{\mathbb{Z}}^\infty$ with $1 \le |l| \le 2$,

$$\operatorname{meas}\{\xi : \langle k,\omega(\xi)\rangle + \langle l,\Omega(\xi)\rangle = 0\} = 0 \tag{2.5}$$

and for $l \in \hat{\mathbb{Z}}^\infty$,

$$\langle l,\Omega(\xi)\rangle \ne 0 \quad \text{on } \mathbb{O}, \tag{2.6}$$

where

$$\hat{\mathbb{Z}}^\infty = \{l = (0,\dots,0,l_{j_0+1},l_{j_0+2},\dots) : l_j \in \mathbb{Z}\} \tag{2.7}$$

and where "meas" ≡ Lebesgue measure for sets, $|l| = \sum_j |l_j|$ for integer vectors, and $\langle\cdot,\cdot\rangle$ is a usual real (or complex) scalar product.

Assumption 2.2 (spectral asymptotic)**.** Assume that $\Omega_j(\xi)$ is real for all $j \ge j_0+1$ and $\xi \in \mathbb{O}$. Moreover, assume that there exist $d \ge 1$ and $\delta < d-1$ such that

$$\Omega_j = j^d + \cdots + O(j^\delta), \quad j \ge j_0+1, \tag{2.8}$$

where the dots stands for fixed lower-order term in j, allowing also negative exponents. More precisely, there exists a fixed, parameter-independent sequence $\bar{\Omega}$ with $\bar{\Omega}_j = j^d + \cdots$ such that the tails $\tilde{\Omega}_j = \Omega_j - \bar{\Omega}_j$ give rise to a Lipschitz map

$$\tilde{\Omega} : \mathbb{O} \longrightarrow \ell_\infty^{-\delta}, \tag{2.9}$$

where ℓ_∞^p is the space of all real sequences with finite norm $|w|_p = \sup_j |w_j| j^p$.

Assumption 2.3 (finite imaginary spectra)**.** There is a constant $\kappa_0 > 0$ such that

$$\Re\Omega_j = 0, \quad \Im\Omega_j \geq \kappa_0, \quad j \leq j_0. \tag{2.10}$$

To give the conditions on the perturbation P, introduce complex $\mathcal{T}_0^n$ neighborhoods

$$D(s,r) := \{(x,y,z,\bar{z}) \in \mathcal{P}^{a,p} : |\operatorname{Im} x| < s,\ |y| < r^2,\ \|z\|_{a,p} + \|\bar{z}\|_{a,p} < r\}, \tag{2.11}$$

where $|\cdot|$ denotes the sup-norm for complex vectors and $\|\cdot\|_{a,p}$ is the norm in the space $\ell_{a,p}$. We define the weighted phase norms

$$|W|_r = |W|_{\bar{p},r} = |x| + \frac{1}{r^2}|y| + \frac{1}{r}\|z\|_{\bar{p}} + \frac{1}{r}\|\bar{z}\|_{a,\bar{p}} \tag{2.12}$$

for $W = (x,y,z,\bar{z}) \in \mathcal{P}^{a,\bar{p}}$ with $\bar{p} \geq p$. For a map $U : D(s,r) \times \mathcal{O} \to \mathcal{P}^{a,\bar{p}}$, define its Lipschitz seminorm $|U|_r^{\mathcal{L}}$,

$$|U|_r^{\mathcal{L}} = \sup_{\xi \neq \zeta} \frac{|\Delta_{\xi\zeta} U|_r}{|\xi - \zeta|}, \tag{2.13}$$

where $\Delta_{\xi\zeta} W = W(\cdot,\xi) - W(\cdot,\zeta)$, and where the supremum is taken over $\mathcal{O}$.

Set

$$|U|_{D(s,r)}^{\mathcal{O},\mathcal{L}} = \sup_{D(s,r)\times\mathcal{O}} \{|U|_r\} + \sup_{D(s,r)} \{|U|_r^{\mathcal{L}}\}. \tag{2.14}$$

For the sup-norm $|\cdot|$ and the operator norm $|||\cdot|||$, the notations $|\cdot|_{D(s,r)}^{\mathcal{O},\mathcal{L}}$ and $|||\cdot|||_{D(s,r)}^{\mathcal{O},\mathcal{L}}$ are defined analogously to $|\cdot|_{D(s,r)}^{\mathcal{O},\mathcal{L}}$.

Assumption 2.4 (regularity)**.** The perturbation $P(x,y,z,\bar{z};\xi)$ is analytic in $(x,y,z,\bar{z}) \in D(s,r)$ for given $s,r > 0$, (not necessary to be real for real arguments), and Lipschitzian in the parameter $\xi \in \mathcal{O}$, and for each $\xi \in \mathcal{O}$, its Hamiltonian vector field $X_P := (P_y, -P_x, P_z, -P_{\bar{z}})^T$ defines on $D(s,r)$ an analytic map

$$X_P : \mathcal{P}^{a,p} \longrightarrow \mathcal{P}^{a,\bar{p}}, \quad \begin{cases} \bar{p} \geq p, & \text{for } d > 1, \\ \bar{p} > p, & \text{for } d = 1. \end{cases} \tag{2.15}$$

By Assumptions 2.1, 2.2, and 2.3, there are two constants M and L such that

$$|\omega|_{\mathcal{O}} + |\Omega|_{-\delta,\mathcal{O}}^{\mathcal{L}} \leq M, \qquad |\omega^{-1}|_{\omega(\mathcal{O})}^{\mathcal{L}} \leq L. \tag{2.16}$$

Following Pöschel [8], introduce notations

$$\begin{aligned} \langle \ell \rangle_d &= \max\Big\{1, \Big|\sum j^d \ell_j\Big|\Big\}, \qquad A_k = |k|^\tau + 1, \\ \mathscr{Z} &= \big\{(k,j) \in \mathbb{Z}^n \times \mathbb{Z}^{\mathbb{Z}} : |k| + |l| \neq 0,\ |l| \leq 2\big\}, \end{aligned} \tag{2.17}$$

where $\tau \geq n+1$ is fixed later.

Theorem 2.5. *Suppose that $H = N + P$ satisfies Assumptions 2.1, 2.2, 2.3, and 2.4, and*

$$\epsilon = \big|X_P\big|_{D(s,r)} + \frac{\alpha}{M}\big|X_P\big|^{\mathscr{L}}_{D(s,r)} \leq \gamma\alpha, \tag{2.18}$$

where $0 < \alpha \leq 1$ is another parameter and γ depends on n, τ, and s. Then there is a Cantor set $\mathcal{O}_\alpha \subset \mathcal{O}$, a Lipschitz continuous family of torus embedding $\Phi : \mathbb{T}^n \times \mathcal{O}_\alpha \to \mathcal{P}^{a,\bar{p}}$, and a map $\omega_* : \mathcal{O}_\alpha \to \mathbb{R}^n$, such that for each $\xi \in \mathcal{O}_\alpha$, the map Φ restricted to $\mathbb{T}^n \times \{\xi\}$ is an analytic embedding of a rational torus with frequencies $\omega_*(\xi)$ for the Hamiltonian H at ξ.

Each embedding is analytic (not necessary being real) on $|\Im x| < s/2$, and

$$\begin{aligned} \big|\Phi - \Phi_0\big|_r + \frac{\alpha}{M}\big|\Phi - \Phi_0\big|^{\mathscr{L}}_r &\leq \frac{c\epsilon}{\alpha}, \\ \big|\omega_* - \omega\big| + \frac{\alpha}{M}\big|\omega_* - \omega\big|^{\mathscr{L}} &\leq c\epsilon \end{aligned} \tag{2.19}$$

uniformly on that domain and $\mathcal{O}_\alpha$, where Φ_0 is the trivial embedding $\mathbb{T}^n \times \mathcal{O} \to \mathbb{T}^n \times \{0\} \times \{0\}$ and $c \leq \gamma^{-1}$ depends on the same parameters as γ.

Moreover, there exist Lipschitz maps ω_ν and Ω_ν on $\mathcal{O}$ for $\nu \geq 0$, satisfying $\omega_0 = \omega$, $\Omega_0 = \Omega$, and

$$\begin{aligned} \big|\omega_\nu - \omega\big| + \frac{\alpha}{M}\big|\omega_\nu - \omega\big|^{\mathscr{L}} &\leq c\epsilon, \\ \big|\Omega_\nu - \Omega\big|_{-\delta} + \frac{\alpha}{M}\big|\Omega_\nu - \Omega\big|^{\mathscr{L}}_{-\delta} &\leq c\epsilon, \end{aligned} \tag{2.20}$$

such that $\mathcal{O} \setminus \mathcal{O}_\alpha \subset \bigcup R^\nu_{k,l}(\alpha)$, where

$$R^\nu_{k,l}(\alpha) = \Big\{\xi \in \mathcal{O} : \big|\langle k, \omega_\nu(\xi)\rangle + \langle l, \Omega_\nu\rangle\big| \leq \alpha\frac{\langle l\rangle_d}{A_k}\Big\}, \tag{2.21}$$

and the union is taken over all $\nu \geq 0$ and $(k,l) \in \mathscr{Z}$ such that $|k| > K_0 2^{\nu-1}$ for $\nu \geq 1$ with a constant $K_0 \geq 1$ depending only on n and τ.

Proof. If all frequency vectors ω and Ω in the zeroth KAM step are real, this theorem is the same as [8, Theorem A]. In our case, however, some normal frequencies Ω's are not real. This gives rise to that both the vectors ω_ν and Ω_ν in νth KAM step are possibly not real. Fortunately, the proof of this theorem does not involve the measure estimate; thus, the argument does not depend on whether or not the frequency vectors ω_ν and Ω_ν are real. Therefore, the proof of [8, Theorem A] due to Pöschel can still be valid. It is worthy to be noted that the frequency map ω_* in our case should be taken as $\omega_* = \Re(\lim_{\nu\to\infty}\omega_\nu)$ instead of $\omega_* = \lim_{\nu\to\infty}\omega_\nu$. □

Theorem 2.6. *Suppose that in Theorem 2.5 the unperturbed frequencies ω and Ω are affine functions of the parameters. Then there is a constant c_0 such that*

$$\operatorname{meas}(\mathbb{O}\setminus\mathbb{O}_\alpha) \le c_0(\operatorname{diam}\mathbb{O})^{n-1}\alpha^\mu, \quad \mu = \begin{cases} 1, & \text{for } d > 1, \\ \dfrac{\kappa}{\kappa+1-(\varpi/4)}, & \text{for } d = 1, \end{cases} \tag{2.22}$$

for all sufficiently small α, where ϖ is any number in $[0, \min(\bar{p}-p, 1))$ and where, in the case $d = 1$, κ is a positive constant such that

$$\frac{\Omega_i - \Omega_j}{i-j} = 1 + O(j^{-k}), \quad i > j > j_0, \tag{2.23}$$

uniformly on $\mathbb{O}$.

Proof. The proof will be given in Section 2.3. □

2.2. The Cantor manifold theorem. In a neighborhood of the origin in $\ell^{a,p}$, we now consider a Hamiltonian $H = \Lambda + Q + R$, where R represents some higher-order perturbation of an integrable normal form $\Lambda + Q$.

Let $z = (\tilde{z}, \hat{z})$ with $\tilde{z} = (z_{j_0+1}, \dots, z_{j_0+n})$, $\hat{z} = (z_1, \dots, z_{j_0}, z_{j_0+n+1}, \dots)$, and

$$\begin{aligned} y &= \frac{1}{2}\left(|z_{j_0+1}|^2, \dots, |z_{j_0+n}|^2\right), \\ Z &= \frac{1}{2}\left(|z_1|^2, \dots, |z_{j_0}|^2, |z_{j_0+n+1}|^2, \dots\right). \end{aligned} \tag{2.24}$$

Assume that

$$\Lambda = \langle \alpha, y\rangle + \langle \beta, Z\rangle, \qquad Q = \frac{1}{2}\langle Ay, y\rangle + \langle By, Z\rangle \tag{2.25}$$

with constant vectors α, β and constant matrices A, B.

The equations of motion of the Hamiltonian $\Lambda + Q$ are

$$\dot{\tilde{z}}_j = \sqrt{-1}(\alpha + Ay + B^T z)_j \tilde{z}_j, \qquad \dot{\hat{z}}_j = \sqrt{-1}(\beta + By)_j \hat{z}_j. \tag{2.26}$$

Thus, the complex n-dimensional manifold $E = \{\hat{z} = 0\}$ is invariant and it is completely filled up to the origin by the invariant tori

$$\mathcal{T}(y) = \{\tilde{z} : |\tilde{z}_j|^2 = 2y_j,\ j_0 + 1 \le j \le j_0 + n\}, \quad y \in \overline{\mathbb{P}^n}. \tag{2.27}$$

On $\mathcal{T}(y)$, the flow is given by the equations

$$\dot{\tilde{z}}_j = \sqrt{-1}\,\omega_j(y)\tilde{z}_j, \quad \omega(y) = \alpha + Ay, \tag{2.28}$$

and on its normal space by

$$\dot{\hat{z}}_j = \sqrt{-1}\Omega_j(y)\hat{z}_j, \quad \Omega(y) = \beta + By. \tag{2.29}$$

They are linear and in a diagonal form. It is worthy noting that since Ω_j ($j = 1,\dots,j_0$) are pure imaginary and Ω_j ($j = j_0+1,\dots$) are real, $\hat{z} = 0$ is a fixed point of hyperbolic-elliptic type. This is different from the elliptic fixed point of [9]. We therefore call $\mathcal{T}(y)$ a hyperbolic-elliptic rational torus with frequencies $\omega(y)$.

Assumption 2.7 (nondegeneracy). (1) For the above constant matrix A, $\det A \neq 0$,

(2) $\langle l,\beta\rangle \neq 0$,

(3) $\langle k,\omega(y)\rangle + \langle l,\Omega(y)\rangle$ does not vanish identically for all $(k,l) \in \mathbb{Z}^n \times \hat{\mathbb{Z}}^\infty$ with $1 \le |l| \le 2$. (See Assumption 2.1 for $\hat{\mathbb{Z}}^\infty$.)

Assumption 2.8 (spectral asymptotic). There exist $d \ge 1$ and $\delta < d-1$ such that

$$\beta_j = j^d + \cdots + O(j^\delta), \quad j \ge j_0 + 1, \tag{2.30}$$

where the dots stands for fixed lower-order term in j. Note that the normalization of the coefficients of j^d can always be achieved by a scaling of time.

Assumption 2.9 (finite imaginary spectra). There is a constant $\kappa > 0$ such that $\Re\Omega_j = 0$ and $\Im\Omega_j \ge \kappa$, $1 \le j \le j_0+1$. In addition, we assume $\Omega_i \neq \Omega_j$ for all $1 \le i$, $j \le j_0$.

Assumption 2.10 (regularity). The vector fields X_Q and X_R corresponding to the Hamiltonians Q and R, respectively, satisfy

$$X_Q, X_R \in \mathcal{A}(\ell^{a,p},\ell^{a,\bar{p}}), \quad \begin{cases} \bar{p} \ge p, & \text{for } d > 1, \\ \bar{p} > p, & \text{for } d = 1, \end{cases} \tag{2.31}$$

where $\mathcal{A}(\ell^{a,p},\ell^{a,\bar{p}})$ denotes the class of all maps from some neighborhood of the origin in $\ell^{a,p}$ into $\ell^{a,\bar{p}}$, which are analytic in the real and imaginary parts of the complex coordinate z.

By the regularity assumption, the coefficients B of the Hamiltonian Q satisfy the estimate $B_{ij} = O(j^{\bar{p}-p})$ uniformly in $j_0 \le i \le j_0+n$. Consequently, for $d = 1$, there is a positive constant κ such that

$$\frac{\Omega_i - \Omega_j}{i-j} = 1 + O(j^{-\kappa}), \quad i > j > j_0, \tag{2.32}$$

uniformly for bounded y. For $d > 1$, set $\kappa = \infty$.

The Cantor manifold theorem. *Suppose that the Hamiltonian $H = \Lambda + Q + R$ satisfy Assumptions 2.7, 2.8, 2.9, 2.10, and*

$$|R| = O\left(\|\hat{z}\|_{a,p}^4\right) + O\left(\|z\|^g\right) \tag{2.33}$$

with

$$g > 4 + \frac{4-\varrho}{\kappa}, \quad \varrho = \min(\bar{p} - p, 1). \tag{2.34}$$

Then, there is a Cantor set $\mathscr{C} \subset \mathbb{P}^n$, a family of n-tori

$$\mathcal{T}_J(\mathscr{C}) = \bigcup_{y \in \mathscr{C}} \mathcal{T}_J(y) \subset E_J \tag{2.35}$$

over $\mathscr{C}$, and a Lipschitz continuous embedding $\Phi : \mathcal{T}_J[\mathscr{C}] \hookrightarrow \mathscr{W}$, which is a higher-order perturbation of the inclusion map $\Phi_0 : E_J \hookrightarrow \mathscr{W}$ restricted to $\mathcal{T}_J[\mathscr{C}]$,

$$\|\Phi - \Phi_0\|_{a,\bar{p};B_r \cap \mathcal{T}[\mathscr{C}]} = O(r^\sigma) \tag{2.36}$$

with some $\sigma > 1$, such that the restriction of Φ to each $\mathcal{T}_J(y)$ in the family is an embedding of a rotational n-torus for the nonlinear equation (1.1). The Cantor set $\mathscr{C}$ has full density at the origin.

Proof. In view of Theorems 2.5 and 2.6, following literally the proof of the Cantor manifold theorem in [6, pages 170–175], we can finish the proof of this theorem. The details are omitted here. $\square$

2.3. Measure estimates and proof of Theorem 2.6. Recall that the unperturbed tangent and normal frequencies are ω and $\Omega = (\Omega_*, \Omega_{**})$, respectively, where $\Omega_* = (\Omega_1, \dots, \Omega_{j_0})$, $\Omega_{**} = (\Omega_{j_0+1}, \dots)$. By Assumptions 2.1, 2.2, and 2.3, we have that $\omega(\xi)$ and $\Omega_{**}(\xi)$ are real for all $\xi \in \mathbb{O}$, and $\Omega_*(\xi)$ are pure imaginary.

Let $\sigma = \min(d, d-1-\delta) > 0$, where $\delta < d-1$ is defined in Assumption 2.8. Set $\Xi = \{l : 1 \le |l| \le 2\}$. Then, $\langle l \rangle_d \ge (2/9)|l|_\sigma |l|_\delta$ for $l \in \Xi$. By Assumption 2.2, there is a positive constant β such that $|\langle l, \Omega_{**} \rangle| \ge 27\beta/2$.

In estimating the measure of the resonance zones, it is not necessary to distinguish between the various perturbations ω_ν and Ω_ν of the frequencies ω and Ω since only the size of the perturbations matters. Therefore, following Pöschel [8], we now write ω' and Ω' for all the perturbed frequencies for which, by Theorem 2.5, the following condition is satisfied.

Condition 2.11. If $\gamma > 0$ is small enough, then

$$\begin{aligned} |\omega' - \omega|, &\qquad |\Omega'_{**} - \Omega_{**}|_{-\delta} \le \alpha, \\ |\omega' - \omega|^{\mathcal{L}}, &\qquad |\Omega'_{**} - \Omega_{**}|^{\mathcal{L}}_{-\delta} \le M\gamma \le \frac{1}{2L}, \\ |\Re\Omega'_{*}| \le \alpha, &\qquad |\Re\Omega'_{*}|^{\mathcal{L}} \le M\gamma \le \frac{1}{2L}. \end{aligned} \tag{2.37}$$

Note that $\Re\Omega_* = 0$.

Note that ω' and Ω' are not necessary real. Set

$$R'_{kl}(\alpha) = \left\{\xi \in \mathbb{O} : |\langle k, \omega'(\xi)\rangle + \langle l, \Omega'\rangle| \le \alpha \frac{\langle l\rangle_d}{A_k}\right\}. \tag{2.38}$$

Let

$$\begin{aligned} \Xi_1 &= \{l \in \Xi : l_j = 0 \text{ for } 1 \le j \le j_0\} \\ \Xi_2 &= \{l \in \Xi : l_j = 0 \text{ for } j \ge j_0 + 1\} \\ \Xi_3 &= \{l \in \Xi : l_{j_1} \ne 0,\ l_{j_2} \ne 0 \text{ for some } 1 \le j_1 \le j_0,\ j_2 \ge j_0 + 1\}. \end{aligned} \tag{2.39}$$

In the following lemmas, we assume that Condition 2.11 and (2.32) are satisfied.

Lemma 2.12. *If ω' and Ω'_{**} are real for all $\xi \in \mathbb{O}$, then there is a constant c_1 such that*

$$\begin{gathered} \operatorname{meas}\left(\bigcup_{l \in \Xi_1} R_{k,l}\right) \le \frac{c_1(\operatorname{diam}\mathbb{O})^{n-1}\alpha^{\mu}|k|^2}{A_k^{\lambda}}, \\ \mu = \begin{cases} 1, & \text{for } d > 1, \\ \dfrac{\kappa}{\kappa + 1 - (\varpi/4)}, & \text{for } d = 1, \end{cases} \qquad \lambda = \begin{cases} 1, & \text{for } d > 1, \\ \dfrac{\kappa}{\kappa + 1 - \varpi}, & \text{for } d = 1, \end{cases} \end{gathered} \tag{2.40}$$

for all sufficiently small α, where ϖ is any number in $[0, \min(\bar{p} - p, 1))$ and where, in the case $d = 1$, κ is a positive constant such that

$$\frac{\Omega_i - \Omega_j}{i - j} = 1 + O(j^{-k}), \quad i > j > j_0, \tag{2.41}$$

uniformly on $\mathbb{O}$.

Proof. The proof of this lemma can be found in [8, Theorem D]. □

Since the frequencies ω' and Ω' are not necessary real for real $\xi \in \mathbb{O}$, we need the following lemma.

Lemma 2.13. *When the frequencies ω' and Ω' are not necessary real for real $\xi \in \mathcal{O}$, then there is a constant $c_2 > 0$ such that*

$$\operatorname{meas}\left(\bigcup_{l\in\Xi_1} R_{k,l}\right) \leq \frac{c_2(\operatorname{diam}\mathcal{O})^{n-1}\alpha^{\mu}|k|^2}{A_k^{\lambda}} \tag{2.42}$$

for all sufficiently small α, where μ is defined in Lemma 2.12.

Proof. Note that the unperturbed frequencies $\omega(\xi)$ and $\Omega_{**}(\xi)$ are real for $\xi \in \mathcal{O}$. By Condition 2.11, we get that

$$\begin{gathered} |\Re\omega' - \omega| \leq |\omega' - \omega| \leq \alpha \\ |\Re\Omega'_{**} - \Omega_{**}|_{-\delta} \leq |\Omega'_{**} - \Omega_{**}|_{-\delta} \leq \frac{1}{2L} \\ |\Re\omega' - \omega|^{\mathcal{L}} \leq |\omega' - \omega|^{\mathcal{L}} \leq \frac{1}{2L} \\ |\Re\Omega'_{**} - \Omega_{**}|^{\mathcal{L}}_{-\delta} \leq |\Omega'_{**} - \Omega_{**}|^{\mathcal{L}}_{-\delta} \leq \frac{1}{2L}. \end{gathered} \tag{2.43}$$

Write

$$\Re(R_{k,l}) := \left\{\xi \in \mathcal{O} : |\langle k, \Re\omega'\rangle + \langle l, \Re\Omega'(\xi)\rangle| \leq \alpha\frac{\langle l\rangle_d}{A_k}\right\}. \tag{2.44}$$

By Lemma 2.12, there is a constant $c_2 > 0$ such that

$$\operatorname{meas}\left(\bigcup_{l\in\Xi_1} \Re(R_{k,l})\right) \leq \frac{c_2(\operatorname{diam}\mathcal{O})^{n-1}\alpha^{\mu}|k|^2}{A_k^{\lambda}}. \tag{2.45}$$

Since $|\langle k, \omega'\rangle + \langle l, \Omega'\rangle| \geq |\langle k, \Re\omega'\rangle + \langle l, \Re\Omega'\rangle|$, we get $R_{k,l} \subset \Re(R_{k,l})$. Thus,

$$\operatorname{meas}\left(\bigcup_{l\in\Xi_1} R_{k,l}\right) \leq \frac{c_2(\operatorname{diam}\mathcal{O})^{n-1}\alpha^{\mu}|k|^2}{A_k^{\lambda}}. \tag{2.46}$$

This finishes the proof. □

Lemma 2.14. *For $l \in \Lambda_2$, there is a constant $c_3 > 0$ such that*

$$\operatorname{meas}\left(\bigcup_{l\in\Xi_2} R_{k,l}\right) \leq \frac{c_3(\operatorname{diam}\mathcal{O})^{n-1}\alpha^{\mu}|k|^2}{A_k^{\lambda}}. \tag{2.47}$$

Proof. We introduce the unperturbed frequencies $\zeta = \omega(\xi)$ as parameters over the domain $\Delta = \omega(\mathcal{O})$ and consider the resonance zones $R^{\Delta}_{k,l} = \omega(R_{k,l})$ in Δ. Write $\omega' = \omega' \circ \omega^{-1}$ and $\Omega' = \Omega' \circ \omega^{-1}$. Then, by Condition 2.11,

$$|\omega' - \mathrm{id}|^{\mathcal{L}} \leq \frac{1}{2}, \qquad |\Re\Omega'_{*}|^{\mathcal{L}} \leq M\gamma. \tag{2.48}$$

Now consider $R^{\Delta}_{k,l}$. Let $\phi(\zeta) = \langle k, \omega'(\zeta)\rangle + \langle l, \Omega'(\zeta)\rangle$ where $l \in \Xi_2$. Choose a vector $v \in \{-1,1\}^n$ such that $\langle k, v\rangle = |k|$ and write $\zeta = rv + w$ with $r \in \mathbb{R}$, $w \in v^{\perp}$. As a function of r, we have, for $t > s$,

$$\begin{aligned}\langle k, \Re\omega'(\zeta)\rangle\big|_s^t &= \langle k, \zeta\rangle\big|_s^t + \langle k, \Re\omega'(\zeta) - \zeta\rangle\big|_s^t \geq |k|(t-s) - \frac{1}{2}|k|(t-s)\\ &= \frac{1}{2}|k|(t-s)\end{aligned} \tag{2.49}$$

and for $l \in \Xi_2$,

$$\Big|\langle l, \Re\Omega'(\zeta)\rangle\big|_s^t\Big| \leq 2j_0\,|\Re\Omega'_*|^{\mathcal{L}}(t-s) \leq 2j_0 M\gamma(t-s). \tag{2.50}$$

Since $2j_0M\gamma < 1/4$ by $\gamma \ll 1$, we get that $\phi(rv+w)|_s^t \geq (1/4)|k|(t-s)$ uniformly in w, when $k \neq 0$. It follows that

$$\operatorname{meas}\left\{r : rv + w \in \Delta,\ |\phi(rv+w)| \leq \alpha\frac{\langle l\rangle_d}{A_k}\right\} \leq 4\alpha|k|^{-1}\frac{\langle l\rangle_d}{A_k} \tag{2.51}$$

and hence

$$\operatorname{meas}(R^{\Delta}_{k,l}) \leq \operatorname{meas}(\Re R^{\Delta}_{k,l}) \leq 4(\operatorname{diam}\Delta)^{n-1}\alpha|k|^{-1}\frac{\langle l\rangle_d}{A_k} \tag{2.52}$$

by Fubini's theorem. Going back to the original parameter domain $\mathbb{O}$ by inverse frequency map ω^{-1}, observing that $\operatorname{diam}\Delta \leq 2M\operatorname{diam}\mathbb{O}$ and $\langle l\rangle_d \leq 2j_0^d$, and noting that $\operatorname{Card}(\Xi_2) \leq 5^{j_0}$, we get that there is a constant c_3 depending on j_0 such that

$$\operatorname{meas}\left(\bigcup_{l\in\Xi_2} R_{k,l}\right) \leq \frac{c_3(\operatorname{diam}\mathbb{O})^{n-1}\alpha^{\mu}|k|^2}{A_k^{\lambda}}, \quad \text{for } k \neq 0. \tag{2.53}$$

When $k = 0$, we have that there are some i and j with $1 \leq i, j \leq j_0$, such that

$$\begin{aligned}|\langle k, \omega'\rangle + \langle l, \Omega'\rangle| &= |\langle l, \Omega'\rangle| \geq |\langle l, \Im\Omega'\rangle|\\ &= |\Omega'_i - \Omega'_j| \geq |\Omega_i - \Omega_j| - 2\alpha.\end{aligned} \tag{2.54}$$

By Assumption 2.9, there is a positive constant c^* such that $|\Omega_i - \Omega_j| \geq c^*$ for all $1 \leq i, j \leq j_0$. Thus, $|\langle k, \omega'\rangle + \langle l, \Omega'\rangle| \geq c^* - 2\alpha \geq c^*/2$ if α is small enough and $k = 0$. Moreover, the set $R_{k,l} = \varnothing$ if α is sufficiently small. This completes the final estimate. □

Lemma 2.15. *There is a constant c_4 such that*

$$\operatorname{meas}\left(\bigcup_{l\in\Xi_3} R_{k,l}\right) \leq \frac{c_4(\operatorname{diam}\mathbb{O})^{n-1}\alpha^{\mu}|k|^2}{A_k^{\lambda}}. \tag{2.55}$$

Proof. For $l \in \Xi_3$, we can write

$$l^i = \Big(0,\dots,0,\overset{i\text{th}}{l_i},0,\dots,\overset{j_0\text{th}}{0},0,\dots,0,l_{j_0+p},0,\dots\Big),\quad i=1,\dots,j_0,\ p=1,2,\dots, \tag{2.56}$$

where $l_i = \pm 1$, $l_{j_0+p} = \pm 1$. For fixed $1 \le i \le j_0$, let

$$\begin{aligned}\tilde{\Omega}_k(i) &= 0,\quad k=1,\dots,j_0,\\ \tilde{\Omega}_{j_0+p}(i) &= \Re\Omega'_{j_0+p} + \frac{l_i}{l_{j_0+p}}\Re\Omega'_i,\quad p=1,2,\dots.\end{aligned} \tag{2.57}$$

Then

$$\begin{aligned}|\langle k,\omega'\rangle + \langle l,\Omega'\rangle| &\ge |\langle k,\Re\omega'\rangle + \langle l,\Re\Omega'\rangle|\\ &= |\langle k,\Re\omega'\rangle + l_i\Re\Omega'_i + l_{j_0+p}\Re\Omega'_{j_0+p}|\\ &= |\langle k,\Re\omega'\rangle + \langle\tilde{l},\Re\tilde{\Omega}(i)\rangle|,\end{aligned} \tag{2.58}$$

where $\tilde{l} = (0,\dots,0,\overset{p\text{th}}{l_{j_0+p}},0,\dots)$ and $\tilde{\Omega}(i) = (\tilde{\Omega}_1(i),\dots,\tilde{\Omega}_p(i),\dots)_{p\in\mathbb{N}}$. We get by Condition 2.11

$$\begin{aligned}|\tilde{\Omega}(i)_{**} - \Omega_{**}|_{-\delta} &\le |\Re\Omega'_{**} - \Omega_{**}|_{-\delta} + \left|\left(\frac{l_i}{l_{j_0+p}}\Re\Omega'_i\right)_{p\in\mathbb{N}}\right|_{-\delta}\\ &\le |\Re\Omega'_{**} - \Omega_{**}|_{-\delta} + |\Re\Omega'_i|\sup_p p^{-\delta}\\ &\le 2\alpha,\\ |\tilde{\Omega}(i)_{**} - \Omega_{**}|^{\mathcal{L}}_{-\delta} &\le |\Re\Omega'_{**} - \Omega_{**}|^{\mathcal{L}}_{-\delta} + |\Re\Omega_*|^{\mathcal{L}} \le 2M\gamma \le \frac{1}{L}.\end{aligned} \tag{2.59}$$

Let

$$\tilde{R}_{k,l}(i) = \left\{\xi \in \mathcal{O} : \langle k,\Re\omega'\rangle + \langle l,\tilde{\Omega}(i)\rangle \le \alpha\frac{\langle l\rangle_d}{|k|^\tau + 1}\right\}. \tag{2.60}$$

By Lemma 2.12, there is a constant c_4 depending on j_0 such that

$$\text{meas}\bigcup_{1\le i\le j_0}\bigcup_{l\in\Xi_1}\tilde{R}_{k,l} \le \frac{c_4(\text{diam}\,\mathcal{O})^{n-1}\alpha^\mu|k|^2}{A_k^\lambda}. \tag{2.61}$$

Observing that by (2.58),

$$\bigcup_{l\in\Xi_3} R_{k,l} \subset \bigcup_{1\le i\le j_0}\bigcup_{l\in\Xi_1}\tilde{R}_{k,l}. \tag{2.62}$$

We finishes the proof of this lemma. □

Lemma 2.16. *For* $0 \ne k \in \mathbb{Z}^n$,

$$\text{meas}\,(R_{k,0}) \le \frac{c_5(\text{diam}\,\mathcal{O})^{n-1}\alpha^\mu|k|^2}{A_k^\lambda}. \tag{2.63}$$

Proof. This proof is the simplest. We omit the details. □

By Lemmas 2.13, 2.14, 2.15, and 2.16, we can give the proof of Theorem 2.6. In fact, we can choose τ sufficiently large but fixed such that

$$\sum_{|k|\geq K}\frac{|k|^2}{A_k^{\lambda}}\leq c_6\frac{1}{1+K}. \tag{2.64}$$

Thus,

$$\operatorname{meas}(\mathbb{O}\setminus\mathbb{O}_\alpha)\leq \operatorname{meas}\bigcup_{\substack{|k|\geq K_0 2^{\nu-1}\\ \nu\geq 0,(k,l)\in\mathcal{Z}}} R_{k,l}^{\nu}\leq c_8\alpha^{\mu}\left(1+K_0 2^{\nu-1}\right)^{-1}\leq c_7\alpha^{\mu}. \tag{2.65}$$

This finishes the proof of Theorem 2.6.

3. Application to NLW equation

3.1. Hamiltonian vector field. We recall that Hamiltonian of our NLW equation is of form (1.5). Write

$$u=\sum_{j\geq 1}\frac{q_j}{\sqrt{\lambda_j}}\phi_j,\qquad v=\sum_{j\geq 1}\sqrt{\lambda_j}p_j\phi_j, \tag{3.1}$$

where $(q,p)\in\ell^{a,p}\times\ell^{a,p}$, and $\phi_j=\sqrt{2/\pi}\sin jx$ for $j=1,2,\ldots$ are the normalized Dirichlet eigenfunctions of the linear differential operator $-d^2/dx^2+m$ with eigenvalues $\lambda_j=\sqrt{j^2+m}$. We obtain the Hamiltonian

$$H=\Lambda+G=\frac{1}{2}\sum_{j\geq 1}\lambda_j\left(p_j^2+q_j^2\right)+\int_0^{\pi}g\left(\sum_{j\geq 1}\frac{q_j}{\sqrt{\lambda_j}}\phi_j\right)dx \tag{3.2}$$

with equations of motion

$$\frac{dq_j}{dt}=\frac{\partial H}{\partial p_j}=\lambda_j p_j,\qquad \frac{dp_j}{dt}=-\frac{\partial H}{\partial q_j}=-\lambda_j q_j-\frac{\partial G}{\partial q_j}. \tag{3.3}$$

These are the Hamiltonian equations of motion with respect to the standard symplectic structure $\sum dq_j\wedge dp_j$ on $\ell^{a,p}\times\ell^{a,p}$.

Lemma 3.1. *Let I be an interval in $\mathbb{R}$. If a curve $I\to\ell^{a,p}\times\ell^{a,p}$, $t\mapsto(q(t),p(t))$ is an analytic solution of (3.3), then*

$$u(t,x)=\sum_{j\geq 1}\frac{q_j(t)}{\sqrt{\lambda_j}}\phi_j(x) \tag{3.4}$$

is a classical solution of (1.1) that is analytic on $I\times[0,\pi]$.

Proof. The proof of [9, Lemma 1] is applicable to our case $m<-1$. □

Lemma 3.2. *The gradient G_q is analytic as a map from some neighborhood of the origin in $\ell^{a,p}$ into $\ell^{a,p+1}$ with*

$$||G_q||_{a,p+1} = O(\|q\|^3_{a,p}). \tag{3.5}$$

Proof. The proof is the same as that of [9, Lemma 3]. □

For the nonlinearity u^3, we find

$$G = \frac{1}{4}\int_0^\pi |u(x)|^4 dx = \frac{1}{4}\sum_{ijkl} G_{i,j,k,l} q_i q_j q_k q_l \tag{3.6}$$

with

$$G_{ijkl} = \frac{1}{\lambda_i \cdots \lambda_l}\int_0^\pi \phi_i \phi_j \phi_k \phi_l \, dx. \tag{3.7}$$

It is not difficult to verify that $G_{ijkl} = 0$ unless $i \pm j \pm k \pm l = 0$ for some combination of plus and minus signs. Thus, the sum extends only over $i \pm j \pm k \pm l = 0$. In particular, we have

$$G_{iijj} = \frac{1}{2\pi}\frac{2+\delta_{ij}}{\lambda_i \lambda_j}. \tag{3.8}$$

From now on, we focus our attention on the nonlinearity u^3 since terms of order five or more will not make any difference. Hence, we are concerned with the Hamiltonian of the form

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geq 1} \lambda_j (p_j^2 + q_j^2) + G, \tag{3.9}$$

where G is defined by (3.6) and (3.7).

3.2. Partial normal form. In order to give the partial normal form for Hamiltonian (3.9), we need the following lemmas.

Lemma 3.3. *Assume that $m \in (-\infty, -1)$ and $m + j^2 \neq 0$ for all $j \in \mathbb{Z}$. If i, j, k, l are nonzero integers such that $(i,j,k,l) \neq (p,-p,q,-q)$ and $\tilde{n} := \min\{|i|,|j|,|k|,|l|\} > \sqrt{|m|}$, then*

$$|\lambda_i \pm \lambda_j \pm \lambda_k \pm \lambda_l| \geq c(m)\left(|\tilde{n}^2 + m|\right)^{-3/2} \tag{3.10}$$

with some positive constant $c = c(m)$ depending on m only.

Proof. We may restrict ourselves to positive integers such that $i \leq j \leq k \leq l$. The condition $i \pm j \pm k \pm l = 0$ then reduces to two possibilities, either $i - j - k + l = 0$ or $i + j + k - l = 0$. We have to study divisors of the form $\delta = \pm\lambda_i \pm \lambda_j \pm \lambda_k \pm \lambda_l$ for all possible combinations of plus and minus signs. To this end, we distinguish them according to their number of minus signs. To shorten notation, we let, for example, $\delta_{++-+} = \lambda_i + \lambda_j - \lambda_k + \lambda_l$. Similarly, for all other combinations of plus and minus signs.

Case 1 (no minus sign)**.** This is trivial since $\delta_{++++} \geq 4\sqrt{i^2+m} \geq 4\tilde{c} > 0$, where $\tilde{c} = \inf\{|j^2+m|^{1/2} : j \in \mathbb{Z}\}$.

Case 2 (one minus sign)**.** The cases δ_{-+++}, δ_{+-++}, and δ_{++-+} are trivial since all of them are larger than $\sqrt{i^2+m} > \tilde{c} \geq c|m|(|\tilde{n}^2+m|)^{-3/2}$. Now we consider δ_{+++-} which is the subtlest.

Case 2.1 (one minus sign and $i+j+k-l=0$)**.** Regard δ as a function of m. Hence

$$\delta(-1) = \sqrt{i^2-1} + \sqrt{j^2-1} + \sqrt{k^2-1} - \sqrt{l^2-1}. \tag{3.11}$$

We need to know whether $\delta(-1) \geq 0$ or not. Noting that $\sqrt{i^2-1} \leq i$, and so forth, we get that

$$\begin{aligned}
\left(\sqrt{i^2-1} + \sqrt{j^2-1} + \sqrt{k^2-1}\right)^2 &= i^2 + j^2 + k^2 + 2\sqrt{i^2-1}\sqrt{j^2-1} + \cdots - 3 \\
&\leq i^2 + j^2 + k^2 + 2ij + 2jk + 2ik - 3 \\
&= l^2 - 3.
\end{aligned} \tag{3.12}$$

This implies that $\delta(-1) < 0$. Differentiating $\delta(m)$ with respect to m and noting that we have assumed $i \leq j \leq k \leq l$,

$$\frac{d}{dm}\delta(m) = \frac{1}{2}\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j} + \frac{1}{\lambda_k} - \frac{1}{\lambda_l}\right) \geq \frac{1}{2}\frac{1}{\lambda_i}. \tag{3.13}$$

Thus,

$$\int_m^{-1} d\delta(m) \geq \int_m^{-1} \frac{1}{2}\frac{1}{\sqrt{i^2+m}}\,dm = \sqrt{i^2-1} - \sqrt{i^2+m}. \tag{3.14}$$

Moreover,

$$\delta(m) \leq \delta(-1) - \left(\sqrt{i^2-1} - \sqrt{i^2+m}\right) \leq -\left(\sqrt{i^2-1} - \sqrt{i^2+m}\right). \tag{3.15}$$

Therefore,

$$|\delta(m)| \geq \sqrt{i^2-1} - \sqrt{i^2+m} = \frac{-1-m}{\sqrt{i^2-1} + \sqrt{i^2+m}}. \tag{3.16}$$

Observe that there are positive constants c_1 and c_2 depending on m such that

$$c_1 < \frac{\sqrt{i^2-1}}{\sqrt{i^2+m}} < c_2. \tag{3.17}$$

Then

$$|\delta(m)| \geq c_3(m)\left(|\tilde{n}^2+m|\right)^{-1/2} \geq c(m)\left(|\tilde{n}^2+m|\right)^{-3/2}. \tag{3.18}$$

Case 2.2 (one minus sign and $i-j-k+l=0$). Let $j-i=l-k:=s\ge 0$. Set

$$\vartheta(s):=\delta(m)=\sqrt{i^2+m}+\sqrt{(i+s)^2+m}+\sqrt{k^2+m}-\sqrt{(k+s)^2+m}. \tag{3.19}$$

Then $\vartheta(0)=2\sqrt{i^2+m}$ and

$$\frac{d}{ds}\vartheta(s)=\frac{i+s}{\sqrt{(i+s)^2+m}}-\frac{k+s}{\sqrt{(k+s)^2+m}}. \tag{3.20}$$

Let $f(\tau)=\tau/\sqrt{\tau^2+m}$. Then, $df/d\tau=m/(\tau^2+m)^{3/2}<0$ in view of $m<-1<0$. This implies that the function $f(\tau)$ is decreasing in $\tau>0$. Thus, $(d/ds)\vartheta(s)\ge 0$ by noting that $i+s\le k+s$. Therefore, $\vartheta(s)\ge\vartheta(0)=2\sqrt{i^2+m}\ge 2\tilde{c}$.

Case 3 (two minus signs). Considering δ_{-+-+}, δ_{--++}, and δ_{+--+}, all other cases deduces to these cases by inverting the signs.

First, we consider the case δ_{-+-+}. By $i\le j\le k\le k\le l$ and $(i,-j,k,-l)\ne(p,-p,q,-q)$, we get that either $k+1\le l$ or $i+1\le j$. Thus

$$\begin{aligned}\delta_{-+-+}&\ge\begin{cases}-\sqrt{k^2+m}+\sqrt{l^2+m}, & \text{if } k+1\le l,\\ -\sqrt{i^2+m}+\sqrt{j^2+m}, & \text{if } i+1\le j,\end{cases}\\ &\ge\begin{cases}\dfrac{k+l}{\sqrt{k^2+m}+\sqrt{l^2+m}}, & \text{if } k+1\le l,\\ \dfrac{i+j}{\sqrt{i^2+m}+\sqrt{j^2+m}}, & \text{if } i+1\le j,\end{cases}\\ &\ge c(m)\left(|\tilde{n}^2+m|\right)^{-3/2}.\end{aligned} \tag{3.21}$$

Secondly, we consider the case δ_{--++}. By $i\le j\le k\le l$ and $(i,-j,k,-l)\ne(p,-p,q,-q)$, we get $i+1\le l$. Thus

$$\begin{aligned}\delta_{--++}&\ge\sqrt{l^2+m}-\sqrt{i^2+m}\\ &\ge\frac{i+l}{\sqrt{i^2+m}+\sqrt{l^2+m}}\\ &\ge c|m|\left(|\tilde{n}^2+m|\right)^{-3/2}.\end{aligned} \tag{3.22}$$

Thirdly, we consider the case δ_{+--+}. This is divided into two subcases.

Case 3.1 (δ_{+--+} and $i+j+k-l=0$). It is very easy to check that

$$\sqrt{l^2+m}-\sqrt{j^2+m}-\sqrt{k^2+m}\ge 0. \tag{3.23}$$

Thus, $\delta_{+--+}\ge\sqrt{i^2+m}\ge\tilde{c}$.

Case 3.2 (δ_{+--+} and $i-j-k+l=0$). Let $j-i=l-k=s$. By $(i,j,k,l)\neq(p,-p,q,-q)$, we get that $s\geq 1$ and $i+1\leq k$. Rewrite δ_{+--+} as

$$\delta_{+--+}=\sqrt{i^2+m}-\sqrt{(i+s)^2+m}-\sqrt{k^2+m}+\sqrt{(k+s)^2+m}:=\vartheta(s). \tag{3.24}$$

Then $\vartheta(0)=0$ and

$$\frac{d}{ds}\vartheta(s)=-\frac{i+s}{\sqrt{(i+s)^2+m}}+\frac{k+s}{\sqrt{(k+s)^2+m}}. \tag{3.25}$$

Noting that $\tau/\sqrt{\tau^2+m}$ is decreasing in $\tau>0$ for $m<-1<0$ and that $i+s\leq k+s$, we get that $(d/ds)\vartheta(s)\leq 0$. Thus, $\vartheta(s)\leq\vartheta(1)$ for $s\geq 1$. Now we are in position to estimate $\vartheta(1)$. Let $g(s)=\sqrt{(s+1)^2+m}-\sqrt{s^2+m}$ for $s^2+m>0$. Observing that the function $\tau/\sqrt{\tau^2+m}$ is decreasing in the variable $\tau>0$ for $m<0$, we get

$$\frac{d}{ds}g(s)=\frac{s+1}{\sqrt{(s+1)^2+m}}-\frac{s}{\sqrt{s^2+m}}<0. \tag{3.26}$$

Thus, the function $g(s)$ is decreasing in s. Moreover, by $i+1\leq k$,

$$\sqrt{(k+1)^2+m}-\sqrt{k^2+m}\leq\sqrt{(i+2)^2+m}-\sqrt{(i+1)^2+m}. \tag{3.27}$$

Thus

$$\vartheta(1)\leq\sqrt{i^2+m}-2\sqrt{(i+1)^2+m}+\sqrt{(i+2)^2+m}. \tag{3.28}$$

Observing that $(d^2/d\tau^2)\sqrt{\tau^2+m}<0$ for $m<0$, we get

$$\begin{aligned}&\sqrt{i^2+m}-2\sqrt{(i+1)^2+m}+\sqrt{(i+2)^2+m}\\&\qquad\leq\frac{d^2}{d\tau^2}\sqrt{\tau^2+m}\,\Big|_{\tau=i}=m\left(i^2+m\right)^{-3/2}<0.\end{aligned} \tag{3.29}$$

Thus

$$\left|\delta_{+--+}\right|\geq\left|\vartheta(1)\right|\geq c(m)\left(\left|\tilde{n}^2+m\right|\right)^{-3/2}. \tag{3.30}$$

$\square$

Lemma 3.4. *Assume that $m\in(-\infty,-1)$ and $m+j^2\neq 0$ for all $j\in\mathbb{Z}$. If i, j, k, l are nonzero integers such that $(i,j,k,l)\neq(p,-p,q,-q)$, $\min\{|i|,|j|,|k|,|l|\}<\sqrt{|m|}$, and $\max\{|i|,|j|,|k|,|l|\}>\sqrt{|m|}$, then*

$$\left|\lambda_i\pm\lambda_j\pm\lambda_k\pm\lambda_l\right|\geq c(m)\left(\left|\tilde{n}^2+m\right|\right)^{-3/2},\quad \tilde{n}=\min\{|i|,|j|,|k|,|l|\}, \tag{3.31}$$

with some constant $c=c(m)$ depending on m.

Proof

Case 1 ($i^2+m \le j^2+m \le k^2+m < 0$ and $l^2+m > 0$)**.** This case is trivial because $|\delta_{\pm\pm\pm\pm}| \ge \sqrt{l^2+m} \ge \tilde{c}$.

Case 2 ($i^2+m \le j^2+m < 0$ and $0 < k^2+m \le l^2+m$)**.** Without loss of generality, assume that, $0 < i \le j \le j \le k \le l$,

$$|\delta_{\pm\pm\pm\pm}| \ge \left(\left(\sqrt{|m|-i^2} \pm \sqrt{|m|-j^2}\right)^2 + \left(\sqrt{k^2+m} \pm \sqrt{l^2+m}\right)^2\right)^{1/2}. \tag{3.32}$$

By the assumption $(i,-j,k,-l) \ne (p,-p,q,-q)$, we get that either $i+1 \le j$ or $k+1 \le l$. If $i+1 \le j$, then

$$|\delta_{\pm\pm\pm\pm}|^2 \ge \left|\sqrt{|m|-i^2} - \sqrt{|m|-j^2}\right| \ge \frac{i+j}{\sqrt{|m|-i^2}+\sqrt{|m|-j^2}} \ge \hat{c}, \tag{3.33}$$

where

$$\hat{c} = \min\left\{\frac{i+j}{\sqrt{|m|-i^2}+\sqrt{|m|-j^2}} : \forall i,j \in \mathbb{Z},\ i^2+m<0,\ j^2+m<0\right\}. \tag{3.34}$$

If $k+1 \le l$, then

$$\begin{aligned} |\delta_{\pm\pm\pm\pm}|^2 &\ge \left|\sqrt{k^2+m} - \sqrt{l^2+m}\right| \ge \frac{k+l}{\sqrt{k^2+m}+\sqrt{l^2+m}} \\ &\ge c(m)\left(|\tilde{n}|^2+m\right)^{-3/2}. \end{aligned} \tag{3.35}$$

Case 3 ($i^2+m < 0$ and $0 < j^2+m \le k^2+m \le l^2+m$)**.** This case is also trivial because $|\delta_{\pm\pm\pm\pm}| \ge \sqrt{|m|-i^2} \ge \tilde{c}$.

This finishes the proof. □

We are now in position to transform the Hamiltonian (3.9) into some Birkhoff normal form of order four. For the rest of this paper, we introduce complex coordinates

$$z_j = \frac{1}{\sqrt{2}}(p_j+q_j), \qquad \bar{z}_j = \frac{1}{\sqrt{2}}(p_j-q_j). \tag{3.36}$$

Then the Hamiltonian (3.9) is of the form

$$H = \Lambda + G = \sum_{j\ge 1} \lambda_j z_j \bar{z}_j + \frac{1}{4}\sum_{ijkl} G_{ijkl}(z_i+\bar{z}_i)(z_j+\bar{z}_j)(z_k+\bar{z}_k)(z_l+\bar{z}_l), \tag{3.37}$$

where H is analytic on the now complex Hilbert space $\ell^{a,p}$ with symplectic structure $\sqrt{-1}\sum_j dz_j \wedge d\bar{z}_j$. Let $j_0 \in \mathbb{Z}^+$ such that $j_0^2+m<0$ and $(j_0+1)^2+m>0$.

Proposition 3.5. *For any given finite $n \geq 1$ and each $m < -1$ with $j^2 + m \neq 0$ for all $j \in \mathbb{Z}$, there is a real symplectic change of coordinate Γ in some neighborhood of the origin in $\ell^{a,p}$ that takes the Hamiltonian (3.37) into*

$$H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K, \tag{3.38}$$

where $X_{\bar{G}}$, $X_{\hat{G}}$, $X_K \in \mathcal{A}(\ell^{a,p}, \ell^{a,p+1})$,

$$\bar{G} = \left(\sum_{\substack{j_0+1 \leq j \leq j_0+n \\ i \leq j_0}} + \frac{1}{2} \sum_{\substack{j_0+1 \leq j \leq j_0+n \\ j_0+1 \leq i \leq j_0+n}} + \sum_{\substack{j \geq j_0+n+1 \\ j_0+1 \leq i \leq j_0+n}} \right) \bar{G}_{ij} \left| z_i \right|^2 \left| z_j \right|^2 \tag{3.39}$$

with uniquely determined coefficient

$$\bar{G}_{ij} = G_{iijj} = \frac{6}{\pi} \cdot \frac{4 - \delta_{ij}}{\lambda_i \lambda_j}, \tag{3.40}$$

$$|\hat{G}| = O\left(\|\hat{z}\|_{a,p}^4\right), \qquad |K| = O\left(\|z\|_{a,p}^6\right), \tag{3.41}$$

where $\hat{z} = (z_1, \ldots, z_{j_0}, z_{j_0+n+1}, \ldots)$.

Proof. Let $\ell_b^{a,p}$ be the Hilbert space consisting of all bi-infinite sequences with finite norm

$$\|q\|_{a,p}^2 = \sum_{j \in \mathbb{Z}} \left| q_j \right|^2 |j|^{2p} e^{2|j|a}. \tag{3.42}$$

Introduce another set of coordinates $(\ldots, w_{-2}, w_{-1}, w_1, w_2, \ldots)$ in $\ell_b^{a,p}$ by setting $z_j = w_j$, $\bar{z}_j = w_{-j}$. The Hamiltonian (3.37) then reads

$$H = \Lambda + G = \sum_{j \geq 1} \lambda_j w_j w_{-j} + \frac{1}{4} \sum_{ijkl}{}' G_{ijkl} w_i w_j w_k w_l, \tag{3.43}$$

where the prime indicates that the subscript indices run through all nonzero integers and the coefficients are defined for arbitrary integers by setting $G_{ijkl} = G_{|i||j||k||l|}$. We recall that the sum is restricted to indices i, j, k, l such that $i \pm j \pm j \pm k \pm l = 0$. This is crucial for the following to hold. In order to find the transformation Γ, we need some extra notations. Let $\lambda_j' = \operatorname{sgn} j \cdot \lambda_{|j|}$,

$$\begin{aligned} \Upsilon_n = {} & \left\{ (i,j,k,l) \in \mathbb{Z}^4 : j_0 < \min \{ |i|, \ldots, |l| \} \leq j_0 + n \right\} \\ & \bigcup \left\{ (i,j,k,l) \in \mathbb{Z}^4 : \min \{ |i|, |j|, |k|, |l| \} < j_0 < \max \{ |i|, \ldots, |l| \} \right\} \end{aligned} \tag{3.44}$$

and $\mathcal{N}_n \subset \Upsilon_n$ is the subset of all $(i,j,k,l) \equiv (p, -p, q, -q)$. Clearly, for $(i,j,k,l) \in \mathcal{N}_n$, the quantity $\lambda_i' + \lambda_j' + \lambda_k' + \lambda_k'$ vanishes identically in m. We now can find the needed transformation Γ. Formally, it is obtained as the time-1-map of the flow of a Hamiltonian vector field X_F given by a Hamiltonian

$$F = \sum_{ijkl}{}' F_{ijkl} w_i w_j w_k w_l \tag{3.45}$$

with coefficients

$$\sqrt{-1}F_{ijkl}=\begin{cases}\dfrac{G_{ijkl}}{\lambda_i'+\lambda_j'+\lambda_k'+\lambda_k'}, & \text{for } (i,j,k,l)\in\Upsilon_n\setminus\mathcal{N}_n,\\ 0, & \text{otherwise.}\end{cases}\tag{3.46}$$

The Hamiltonian F is thus well defined by Lemmas 3.3 and 3.4. Now the following proof is all the same as that of the main proposition in Pöschel [9, pages 272–273]. We omit the details. □

3.3. Proof of Theorem 1.1. Consider the Hamiltonian (3.38). All what we need to do is to check that the Hamiltonian (3.38) satisfies Assumptions 2.7, 2.8, 2.9, and 2.10. Let $\alpha=(\lambda_{j_0+1},\dots,\lambda_{j_0+n})$, $\beta=(\beta_*,\beta_{**})=(\lambda_1,\dots,\lambda_{j_0},\lambda_{j_0+n+1},\dots)$ $A=(\bar{G}_{ij})_{j_0+1\le i,j\le j_0+n}$, $B=(B_1,B_2)^T$, $B_1=(\bar{G}_{ij})_{i\le j_0,j_0+1\le j\le j_0+n}$, and $B_2=(\bar{G}_{ij})_{i\ge j_0+n+1,j_0+1\le j\le j_0+n}$. Let

$$\begin{gathered}y=\frac{1}{2}\Big(|z_{j_0+1}|^2,\dots,|z_{j_0+n}|^2\Big),\\ Z=(Z_*;Z_{**})=\frac{1}{2}\Big(|z_1|^2,\dots,|z_{j_0}|^2;|z_{j_0+n+1}|^2,\dots\Big).\end{gathered}\tag{3.47}$$

Then the Hamiltonian (3.38) is of the form $H=\Lambda+\bar{G}+\hat{G}+K$, where

$$\begin{gathered}\Lambda=\langle\alpha,y\rangle+\langle\beta,Z\rangle,\\ \bar{G}=\frac{1}{2}\langle Ay,y\rangle+\langle By,Z\rangle=\frac{1}{2}\langle Ay,y\rangle+\langle B_1y,Z_*\rangle+\langle B_2y,Z_{**}\rangle,\end{gathered}\tag{3.48}$$

$|\hat{G}|=O(\|\hat{z}\|^4_{a,p})$, and $|K|=O(\|z\|^6_{a,p})$. According to the notations in Section 2.2,

$$\begin{gathered}\omega(y)=\alpha+Ay,\qquad \Omega(y)=\beta+By,\\ \Omega_*(y)=\beta_*+B_1y,\qquad \Omega_{**}(y)=\beta_{**}+B_2y.\end{gathered}\tag{3.49}$$

Observing that λ_j is pure imaginary for $1\le j\le j_0$ and λ_j is real for $j\ge j_0+1$ and

$$(B_1)_{ij}=(\bar{G})_{ij}=\frac{6}{\pi}\cdot\frac{4-\delta_{ij}}{\lambda_i\lambda_j}\quad\text{with } i\le j_0,\ j_0+1\le j\le j_0+n,\tag{3.50}$$

we get that $\Re B_1=0$; hence, $\Re\Omega_*=0$. Namely, Assumption 2.9 is satisfied. Let $Q=\bar{G}$ and $R=\hat{G}+K$. Then $H=\Lambda+Q+R$ which is of the form required by the Cantor manifold theorem. By Proposition 3.5, $X_Q,X_R\in A(\ell^{a,p},\ell^{a,p+1})$ with $|R|=O(\|\hat{z}\|^4_{a,p})+O(\|z\|^6_{a,p})$. On the other hand, we have

$$\lambda_j=\sqrt{j^2+m}=j+\frac{m}{2j}+O(j^{-3}),\quad j\ge j_0+1.\tag{3.51}$$

So Assumptions 2.8 and 2.10 are satisfied with $d=1$, $\delta=-1$ and $\bar{p}=p+1>p$.

Moreover, by (3.40) and the definition of B_2,

$$(\Omega_{**})_l = (\beta_{**} + B_2 y)_l = \lambda_l + \frac{\langle v, y\rangle}{\lambda_l} \tag{3.52}$$

with $v = (24/\pi)\cdot(\lambda_{j_0+1}^{-1}, \dots, \lambda_{j_0+n}^{-1})$. This gives the asymptotic expansion

$$\Omega_l = l + \frac{m}{2l} + \frac{\langle v, y\rangle}{l} + O(l^{-3}) = j + \frac{m_y}{l} + O(l^{-3}), \tag{3.53}$$

where $m_y = (1/2)m + \langle v, y\rangle$. Thus, for $i > j > j_0 + n$,

$$\frac{\Omega_i - \Omega_j}{i-j} = 1 - \frac{m_y}{ij} + O(j^{-3}) = 1 + O(j^2) \tag{3.54}$$

uniformly for bounded y. This gives $\kappa = 2$ in (2.32). Consequently, also the smallness condition in the Cantor manifold theorem is satisfied since

$$g > 4 + \frac{4-\varrho}{\kappa} \tag{3.55}$$

for $g = 6$, $\kappa = 2$, and $\varrho = 1$.

Finally, we verify that Assumption 2.7 is satisfied. Item (1) is satisfied in view of the following lemma.

Lemma 3.6. *For all $n \geq 1$ and all $m < -1$ with $j^2 + m \neq 0$ for all $j \in \mathbb{Z}$, the matrix $A = (\bar{G}_{ij})_{j_0+1\leq i,j\leq j_0+n}$ is nonsingular.*

Proof. Following the argument of [9, Lemma 5], we get that

$$\det A = (-1)^n \left(\frac{\pi}{6}\right)^n (1-4n) \prod_{j_0+1\leq j\leq j_0+n} \frac{1}{\lambda_j^2} \neq 0. \tag{3.56}$$

□

Lemma 3.7. *For the index set $J = \{j_0+1, \dots, j_0+n\}$ with $n \geq 5$, $\langle k, \omega(y)\rangle + \langle l, \Omega(y)\rangle$ does not vanish identically for all $(k,l) \in \mathbb{Z}^n \times \hat{\mathbb{Z}}^\infty$ and all $m < -1$ with $j^2 + m \neq 0$ for all $j \in \mathbb{Z}$.*

Proof. The argument of [9, Lemma 6] is applicable provided that we take slight modification. It suffices to show that either $\langle \alpha, k\rangle \neq \langle \beta, l\rangle$ or $Ak \neq B_2^T l$ for $(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$.

Suppose to the contrary that $\langle \alpha, k\rangle = \langle \beta, l\rangle$ and $Ak = B_2^T l$. Multiplying A and B_2 by $\pi/6$ and letting D be the n-dimensional diagonal matrix with diagonal elements $D_j = \lambda_{j_0+j}^{-2}$, and C is the rank one $n \times n$ matrix with elements $C_{ij} = 4\lambda_{j_0+i}^{-1}\lambda_{j_0+j}^{-1}$ where $i, j = 1, 2, \dots, n$, we then have $Dk = Ck - B_2^T l$ or

$$\frac{k_i}{\lambda_i} = 4\langle v, k\rangle - 4\langle w, l\rangle, \tag{3.57}$$

where $v = (\lambda_{j_0+1}^{-1}, \dots, \lambda_{j_0+n}^{-1})$ and $w = (\lambda_{j_0+n+1}^{-1}, \lambda_{j_0+n+2}^{-1}, \dots)$. Thus, $k_i \lambda_{j_0+i}^{-1}$ is independent of i, whence $\langle v, k\rangle = n k_i \lambda_{j_0+i}^{-1}$ and thus

$$k_i = \frac{4}{4n-1} \lambda_{j_0+i} \langle w, l\rangle, \quad 1 \le i \le n. \tag{3.58}$$

The assumption $\langle \alpha, k\rangle = \langle \beta, l\rangle$, then further implies that

$$\frac{4}{4n-1} \sum_{1 \le i \le n} \lambda_{j_0+i}^2 = \frac{\langle \beta, l\rangle}{\langle w, l\rangle}. \tag{3.59}$$

We first show that for $|l| = 1$, this is not possible. In fact, we then have that $\langle \beta, l\rangle = \pm \lambda_{j_0+n+\nu} = \langle w, l\rangle^{-1}$ for some $\nu \in \mathbb{N}$, so (3.58) and (3.59) combined give

$$k_i^2 = \frac{4}{4n-1} \cdot \frac{\lambda_{j_0+i}^2}{\sum_i \lambda_{j_0+i}^2}, \quad 1 \le i \le n. \tag{3.60}$$

But this equation cannot have an integer solution for any $n \ge 5$ and any $1 \le i \le n$.

So now consider the case $|l| = 2$. To show this case is also ridiculous, we need an inequality. Let

$$g(x) = \sqrt{(x+1)^2 + m} - \sqrt{x^2 + m}, \quad x \ge \sqrt{|m|}. \tag{3.61}$$

It is very easy to verify that the function g is positive and decreasing for $x \ge j_0 + 1$. Hence, we get that for $x \ge j_0 + 1$,

$$g(x) \le f(j_0+1) = \sqrt{(j_0+2)^2 + m} - \sqrt{(j_0+1)^2 + m} \le \sqrt{2j_0+3}. \tag{3.62}$$

If we had

$$|\langle w, l\rangle| < \frac{1}{\sqrt{2j_0+3}} \cdot \frac{4n-1}{4n-4}, \tag{3.63}$$

then (3.58) and (3.62) imply

$$0 \ne \min_{1 \le i \le n} |k_{i+1} - k_i| < \min_{1 \le i \le n} \frac{|\lambda_{j_0+i+1} - \lambda_{j_0+i}|}{(n-1)\sqrt{2j_0+3}} \le \frac{1}{n-1} \tag{3.64}$$

which is not possible. On the other hand, the case

$$|\langle w, l\rangle| \ge \frac{1}{\sqrt{2j_0+3}} \cdot \frac{4n-1}{4n-4} \tag{3.65}$$

is ridiculous since

$$
\begin{aligned}
|\langle w, l\rangle| \leq 2\lambda_{j_0+n+1}^{-1} &= \frac{2}{\sqrt{(j_0+n+1)^2+m}} \\
&\leq \frac{2}{\sqrt{2(j_0+1)n+n^2}} < \frac{1}{\sqrt{2j_0+3}} \cdot \frac{4n-1}{4n-4}
\end{aligned}
\tag{3.66}
$$

if $n \geq 5$.

This completes the proof. □

This lemma shows that Assumption 2.7(3) is satisfied. Assumption 2.7(2) is very easy to check. Finally, we finish the proof of Theorem 1.1.

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