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CAUCHY APPROXIMATION FOR SUMS OF INDEPENDENT RANDOM VARIABLES

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We use Stein's method to find a bound for Cauchy approximation. The random variables which are considered need to be independent.

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1. Introduction. In Stein's work [19], the aim was to show convergence in distribution to the normal. His technique was novel. Stein's technique was free from Fourier methods and relied instead on the elementary differential equation

$$f'(w)-wf(w)=h(x)-Nh \quad (w\in\mathbb{R}), \tag{1.1}$$

where $h:\mathbb{R}\to\mathbb{R}$ is such that

$$\int_{-\infty}^{\infty}|h(x)|e^{-(1/2)x^2}dx<\infty \tag{1.2}$$

and $Nh=E(h(Z))$, where $Z\sim N(0,1)$.

Stein's method was extended from normal distribution to the Poisson distribution by Chen [9]. Stein's equation for Poisson with parameter λ is

$$\lambda f(w+1)-wf(w)=h(w)-P_\lambda h \quad (w\in\mathbb{Z}^+), \tag{1.3}$$

where $P_\lambda h=E(h(Z))$, $Z\sim\text{Poi}(\lambda)$.

Since then, Stein's method has found considerable applications in combinatorics, probability, and statistics. Recent literature pertaining to this method includes Arratia et al. [1, 2], Baldi and Rinott [3], Barbour [4, 5], Barbour et al. [6], Bolthausen and Götze [7], Chen [10, 11], Goldstein and Reinert [12], Goldstein and Rinott [13], Götze [14], and Green [15]; the work of Holst and Janson [16] gives an excellent account of this method. In this paper, we further develop the Stein technique to bound errors for a Cauchy approximation to the distribution of W, the sum of independent random variables. In fact, there are some literatures (e.g., Boonyasombut and Shapiro [8], Neammanee [17], and Shapiro [18]) give a bound of Cauchy approximation in some kind of random variables. But they used Fourier methods.

This paper is organized as follows. Main results are stated in Section 2. Proof of main results is in Section 3, while an example is given in Section 4.

2. Main results. At the heart of Stein's method lies a Stein equation. For example,

$$\begin{aligned} f'(w)-wf(w)&=g(w), \quad w\in\mathbb{R},\\ \lambda f(w+1)-wf(w)&=g(w), \quad w\in\mathbb{Z}^+ \end{aligned} \tag{2.1}$$

are Stein equations for normal and Poisson distribution, respectively.

Let $\mathcal{H}=\{h:\mathbb{R}\to\mathbb{R} \mid \int_{-\infty}^{\infty}(|h(x)|/(1+x^2))dx<\infty\}$, and for each $h\in\mathcal{H}$,

$$\text{Cau}(h)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h(x)}{1+x^2}dx. \tag{2.2}$$

The Stein equation for Cauchy distribution F

$$F(x)=\frac{1}{\pi}\int_{-\infty}^{x}\frac{1}{1+t^2}dt \tag{2.3}$$

is

$$f'(w)-\frac{2wf(w)}{1+w^2}=h(w)-\text{Cau}(h). \tag{2.4}$$

It is easy to check that a solution of (2.4) is $U_h:\mathbb{R}\to\mathbb{R}$ defined by

$$U_h(w)=(1+w^2)\int_{-\infty}^{w}\frac{h(x)-\text{Cau}(h)}{1+x^2}dx. \tag{2.5}$$

Fix $w_0\in\mathbb{R}$, and choose h to be the indicator function $I_{(-\infty,w_0]}$ which is defined by

$$I_{(-\infty,w_0]}(w)=\begin{cases}1 & \text{if } w\le w_0,\\ 0 & \text{if } w>w_0.\end{cases} \tag{2.6}$$

Let $f_{w_0}=U_{I_{(-\infty,w_0]}}$. Then, by (2.2), (2.3), and (2.5), we see that

$$f_{w_0}(w)=\begin{cases}\pi(1+w^2)F(w)(1-F(w_0)) & \text{if } w\le w_0,\\ \pi(1+w^2)F(w_0)(1-F(w)) & \text{if } w\ge w_0.\end{cases} \tag{2.7}$$

The broad idea of Stein's argument is as follows. First, for any $w_0\in\mathbb{R}$, a function $f_{w_0}:\mathbb{R}\to\mathbb{R}$ is constructed to solve (2.4) when h is the indicator function $I_{(-\infty,w_0]}$. Replacing w by W, for any random variable W, it therefore follows that the difference between $P(W\le w_0)$ and $F(w_0)$ can be expressed as

$$E\left\{f'_{w_0}(W)-\frac{2Wf_{w_0}(W)}{1+W^2}\right\}. \tag{2.8}$$

The main results are the following.

Theorem 2.1. *Let $X_1, X_2, \dots, X_n$ be independent random variables with $EX_i = 0$, $EX_i^2 = \sigma_i^2$, and $E|X_i|^4 < \infty$. Then,*

$$\begin{aligned} |P(W \le w_0) - F(w_0)| &\le 3\sqrt{E\left[1 - \sum_{i=1}^{n} \frac{\sigma_i^2 + X_i^2}{1+W^2}\right]^2} \\ &\quad + 4\pi \min\left\{\sum_{i=1}^{n} \sigma_i^2, 2\sqrt{n\sum_{i=1}^{n}\sigma_i^2 \sum_{i=1}^{n} E|X_i|^4}\right\} F(w_0)(1-F(w_0)) \\ &\quad + C\sum_{i=1}^{n} E|X_i|^3, \end{aligned} \tag{2.9}$$

when $W = X_1 + X_2 + \cdots + X_n$.

Corollary 2.2. *Let $Y_1, Y_2, \dots, Y_n$ be identically independent random variables with zero means $EY_i^2 = 1/2$ and $E|Y_i|^5 < \infty$. Let $X_i = Y_i/\sqrt{n}$ and $W = X_1 + X_2 + \cdots + X_n$. Then,*

$$|P(W \le w_0) - F(w_0)| < \frac{C}{\sqrt[4]{n}} + C\min\left\{\frac{1}{2}, \sqrt{2}\sqrt{EY_i^4}\right\} F(w_0)(1-F(w_0)). \tag{2.10}$$

Throughout this paper, C stands for an absolute constant with possibly different values in different places.

3. Proof of main results. Before we prove the main results, we need the following lemmas.

Lemma 3.1. *For any real numbers w_0 and w,*

(1) $|f_{w_0}(w)/(1+w^2)| \le \pi F(w_0)(1-F(w_0))$
(2) $|f'_{w_0}(w)| \le 3$
(3) $|f''_{w_0}(w)| \le 3 + 2\pi$
(4) $|(f'_{w_0}(w)/(1+w^2))'| \le 6 + 2\pi$
(5) $|(w f_{w_0}(w)/(1+w^2)^2)'| \le 3 + 5\pi$.

Proof. (1) follows directly from (2.7).

(2) Before we start the proof, we need the following inequalities:

$$-\frac{1}{\pi} \le wF(w) \le 0 \quad \text{for } w \le 0, \tag{3.1}$$

$$0 \le w(1-F(w)) \le \frac{1}{\pi} \quad \text{for } w > 0. \tag{3.2}$$

To show (3.1), we define g on $(-\infty,0]$ by $g(w) = wF(w)$. Since $g''(w) = 2/\pi(1+w^2)^2 > 0$, g' is increasing. From this fact and the fact that

$$\lim_{w\to-\infty} g'(w) = \lim_{w\to-\infty} \frac{1}{\pi}\left(\frac{w}{1+w^2} + \arctan w + \frac{\pi}{2}\right) = 0, \tag{3.3}$$

we have $g' \geq 0$. Hence, g is increasing and

$$-\frac{1}{\pi} = \lim_{t\to-\infty} g(t) \leq g(w) \leq g(0) = 0 \tag{3.4}$$

for any $w \leq 0$. So (3.1) holds. To show (3.2), we can apply the same argument to the function $\tilde{g}$ on $[0,\infty)$ which is defined by $\tilde{g}(w) = w(1-C(w))$. Since $f_{w_0}(w) = f_{-w_0}(-w)$, it suffices to prove the lemma in the case where $w_0 \geq 0$. By (2.7), we have

$$\begin{aligned}
|f'_{w_0}(w)| &= \begin{cases} |(1-F(w_0))(1+2\pi wF(w))| & \text{if } w \leq 0, \\ |F(w_0)(-1+2\pi w(1-F(w)))| & \text{if } w \geq w_0 \end{cases} \\
&\leq \begin{cases} 1+2\pi|wF(w)| & \text{if } w \leq 0, \\ 1+2\pi|w(1-F(w))| & \text{if } w \geq w_0 \end{cases} \\
&\leq \begin{cases} 3 & \text{if } w \leq 0, \\ 3 & \text{if } w \geq w_0, \end{cases}
\end{aligned} \tag{3.5}$$

where we have used the fact that $0 \leq F(w) \leq 1$ in the first inequality and (3.1) and (3.2) in the second inequality. In the case where $0 \leq w \leq w_0$, by monotonicity of F and (3.2), we see that

$$\begin{aligned}
0 &\leq f'_{w_0}(w) \\
&= (1-F(w_0)) + 2\pi(1-F(w_0))wF(w) \\
&\leq 1 + 2\pi(1-F(w))w \leq 3.
\end{aligned} \tag{3.6}$$

Hence, (2) follows from (3.5) and (3.6).

(3) follows immediately from (2) and the fact that

$$f''_{w_0}(w) = \frac{2w}{1+w^2} f'_{w_0}(w) + \frac{2(1-w^2)}{(1+w^2)^2} f_{w_0}(w). \tag{3.7}$$

(4) and (5) follow from (2) and (3) and the facts that

$$\begin{aligned}
\left(\frac{f'_{w_0}(w)}{1+w^2}\right)' &= \frac{f''_{w_0}(w)}{1+w^2} - \frac{2wf'_{w_0}(w)}{(1+w^2)^2}, \\
\left(\frac{wf_{w_0}(w)}{(1+w^2)^2}\right)' &= \frac{wf'_{w_0}(w) + f_{w_0}(w)}{(1+w^2)^2} - \frac{4w^2 f_{w_0}(w)}{(1+w^2)^3}.
\end{aligned} \tag{3.8}$$

□

Lemma 3.2. *Let $(W,\widetilde{W})$ be an exchangeable pair of random variables, that is,*

$$P(W \in B, \widetilde{W} \in \widetilde{B}) = P(W \in \widetilde{B}, \widetilde{W} \in B) \tag{3.9}$$

for any Borel sets B and $\widetilde{B}$ on $\mathbb{R}$, and there exists $\lambda > 0$ such that

$$E^W \widetilde{W} = (1-\lambda)W, \qquad E|\widetilde{W} - W|^2 < \infty, \tag{3.10}$$

where $E^W \widetilde{W}$ is the conditional expectation of $\widetilde{W}$ with respect to W. Then,

$$E\left[\frac{2Wf(W)}{1+W^2} - \frac{1}{\lambda}(\widetilde{W}-W)\left(\frac{f(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f(W)}{1+W^2}\right)\right] = 0 \tag{3.11}$$

for any function $f : \mathbb{R} \to \mathbb{R}$, for which there exists $C > 0$ such that for all $w \in \mathbb{R}$,

$$|f(w)| \le C(1+w^2). \tag{3.12}$$

Moreover,

$$P(W \le w_0) = C(w_0) + E\left[f'_{w_0}(W) - \frac{1}{\lambda}(\widetilde{W}-W)\left(\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f_{w_0}(W)}{1+W^2}\right)\right] \tag{3.13}$$

for any $w_0 \in \mathbb{R}$.

Proof. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(w,\widetilde{w}) = (\widetilde{w} - w)\left[\frac{f(\widetilde{w})}{1+\widetilde{w}^2} + \frac{f(w)}{1+w^2}\right]. \tag{3.14}$$

Then, F is antisymmetric, that is, $F(w,\widetilde{w}) = -F(\widetilde{w},w)$. By Stein [20, pages 9–10], we have $EF(W,\widetilde{W}) = 0$, which implies that

$$\begin{aligned}
0 &= E(\widetilde{W}-W)\left[\frac{f(\widetilde{W})}{1+\widetilde{W}^2} + \frac{f(W)}{1+W^2}\right] \\
&= E(\widetilde{W}-W)\left\{\frac{2f(W)}{1+W^2} + \left[\frac{f(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f(W)}{1+W^2}\right]\right\} \\
&= 2E(E^W\widetilde{W}-W)\frac{f(W)}{1+W^2} + E(\widetilde{W}-W)\left[\frac{f(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f(W)}{1+W^2}\right] \\
&= -\lambda E\left(\frac{2Wf(W)}{1+W^2}\right) + E(\widetilde{W}-W)\left[\frac{f(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f(W)}{1+W^2}\right] \\
&= E\left[\frac{2Wf(W)}{1+W^2} - \frac{1}{\lambda}(\widetilde{W}-W)\left[\frac{f(\widetilde{W})}{1+\widetilde{W}^2} - \frac{f(W)}{1+W^2}\right]\right].
\end{aligned} \tag{3.15}$$

Then, (3.11) holds and (3.13) follows from (3.11) and (2.4) when $h = I_{(-\infty,w_0]}$. □

Lemma 3.3. *Let $(W,\widetilde{W})$ be an exchangeable pair of random variables such that*

$$E^W\widetilde{W}=(1-\lambda)W,\qquad E|\widetilde{W}-W|^2<\infty \tag{3.16}$$

with $\lambda>0$. Then, for any $w_0\in\mathbb{R}$,

$$\begin{aligned}
&P(W\le w_0)\\
&\quad=C(w_0)+Ef'_{w_0}(W)\left[1-\frac{1}{\lambda}E^W\frac{(\widetilde{W}-W)^2}{1+W^2}\right]+\frac{2}{\lambda}\frac{E(\widetilde{W}-W)^2Wf_{w_0}(W)}{(1+W^2)^2}\\
&\qquad+\frac{1}{\lambda}\int_{-\infty}^{\infty}E(\widetilde{W}-W)\left(w-\frac{W+\widetilde{W}}{2}\right)\\
&\qquad\qquad\times[I(w\le\widetilde{W})-I(w\le W)]\left(\frac{f'_{w_0}(w)}{1+w^2}\right)'dw\\
&\qquad-\frac{2}{\lambda}\int_{-\infty}^{\infty}E(\widetilde{W}-W)\left(w-\frac{W+\widetilde{W}}{2}\right)\\
&\qquad\qquad\times[I(w\le\widetilde{W})-I(w\le W)]\left(\frac{wf_{w_0}(w)}{(1+w^2)^2}\right)'dw.
\end{aligned}\tag{3.17}$$

Proof. Let $w_0\in\mathbb{R}$. For $W<\widetilde{W}$, we see that

$$\begin{aligned}
&\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}-\frac{f_{w_0}(W)}{1+W^2}-\frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2}+\frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\\
&\quad=\int_W^{\widetilde{W}}\left[\left(\frac{f_{w_0}(w)}{1+w^2}\right)'-\frac{f'_{w_0}(W)}{1+W^2}+\frac{2Wf_{w_0}(W)}{(1+W^2)^2}\right]dw\\
&\quad=\int_W^{\widetilde{W}}\left[\frac{f'_{w_0}(w)}{1+w^2}-\frac{2wf_{w_0}(w)}{(1+w^2)^2}-\frac{f'_{w_0}(W)}{1+W^2}+\frac{2Wf_{w_0}(W)}{(1+W^2)^2}\right]dw\\
&\quad=\int_W^{\widetilde{W}}\int_W^{w}\left(\frac{f'_{w_0}(y)}{1+y^2}\right)'dy\,dw-2\int_W^{\widetilde{W}}\int_W^{w}\left(\frac{yf_{w_0}(y)}{(1+y^2)^2}\right)'dy\,dw\\
&\quad=\int_W^{\widetilde{W}}\int_y^{\widetilde{W}}\left(\frac{f'_{w_0}(y)}{1+y^2}\right)'dw\,dy-2\int_W^{\widetilde{W}}\int_y^{\widetilde{W}}\left(\frac{yf_{w_0}(y)}{(1+y^2)^2}\right)'dw\,dy\\
&\quad=\int_W^{\widetilde{W}}(\widetilde{W}-y)\left(\frac{f'_{w_0}(y)}{1+y^2}\right)'dy-2\int_W^{\widetilde{W}}(\widetilde{W}-y)\left(\frac{yf_{w_0}(y)}{(1+y^2)^2}\right)'dy,
\end{aligned}\tag{3.18}$$

and by the same argument we can show that

$$\begin{aligned}
&\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}-\frac{f_{w_0}(W)}{1+W^2}-\frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2}+\frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\\
&\quad=\int_{\widetilde{W}}^{W}(w-\widetilde{W})\left(\frac{f'_{w_0}(w)}{1+w^2}\right)'dw-2\int_{\widetilde{W}}^{W}(w-\widetilde{W})\left(\frac{wf_{w_0}(w)}{(1+w^2)^2}\right)'dw
\end{aligned}\tag{3.19}$$

for $\widetilde{W}<W$.

So,

$$\begin{aligned}
&\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}-\frac{f_{w_0}(W)}{1+W^2}-\frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2}+\frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\\
&\quad=\int_{-\infty}^{\infty}(\widetilde{W}-w)[I(w\le\widetilde{W})-I(w\le W)]\left(\frac{f'_{w_0}(w)}{1+w^2}\right)'dw\\
&\qquad-2\int_{-\infty}^{\infty}(\widetilde{W}-w)[I(w\le\widetilde{W})-I(w\le W)]\left(\frac{wf_{w_0}(w)}{(1+w^2)^2}\right)'dw.
\end{aligned}\tag{3.20}$$

By Lemma 3.2, we have

$$\begin{aligned}
&P(W\le w_0)\\
&\quad=C(w_0)+E\bigg[f'_{w_0}(W)-\frac{1}{\lambda}\frac{f'_{w_0}(W)(\widetilde{W}-W)^2}{1+W^2}+\frac{1}{\lambda}\frac{f'_{w_0}(W)(\widetilde{W}-W)^2}{1+W^2}\\
&\qquad\qquad+\frac{2}{\lambda}\frac{(\widetilde{W}-W)^2Wf_{w_0}(W)}{(1+W^2)^2}-\frac{2}{\lambda}\frac{(\widetilde{W}-W)^2Wf_{w_0}(W)}{(1+W^2)^2}\\
&\qquad\qquad-\frac{1}{\lambda}(\widetilde{W}-W)\left[\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}-\frac{f_{w_0}(W)}{1+W^2}\right]\bigg]\\
&\quad=C(w_0)+Ef'_{w_0}(W)-\frac{1}{\lambda}EE^W\frac{f'_{w_0}(W)(\widetilde{W}-W)^2}{1+W^2}\\
&\qquad+\frac{2}{\lambda}\frac{E(\widetilde{W}-W)^2Wf_{w_0}(W)}{(1+W^2)^2}-\frac{1}{\lambda}E(\widetilde{W}-W)\\
&\qquad\times\left[\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}-\frac{f_{w_0}(W)}{1+W^2}-\frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2}+\frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\right]\\
&\quad=C(w_0)+E\left[f'_{w_0}(W)\left\{1-\frac{1}{\lambda}E^W\frac{(\widetilde{W}-W)^2}{1+W^2}\right\}\right]\\
&\qquad+\frac{2}{\lambda}\frac{E(\widetilde{W}-W)^2Wf_{w_0}(W)}{(1+W^2)^2}-\frac{1}{\lambda}E(\widetilde{W}-W)\\
&\qquad\times\left[\frac{f_{w_0}(\widetilde{W})}{1+\widetilde{W}^2}-\frac{f_{w_0}(W)}{1+W^2}-\frac{(\widetilde{W}-W)f'_{w_0}(W)}{1+W^2}+\frac{2(\widetilde{W}-W)Wf_{w_0}(W)}{(1+W^2)^2}\right]\\
&\quad=C(w_0)+E\left[f'_{w_0}(W)\left\{1-\frac{1}{\lambda}E^W\frac{(\widetilde{W}-W)^2}{1+W^2}\right\}\right]+\frac{2}{\lambda}\frac{E(\widetilde{W}-W)^2Wf_{w_0}(W)}{(1+W^2)^2}\\
&\qquad-\frac{1}{\lambda}E(\widetilde{W}-W)\int_{-\infty}^{\infty}(\widetilde{W}-w)[I(w\le\widetilde{W})-I(w\le W)]\left(\frac{f'_{w_0}(w)}{1+w^2}\right)'dw\\
&\qquad+\frac{2}{\lambda}E(\widetilde{W}-W)\int_{-\infty}^{\infty}(\widetilde{W}-w)[I(w\le\widetilde{W})-I(w\le W)]\left(\frac{wf_{w_0}(w)}{(1+w^2)^2}\right)'dw,
\end{aligned}\tag{3.21}$$

where we have used (3.20) in the last equality.

For fixed w, we define $F:\mathbb{R}^2\to\mathbb{R}$ by

$$F(x,\tilde{x}) = (x-\tilde{x})\left(\frac{x-\tilde{x}}{2}\right)[I(w\le\tilde{x})-I(w\le x)]. \tag{3.22}$$

Then, F is antisymmetric. Since W and $\widetilde{W}$ are exchangeable, $EF(W,\widetilde{W}) = 0$. Thus,

$$\begin{aligned}
&E(\widetilde{W}-W)(w-\widetilde{W})[I(w\le\widetilde{W})-I(w\le W)]\\
&\quad= E(\widetilde{W}-W)\left(w-\frac{W+\widetilde{W}}{2}+\frac{W-\widetilde{W}}{2}\right)[I(w\le\widetilde{W})-I(w\le W)]\\
&\quad= E(\widetilde{W}-W)\left(w-\frac{W+\widetilde{W}}{2}\right)[I(w\le\widetilde{W})-I(w\le W)]-EF(W,\widetilde{W})\\
&\quad= E(\widetilde{W}-W)\left(w-\frac{W+\widetilde{W}}{2}\right)[I(w\le\widetilde{W})-I(w\le W)].
\end{aligned} \tag{3.23}$$

By (3.21) and (3.23), the lemma is proved. □

Proof of Theorem 2.1. Let $X_1,X_2,\dots,X_n$ be independent random variables and $W = X_1+X_2+\cdots+X_n$. In order to prove the theorem, we introduce additional random variables I, $\widetilde{X}_1,\widetilde{X}_2,\dots,\widetilde{X}_n$, and $\widetilde{W}$ defined in the following way. The random variables I, $X_1,X_2,\dots,X_n$, $\widetilde{X}_1,\widetilde{X}_2,\dots,\widetilde{X}_n$ are independent, I is uniformly distributed over the index set $\{1,2,\dots,n\}$, each $\widetilde{X}_i$ has the same distribution as the corresponding X_i and $\widetilde{W} = W+(\widetilde{X}_I-X_I)$. Then, $(W,\widetilde{W})$ is an exchangeable pair. We note that

$$\begin{aligned}
E^W\widetilde{W} &= W+E^W\widetilde{X}_I-E^WX_I = W-\frac{1}{n}\sum_{i=1}^{n}X_i = \left(1-\frac{1}{n}\right)W,\\
E|\widetilde{W}-W|^2 &= E\left|\widetilde{X}_I-X_I\right|^2 = \frac{1}{n}\sum_{i=1}^{n}E\left|\widetilde{X}_i-X_i\right|^2 = \frac{2}{n}\sum_{i=1}^{n}\sigma_i^2.
\end{aligned} \tag{3.24}$$

Then, the assumptions of Lemma 3.3 are satisfied with $\lambda = 1/n$. Moreover, we know that

$$E|\widetilde{W}-W|^3 = E\left|\widetilde{X}_I-X_I\right|^3 = \frac{1}{n}\sum_{i=1}^{n}E\left|\widetilde{X}_i-X_i\right|^3 \le \frac{8}{n}\sum_{i=1}^{n}E\left|X_i\right|^3, \tag{3.25}$$

$$E|\widetilde{W}-W|^4 = E\left|\widetilde{X}_I-X_I\right|^4 = \frac{1}{n}\sum_{i=1}^{n}E\left|\widetilde{X}_i-X_i\right|^4 \le \frac{16}{n}\sum_{i=1}^{n}E\left|X_i\right|^4. \tag{3.26}$$

To prove the theorem, let $w_0 \in \mathbb{R}$. By Lemma 3.3, we obtain

$$
\begin{aligned}
&|P(W \le w_0) - C(w_0)| \\
&\quad \le \sup_{w\in\mathbb{R}} |f'_{w_0}(w)| E\left|1 - nE^W \frac{(\widetilde{W}-W)^2}{1+W^2}\right| + 2n\left|\frac{E(\widetilde{W}-W)^2 W f_{w_0}(W)}{(1+W^2)^2}\right| \\
&\qquad + n\sup_{w\in\mathbb{R}} \left|\left(\frac{f'_{w_0}(w)}{1+w^2}\right)'\right| E\int_{-\infty}^{\infty} |\widetilde{W}-W| \left|\left(w - \frac{W+\widetilde{W}}{2}\right)\right| \\
&\qquad\qquad \times |[I(w \le \widetilde{W}) - I(w \le W)]| \, dw \\
&\qquad + 2n\sup_{w\in\mathbb{R}} \left|\left(\frac{w f_{w_0}(w)}{(1+w^2)^2}\right)'\right| E\int_{-\infty}^{\infty} |\widetilde{W}-W| \left|\left(w - \frac{W+\widetilde{W}}{2}\right)\right| \\
&\qquad\qquad \times |[I(w \le \widetilde{W}) - I(w \le W)]| \, dw \\
&\quad \le \sup_{w\in\mathbb{R}} |f'_{w_0}(w)| E\left|1 - nE^W \frac{(\widetilde{W}-W)^2}{1+W^2}\right| + 2n\left|\frac{E(\widetilde{W}-W)^2 W f_{w_0}(W)}{(1+W^2)^2}\right| \\
&\qquad + \left(n\sup_{w\in\mathbb{R}} \left|\left(\frac{f'_{w_0}(w)}{1+w^2}\right)'\right| + 2n\sup_{w\in\mathbb{R}} \left|\left(\frac{w f_{w_0}(w)}{(1+w^2)^2}\right)'\right|\right) E \\
&\qquad \times \int_{W\wedge\widetilde{W}}^{W\vee\widetilde{W}} |\widetilde{W}-W| \left|w - \frac{W+\widetilde{W}}{2}\right| dw \\
&\quad \le \sup_{w\in\mathbb{R}} |f'_{w_0}(w)| \sqrt{E\left[1 - nE^W \frac{(\widetilde{W}-W)^2}{1+W^2}\right]^2} + 2n\left|\frac{E(\widetilde{W}-W)^2 W f_{w_0}(W)}{(1+W^2)^2}\right| \\
&\qquad + \left(\frac{n}{2}\sup \left|\left(\frac{f'_{w_0}(w)}{1+w^2}\right)'\right| + n\sup \left|\left(\frac{w f_{w_0}(w)}{(1+w^2)^2}\right)'\right|\right) E|\widetilde{W}-W|^3 \\
&\quad \le 3\sqrt{E\left[1 - nE^W \frac{(\widetilde{W}-W)^2}{1+W^2}\right]^2} + 2n\left|\frac{E(\widetilde{W}-W)^2 W f_{w_0}(W)}{(1+W^2)^2}\right| \\
&\qquad + 6n(\pi+1)E|\widetilde{W}-W|^3 \\
&\quad \le 3\sqrt{E\left[1 - nE^W \frac{(\widetilde{W}-W)^2}{1+W^2}\right]^2} + 2n\left|\frac{E(\widetilde{W}-W)^2 W f_{w_0}(W)}{(1+W^2)^2}\right| \\
&\qquad + C\sum_{i=1}^{n} E|X_i|^3,
\end{aligned}
\tag{3.27}
$$

where the fourth inequality comes from (4) and (5) of Lemma 3.1 and the last inequality comes from (3.25). Since X_i and $\tilde{X}_i$ are independent and have the same distribution,

$$
E^W(\widetilde{W}-W)^2 = E^W(\tilde{X}_I - X_I)^2 = \frac{1}{n}\sum_{i=1}^{n}(\tilde{X}_i - X_i)^2 = \frac{1}{n}\left(\sum_{i=1}^{n}\sigma_i^2 + \sum_{i=1}^{n}X_i^2\right). \tag{3.28}
$$

Hence,

$$E\left[1-nE^W\frac{(\widetilde{W}-W)^2}{1+W^2}\right]^2 = E\left[1-nE^W\frac{(\widetilde{W}-W)^2}{1+W^2}\right]^2 = E\left[1-\sum_{i=1}^{n}\frac{\sigma_i^2+X_i^2}{1+W^2}\right]^2. \tag{3.29}$$

Next, we will give a bound of $2nE(\widetilde{W}-W)^2(Wf_{w_0}(W)/(1+W^2)^2)$.

From Lemma 3.1(1),

$$\begin{aligned}
\left|2nE(\widetilde{W}-W)^2\frac{Wf_{w_0}(W)}{(1+W^2)^2}\right| &\le 2\pi F(w_0)(1-F(w_0))\sum_{i=1}^{n}E|\widetilde{X}_i-X_i|^2 \\
&= 4\pi F(w_0)(1-F(w_0))\sum_{i=1}^{n}\sigma_i^2, \\
\left|2nE(\widetilde{W}-W)^2\frac{Wf_{w_0}(W)}{(1+W^2)^2}\right| &\le 2n\pi F(w_0)(1-F(w_0))E|\widetilde{W}-W|^2|W| \\
&\le 2n\pi F(w_0)(1-F(w_0))\sqrt{E|\widetilde{X}_I-X_I|^4}\sqrt{EW^2} \\
&= 8\pi F(w_0)(1-F(w_0))\sqrt{n\sum_{i=1}^{n}\sigma_i^2\sum_{i=1}^{n}E|X_i|^4}.
\end{aligned} \tag{3.30}$$

Hence,

$$\left|2nE(\widetilde{W}-W)^2\frac{Wf_{w_0}(W)}{(1+W^2)^2}\right| \le 4\pi\min\left\{\sum_{i=1}^{n}\sigma_i^2, 2\sqrt{n\sum_{i=1}^{n}\sigma_i^2\sum_{i=1}^{n}E|X_i|^4}\right\}F(w_0)(1-F(w_0)). \tag{3.31}$$

This completes the proof. □

4. Proof of Corollary 2.2. Using Taylor's formula, we see that

$$\begin{aligned}
\frac{1}{1+W^2} &= 1-W^2+CW^3 \quad \text{for some } |C|<1, \\
\frac{1}{(1+W^2)^2} &= 1-2W^2+CW^3 \quad \text{for some } |C|<1.
\end{aligned} \tag{4.1}$$

Hence,

$$
\begin{aligned}
&E\left(\frac{1}{1+W^2}\right) \le \frac{1}{2}+\frac{C}{\sqrt{n}}, \qquad E\left(\frac{1}{(1+W^2)^2}\right) \le \frac{C}{\sqrt{n}},\\
&E\left(\frac{\sum_{i=1}^n X_i^2}{1+W^2}\right) = E\left(\sum_{i=1}^n X_i^2\right) - E\left(\sum_{i=1}^n X_i^2\right)W^2 + C_1 E\left(\sum_{i=1}^n X_i^2\right)W^3\\
&\qquad \le \frac{1}{4}+\frac{C}{\sqrt{n}},\\
&E\frac{\sum_{i=1}^n X_i^2}{(1+W^2)^2} \le \frac{C}{\sqrt{n}}, \qquad E\left(\frac{\sum_{i=1}^n X_i^2}{1+W^2}\right)^2 \le \frac{C}{n},
\end{aligned}
\tag{4.2}
$$

which implies that

$$
\begin{aligned}
&E\left[1-\frac{1}{1+W^2}\left(\frac{1}{2}+\sum_{i=1}^n X_i^2\right)\right]^2\\
&\quad = 1 - E\left(\frac{1}{1+W^2}\right) - 2E\left(\frac{\sum_{i=1}^n X_i^2}{1+W^2}\right) + \frac{1}{4}E\left(\frac{1}{(1+W^2)^2}\right)\\
&\quad + E\left(\frac{\sum_{i=1}^n X_i^2}{(1+W^2)^2}\right) + E\left(\frac{\sum_{i=1}^n X_i^2}{1+W^2}\right)^2\\
&\quad \le \frac{C}{\sqrt{n}}.
\end{aligned}
\tag{4.3}
$$

Clearly, that

$$
\begin{aligned}
&C\sum_{i=1}^n E\left|X_i\right|^3 \le \frac{C}{\sqrt{n}},\\
&4\pi \min\left\{\sum_{i=1}^n \sigma_i^2, 2\sqrt{n\sum_{i=1}^n \sigma_i^2 \sum_{i=1}^n EX_i^4}\right\}F(w_0)(1-F(w_0))\\
&\qquad \le C\min\left\{\frac{1}{2}, \sqrt{2}\sqrt{EY_i^4}\right\}F(w_0)(1-F(w_0)).
\end{aligned}
\tag{4.4}
$$

Hence, by (4.3) and (4.4), the example is proved.

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