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ON A CRITERION OF YAKUBOVICH TYPE FOR THE ABSOLUTE STABILITY OF NONAUTONOMOUS CONTROL PROCESSES

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We generalize a criterion of Yakubovich for the absolute stability of control processes with periodic coefficients to the case when the coefficients are bounded and uniformly continuous functions.

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1. Introduction. In the past two decades, the study of differential equations with time-varying coefficients (nonautonomous differential equations) has benefited from the use of methods of the ergodic theory and topological dynamics. The well-known paper of Oseledets [15] showed how basic methods of the ergodic theory can be applied to the theory of the classical Lyapunov exponents of linear and nonautonomous differential systems. Then, Sacker and Sell [17] introduced the important concept of dichotomy spectrum for linear systems with time-varying coefficients. Using the dichotomy spectrum, we can study the basic notion of exponential dichotomy for such systems using the methods of topological dynamics. In more recent years, the concepts of rotation number [8] and pullback attractor [19] have been used with profit by workers in the field of nonautonomous differential systems. In the study of rotation numbers and exponential dichotomies, we use techniques of the ergodic theory and topological dynamics. It is, therefore, not surprising that such techniques have had an increasing impact in the study of control-theoretic problems. To illustrate this point, the work of Colonius and Kliemann [3] on the reachability theory for an exposition of many results concerning their concept of control set. We also mention the monograph [9], where it is shown how exponential dichotomies and the rotation number for linear and nonautonomous Hamiltonian systems can be used to a good effect to study the nonautonomous versions of the classical linear regulator problem and the classical feedback control problem.

In this paper, we study the absolute stability problem for linear control processes with an integral quadratic constraint, when the coefficients are aperiodic and bounded functions of time. To orient the discussion, consider the control system

$$\dot{x} = A(t)x + B(t)\xi, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, and A and B are bounded and uniformly continuous functions with values in the appropriate sets of matrices. Let $m = 1$ for the time being. Let $\varphi = \varphi(\sigma, t)$ be a continuous and real-valued function of the real variable σ and of the time t. Suppose that φ satisfies a sector condition of the form

$$-k_1(t) \leq \frac{\varphi(\sigma, t)}{\sigma} \leq k_2(t), \tag{1.2}$$

where $k_1(t)$ and $k_2(t)$ are bounded and uniformly continuous real functions. Further, put $\sigma = C(t)x$, where C is a $1 \times n$ matrix-valued function with the same properties as A and B in (1.1).

Substitute $\xi(t) = \varphi(C(t)x, t)$ for ξ on the right-hand side of (1.1). We wish to determine conditions sufficient to guarantee that if $x(t)$ is any solution of (1.1), then $x(t) \to 0$ as $t \to \infty$. This is one basic version of the absolute stability problem. It amounts to a reformulation, in the context of nonautonomous control processes, of a problem posed by Aizermann and Kalman ([1, 10]; see [21] for a discussion). When the functions A, B, C, φ, k_1, and k_2 do not depend on time, satisfactory criteria for absolute stability based on the well-known Kalman-Yakubovich lemma [11] have been found. Such one is the famous circle criterion; for a discussion, again see [21].

When the functions A, B, C, φ, k_1, and k_2 are all T-periodic functions of time, Yakubovich [23, 25] has shown how the absolute stability of (1.1) relative to inputs $\xi(t) = \varphi(C(t)x, t)$, satisfying the sector condition (1.2), may be derived from hypotheses imposed on a certain Hamiltonian system of linear differential equations with T-periodic coefficients. The Hamiltonian system is obtained by formally applying the Pontryagin maximum principle [16] to (1.1), together with a certain integral quadratic cost function.

We briefly explain the relevant constructions. Introduce the quadratic form

$$\mathcal{Q}(t, x, \xi) = (\xi + k_1(t)\sigma)(k_2(t)\sigma - \xi) \tag{1.3}$$

with $\sigma = C(t)x$. Then, the condition of (1.2) can be expressed as the local quadratic constraint

$$\mathcal{Q}(t, x, \xi) \geq 0. \tag{1.4}$$

We wish to determine conditions sufficient to guarantee that if $x(t)$ and $\xi(t)$ satisfy (1.1) and (1.4), then $x(t) \to 0$ as $t \to \infty$. More generally, we can consider the pairs (x, ξ) such that the following integral quadratic constraint is satisfied:

$$\limsup_{t \to \infty} \int_0^t \mathcal{Q}(s, x(s), \xi(s))\, ds > -\infty. \tag{1.5}$$

Clearly, pairs (x, ξ) which satisfy (1.4) also satisfy (1.5). Suppose that conditions which have been found ensure that whenever the pair (x, ξ) satisfies

(1.1) and (1.5), then, $x(t) \to 0$ as $t \to \infty$. Then, such conditions will certainly be sufficient for the absolute stability of (1.1) relative to inputs $\xi(t)$ satisfying (1.4).

It is convenient to generalize the preceding considerations. Now, let $m \geq 1$ be an integer. Introduce the quadratic form

$$\mathcal{Q}(t,x(t),\xi(t)) = \frac{1}{2}(\langle x,G(t)x\rangle + 2\langle x,g(t)\xi\rangle + \langle \xi,R(t)\xi\rangle), \tag{1.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$ ($\mathbb{R}^m$), and where G, g, and R are bounded and uniformly continuous matrix functions of appropriate dimensions. Assume that $R(t)$ is strictly negative definite for all $t \in \mathbb{R}$. We will also assume that $G(t)$ is positive semidefinite for all $t \in \mathbb{R}$, as this gives rise to the *hardest case* in the theory. Introduce the functional

$$\mathcal{J}(x,\xi) = -\int_0^\infty \mathcal{Q}(s,x(s),\xi(s))\,ds. \tag{1.7}$$

Although this functional is not, in the first instance, directly related to the absolute stability problem, it turns out to be useful to study the problem of minimizing $\mathcal{J}$ with respect to pairs $(x,\xi) \in L^2([0,\infty),\mathbb{R}^n) \times L^2([0,\infty),\mathbb{R}^m)$, which are solutions of (1.1), such that $x(0) = x_0$ for fixed $x_0 \in \mathbb{R}^n$. This optimization problem leads in a well-known way to the system of the Hamiltonian equations

$$J\frac{dz}{dt} = H(t)z = \begin{bmatrix} G - gR^{-1}g^* & (A - BR^{-1}g^*)^* \\ A - BR^{-1}g^* & -BR^{-1}B^* \end{bmatrix} z. \tag{1.8}$$

Here, $z = [x^T,y^T]^T \in \mathbb{R}^{2n}$ with $y \in \mathbb{R}^n$ being a variable dual to x, and

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \tag{1.9}$$

is the standard $2n \times 2n$ antisymmetric matrix. Briefly, (1.8) is obtained by applying the Pontryagin maximum principle to the Hamiltonian

$$\mathcal{H}(t,x,\xi) = \langle y, A(t)x + B(t)\xi\rangle + \mathcal{Q}(t,x,\xi), \tag{1.10}$$

which leads formally, via the relation $\partial\mathcal{H}/\partial\xi = 0$, to the feedback rule

$$\xi = -R^{-1}(t)(B^*(t)y + g^*(t)x), \tag{1.11}$$

and hence to (1.8).

Now, Yakubovich [23, 25] has shown that when A, B, G, g, and R are all T-periodic functions of t, then the absolute stability of (1.1) relative to pairs (x,ξ) satisfying (1.5) is equivalent to the validity of certain conditions on the

Hamiltonian system (1.8); namely, the Frequency condition and the Nonoscillation condition. We generalize Yakubovich's results to the case when A, B, G, g, and R are bounded and uniformly continuous functions. To do this, it is, first, necessary to reformulate the Frequency condition and the Nonoscillation condition in a way appropriate for the study of the nonperiodic equation (1.8). As was shown in [5], this can be done using the concept of exponential dichotomy and of rotation number for linear and nonautonomous Hamiltonian systems. We summarize part of the discussion of [5] in Section 2. In Section 3, we formulate, in a way appropriate to the study of nonautonomous control processes, the concept of the absolute stability of (1.1) relative to pairs (x,ξ) satisfying (1.5). We then prove that the absolute stability is equivalent to the validity of the Frequency condition 2.5 and the Nonoscillation condition 2.6 for (1.8); these are the nonautonomous versions of Yakubovich's conditions. Thus, we obtain a quite direct generalization of Yakubovich's results for periodic control processes to the case when the coefficients A, B, G, g, and R are merely bounded and uniformly continuous. Finally, in Section 4, we give examples illustrating our results for control processes with almost periodic coefficients. Such processes arise, for example, when the coefficient functions A, B, G, g, and R are all periodic, but at least two among them have incommensurate periods. The examples also illustrate the power of the roughness theorems available for differential systems exhibiting an exponential dichotomy.

Our work was stimulated by the study of the paper [23] by Yakubovich, and we wish to express our respect for that contribution to the theory of absolute stability.

We finish this introduction by listing a notation used in this paper. As already stated, the symbol $\langle\cdot,\cdot\rangle$ indicates the Euclidean inner product on $\mathbb{R}^n$. Let $|\cdot|$ indicate the Euclidean norm on $\mathbb{R}^n$ and also on finite-dimensional vector spaces of matrices. For integers $k \geq 1$ and $\ell \geq 1$, let

$$\mathcal{M}_{k\ell} = \{M \mid M \text{ is a } k\times\ell \text{ real matrix}\}. \tag{1.12}$$

2. Preliminaries. Let $G : \mathbb{R} \to \mathcal{M}_{nn}$, $g : \mathbb{R} \to \mathcal{M}_{nm}$, and $R : \mathbb{R} \to \mathcal{M}_{mm}$ be uniformly-bounded and uniformly-continuous matrix functions. Consider the corresponding quadratic form

$$\mathcal{Q}(t,x,\xi) = \frac{1}{2}\left(\langle x,G(t)x\rangle + 2\langle x,g(t)\xi\rangle + \langle \xi,R(t)\xi\rangle\right) \tag{2.1}$$

for $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$.

2.1. Hypotheses. Assume that G and R are symmetric matrix-valued functions. Assume further that $R(t) < 0$ and $G(t) \geq 0$ for each $t \in \mathbb{R}$.

Let $A : \mathbb{R} \to \mathcal{M}_{nn}$ and $B : \mathbb{R} \to \mathcal{M}_{nm}$ be bounded and uniformly continuous matrix functions. If $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, and if $\xi : [t_0,\infty) \to \mathbb{R}^m$ is a locally integrable function, let $x(t)$ be the solution of (1.1) which satisfies $x(t_0) = x_0$.

We say that the pair (x,ξ) is admissible if there exist $\gamma \geq 0$ and a sequence $t_k \to \infty$ such that

$$\lim_{k\to\infty}\int_{t_0}^{t_k}\mathfrak{Q}(s,x(s),\xi(s))\,ds \geq -\gamma. \tag{2.2}$$

We look for conditions necessary and sufficient for the validity of the following criterion.

Absolute stability criterion 2.1. There exists a constant $\kappa > 0$ such that, for each admissible pair (x,ξ), there holds

$$\int_{t_0}^{\infty}\left(|x(s)|^2+|\xi(s)|^2\right)ds \leq \kappa\left(|x(t_0)|^2+\gamma\right), \tag{2.3}$$

where γ is the number in (2.2).

As stated in the Introduction, we formulate such necessary and sufficient conditions in terms of properties of the solutions of the linear and nonautonomous Hamiltonian system given previously in (1.8). The relevant properties of the solutions of (1.8) were stated by Yakubovich [22, 24] when the coefficients A, B, G, g, and R are T-periodic functions of time. Those properties are summarized in Frequency condition 2.5 and Nonoscillation condition 2.6. We state the versions of these conditions appropriate to the case when A, B, G, g, and R are merely bounded and uniformly continuous. To do this, we need to apply the well-known Bebutov construction to the matrix functions A, B, G, g, and R.

Let $k \geq 1$ and $\ell \geq 1$ be integers, and let

$$\mathcal{F} = \mathcal{F}_{k\ell} = \{f : \mathbb{R} \longrightarrow \mathcal{M}_{k\ell} \mid f \text{ is bounded and continuous}\}. \tag{2.4}$$

We endow $\mathcal{F}$ with the compact-open topology. We can define a topological flow, called the Bebutov flow, on $\mathcal{F}$ using the natural time translations. Thus, if $f \in \mathcal{F}$ and $t \in \mathbb{R}$, define

$$\tau_t(f)(s) = f(t+s), \tag{2.5}$$

where $s \in \mathbb{R}$. It is easy to check that $\{\tau_t \mid t \in \mathbb{R}\}$ satisfies the following three conditions:

(i) τ_0 is the identity map on $\mathcal{F}$,
(ii) the map $\tau : \mathcal{F}\times\mathbb{R} \to \mathcal{F} : (f,t) \to \tau_t(f)$ is jointly continuous,
(iii) $\tau_t \circ \tau_s = \tau_{t+s}$ for all $(t,s) \in \mathbb{R}$.

Thus, $(\mathcal{F},\{\tau_t\})$ is indeed a topological flow [13].

Now, let $f \in \mathcal{F}$ be uniformly continuous. Then, the hull

$$\Omega_f = \text{closure}\{\tau_t(f) \mid t \in \mathbb{R}\} \subset \mathcal{F} \tag{2.6}$$

is a compact subset of $\mathcal{F}$ (the closure is taken in the compact-open topology). Moreover, Ω_f is invariant in the sense that if $f' \in \Omega_f$, then $\tau_t(f') \in \Omega_f$ for all

$t \in \mathbb{R}$. Next, set

$$\mathcal{F}_* = \mathcal{F}_{nn} \times \mathcal{F}_{nm} \times \mathcal{F}_{nn} \times \mathcal{F}_{nm} \times \mathcal{F}_{mm}. \tag{2.7}$$

Then, $\omega_0 = (A, B, G, g, R) \in \mathcal{F}_*$. We can define a Bebutov flow on $\mathcal{F}_*$ by setting (with various abuses of notation)

$$\tau_t(A', B', G', g', R') = (\tau_t(A'), \tau_t(B'), \tau_t(G'), \tau_t(g'), \tau_t(R')) \tag{2.8}$$

for all quintuples in $\mathcal{F}_*$ and all $t \in \mathbb{R}$. Let $\Omega = \text{closure}\{\tau_t(\omega_0) \mid t \in \mathbb{R}\}$. Note that there are continuous mappings $\tilde{A}$, $\tilde{B}$, $\tilde{G}$, $\tilde{g}$, and $\tilde{R}$ defined on Ω as follows: if $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) \in \Omega$, then $\tilde{A}(\omega) = \omega_1(0)$, $\tilde{B}(\omega) = \omega_2(0)$, $\tilde{G}(\omega) = \omega_3(0)$, $\tilde{g}(\omega) = \omega_4(0)$, and $\tilde{R}(\omega) = \omega_5(0)$. It is clear that $\tilde{A}(\tau_t(\omega_0)) = A(t)$, $\tilde{B}(\tau_t(\omega_0)) = B(t)$, $\tilde{G}(\tau_t(\omega_0)) = G(t)$, $\tilde{g}(\tau_t(\omega_0)) = g(t)$, and $\tilde{R}(\tau_t(\omega_0)) = R(t)$, where $t \in \mathbb{R}$. It is convenient to abuse the notation again, and to write A, B, G, g, and R instead of $\tilde{A}$, $\tilde{B}$, $\tilde{G}$, $\tilde{g}$, and $\tilde{R}$, respectively. Introducing the family of control systems

$$\dot{x} = A(\tau_t(\omega))x + B(\tau_t(\omega))\xi, \tag{2.9}$$

where $\omega \in \Omega$, we see that system (1.1) coincides with system (2.9) for $\omega = \omega_0$. We also introduce the family of quadratic forms

$$\begin{aligned} \mathcal{Q}_\omega(t, x, \xi) = \frac{1}{2}\big(&\langle x, G(\tau_t(\omega))x\rangle + 2\langle x, g(\tau_t(\omega))\xi\rangle \\ &+ \langle \xi, R(\tau_t(\omega))\xi\rangle\big), \end{aligned} \tag{2.10}$$

where $\omega \in \Omega$. It is clear that for $\omega = \omega_0$, this expression coincides with the form $\mathcal{Q}$ of (2.2).

Now, there is no particular reason to insist that the compact invariant subset of Ω of $\mathcal{F}_*$ is the hull of a fixed quintuple ω_0. In what follows, we let Ω denote an arbitrary compact, translation-invariant subset of $\mathcal{F}_*$. We let A, B, G, g, and R be the continuous functions on Ω defined, respectively, by $A(\omega) = \omega_1(0)$, $B(\omega) = \omega_2(0)$, $G(\omega) = \omega_3(0)$, $g(\omega) = \omega_4(0)$, and $R(\omega) = \omega_5(0)$ for each quintuple $\omega = (\omega_1, \dots, \omega_5) \in \Omega$. Unless otherwise specified, we will always assume that G and R have symmetric values, and that $R(\omega) < 0$ and $G(\omega) \geq 0$ for all $\omega \in \Omega$.

If $\omega \in \Omega$ and $x_0 \in \mathbb{R}^n$, and if $\xi : [0, \infty) \to \mathbb{R}^m$ is a locally integrable function, let $x(t)$ be the solution of (2.9), which satisfies $x(0) = x_0$. We say that (x, ξ) is an admissible pair for ω if there is a number $\gamma \geq 0$ and a sequence $t_k \to \infty$ such that

$$\lim_{k \to \infty} \int_0^{t_k} \mathcal{Q}_\omega(s, x(s), \xi(s))\, ds \geq -\gamma. \tag{2.11}$$

Here, t_k and γ may depend on the pair (x, ξ), hence also on $\omega \in \Omega$. Comparing (2.11) with (2.2), we set $t_0 = 0$ because of the freedom in choosing $\omega \in \Omega$.

We now formulate our nonautonomous version of the absolute stability criterion.

Absolute stability criterion 2.2. There is a constant κ such that, if $\omega \in \Omega$ and if (x,ξ) is an admissible pair for ω, then

$$\int_0^\infty \left(|x(s)|^2 + |\xi(s)|^2\right) ds \le \kappa\left(|x(0)|^2 + \gamma\right). \tag{2.12}$$

The constant γ is the one appearing in (2.11). The constant κ is independent of the admissible pair and of $\omega \in \Omega$.

We also have a family of Hamiltonian equations

$$J\frac{dz}{dt} = H(\tau_t(\omega))z = \begin{bmatrix} G - gR^{-1}g^* & (A - BR^{-1}g^*)^* \\ A - BR^{-1}g^* & -BR^{-1}B^* \end{bmatrix} z, \tag{2.13}$$

where, as indicated, each of the entries in the matrix function H has argument $\tau_t(\omega)$ with $\omega \in \Omega$. We now formulate our nonautonomous version of the Frequency condition and the Nonoscillation condition of Yakubovich. To do so, we require some standard definitions and facts. Let $\Phi_\omega(t)$ be the fundamental matrix solution in $t = 0$ of (2.13) with $\omega \in \Omega$. Also let $\mathcal{P}$ be the set of all linear projections $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$; we give $\mathcal{P}$ its natural topology.

Definition 2.3. Say that (2.13) have an *exponential dichotomy over* Ω if there are positive constants K and k, and a continuous map $P : \Omega \to \mathcal{P} : \omega \to P_\omega$, such that

$$\begin{aligned} \left|\Phi_\omega(t)P_\omega\Phi_\omega^{-1}(s)\right| &\le Ke^{-k(t-s)} \quad (t \ge s), \\ \left|\Phi_\omega(t)(I - P_\omega)\Phi_\omega^{-1}(s)\right| &\le Ke^{k(t-s)} \quad (t \le s). \end{aligned} \tag{2.14}$$

Definition 2.4. Let Ω be a compact metrizable space and let $(\Omega, \{\tau_t\})$ be a topological flow. The flow is said to be minimal if for each $\omega \in \Omega$, the orbit $\{\tau_t(\omega) \mid t \in \mathbb{R}\}$ is dense in Ω. We usually speak, with a slight inaccuracy, of a *minimal set* Ω.

We now state the following condition.

Frequency condition 2.5. For each $\omega \in \Omega$, the only solution $z(t)$ of (2.13) which is bounded on all of $\mathbb{R}$ is the identically zero solution $z(t) \equiv 0$.

Next, let $(\Omega, \{\tau_t\})$ be a topological flow, not necessarily minimal. In [5], it is explained how the Hamiltonian nature of (2.13) implies that if the Frequency condition 2.5 is satisfied, (2.13) have an exponential dichotomy over all of Ω. The discussion in [5] is based on a basic result of Sacker and Sell [17] and Selgrade [20]. The key point in the proof of this fact is the observation that for each minimal subset M of Ω, the dichotomy projection P_ω satisfies $\dim(\mathrm{Im}[P_\omega]) = n$, where $\omega \in M$ and $\mathrm{Im}[\cdot]$ denotes the image of its argument. We then use another result of Sacker and Sell [18] to verify that the dichotomy property *extends* from the minimal subsets of Ω to all of Ω.

Suppose now that the family (2.13) satisfies the Frequency condition 2.5, hence has an exponential dichotomy over all of Ω with family of projections $\{P_\omega \mid \omega \in \Omega\}$. We formulate the Nonoscillation condition 2.6 in a geometric way. For this, recall that an n-dimensional vector subspace $\lambda \subset \mathbb{R}^{2n}$ is called a *Lagrange plane* if $\langle z_1, Jz_2\rangle = 0$ for all $(z_1,z_2) \in \lambda$. The set $\Lambda = \{\lambda\}$ of all Lagrange planes in $\mathbb{R}^{2n}$ can be endowed in a natural way with the structure of a C^∞-smooth, $n(n+1)/2$-dimensional manifold. The subspace $\lambda_o = \operatorname{Span}\{e_{n+1},\dots,e_{2n}\} \subset \mathbb{R}^{2n}$ is a Lagrange plane. Define the (vertical) Maslov cycle $\mathscr{C}$ to be $\{\lambda \in \Lambda \mid \dim(\lambda \cap \lambda_o) \geq 1\}$. It is known that $\mathscr{C}$ is two-sided and $\mathbb{Z}_2$-cycle in Λ of codimension 1 [2].

It can be shown [14] that for each $\omega \in \Omega$, $\operatorname{Im}[P_\omega]$ is a Lagrange plane. Write $\lambda_\omega = \operatorname{Im}[P_\omega]$, where $\omega \in \Omega$.

Nonoscillation condition 2.6. For each $\omega \in \Omega$, λ_ω does not belong to the vertical Maslov cycle $\mathscr{C}$.

According to this formulation, the Nonoscillation condition 2.6 automatically implies the Frequency condition 2.5. In [5], it is explained how certain hypotheses involving the rotation number [6, 7, 8, 14] of (2.13) permit the *simultaneous* verification of the Frequency condition 2.5 and the Nonoscillation condition 2.6. As the rotation number has no direct role in the sequel, we do not discuss it here.

3. Absolute stability.

We begin the discussion with a simple observation. Suppose that the Absolute stability criterion 2.2 is valid. Let $\omega_0 \in \Omega$ and $x_0 \in \mathbb{R}^n$, and set $\xi(t) \equiv 0$ in (2.9) with $\omega = \omega_0$, then let $x(t)$ be the solution of (2.9) with $\omega = \omega_0$ satisfying $x(0) = x_0$. Since $G(\omega) \geq 0$ for all $\omega \in \Omega$, the pair $(x,0)$ is admissible for ω_0. It follows that $x(t) \to 0$ as $t \to \infty$. Thus, $x(t) \equiv 0$ is an asymptotically stable solution of

$$\dot{x} = A(\tau_t(\omega))x \tag{3.1}$$

for each $\omega \in \Omega$.

The following result is well known (see, e.g., [5, 17]).

Theorem 3.1. *Let $\Psi_\omega(t)$ be the fundamental matrix solution in $t = 0$ of (3.1) with $\omega \in \Omega$. Then, there are positive constants K' and k', which do not depend on $\omega \in \Omega$ such that*

$$|\Psi_\omega(t)| \leq K' e^{-k'(t-s)} \quad (t \geq s) \tag{3.2}$$

for all $\omega \in \Omega$.

Clearly, (3.2) implies that $x(t) \equiv 0$ is uniformly exponentially asymptotically stable for each $\omega \in \Omega$. When (3.2) holds for all $\omega \in \Omega$, we say that the family of (2.9), together with the family of constraints (2.11), are minimally stable (see also Remark 3.3).

We now formulate the main result of this paper.

Theorem 3.2. *The Absolute stability criterion 2.2 holds if and only if the family of Hamiltonian systems (2.13) satisfies the Frequency condition 2.5 and the Nonoscillation condition 2.6.*

Proof. First, assume that the Frequency condition 2.5 and the Nonoscillation condition 2.6 are valid. We prove that the Absolute stability criterion 2.2 is valid. The arguments, which follow, generalize those given by Yakubovich [23] in the case when the coefficients are T-periodic.

First, we apply [5, Theorem 4.3] to draw the following conclusion: for each sufficiently small $\delta > 0$, there is a continuous function $m^\delta : \Omega \to \mathcal{M}_{nn}$ with the values in the set of symmetric $n \times n$ matrices such that, if

$$V_\omega(t,x) = \langle x, m^\delta(\tau_t(\omega))x\rangle, \tag{3.3}$$

where $x \in \mathbb{R}^n$ and $\omega \in \Omega$, then

$$\frac{d}{dt}V_\omega(t,x) \le -2\mathcal{Q}_\omega(t,x(t),\xi(t)) - \delta\left(|x(t)|^2 + |\xi(t)|^2\right) \tag{3.4}$$

for each continuous function $\xi : [0,\infty) \to \mathbb{R}^n$. Here, $x(t)$ is obtained by solving (2.9) after the substitution of $\xi(\cdot)$ on the right-hand side.

Following Yakubovich [23], we prove that $m^\delta(\omega)$ is positive semidefinite for each $\omega \in \Omega$. Let $x_0 \in \mathbb{R}^n$, and set $\xi(t) \equiv 0$. By Theorem 3.1, the solution $x(t)$ of (3.1) with $x(0) = x_0$ tends to zero as $t \to \infty$. Now, integrate (3.4) from 0 to t to obtain

$$V_\omega(t,x(t)) - V_\omega(0,x_0) \le -2\int_0^t \mathcal{Q}_\omega(s,x(s),0)\,ds \le 0. \tag{3.5}$$

Letting $t \to \infty$, we see that $V_\omega(0,x_0) \ge 0$. This implies that $m^\delta(\omega)$ is positive semidefinite for $\omega \in \Omega$.

We now prove that the Absolute stability criterion 2.2 is valid. Let $\omega \in \Omega$, and let (x,ξ) be an admissible pair for ω. Thus, there is a number $\gamma \ge 0$ and a sequence $t_k \to \infty$ such that (2.11) holds. Writing $x(0) = x_0$ and integrating (3.4) from 0 to t_k, we get

$$\begin{aligned} V_\omega(t_k,x(t_k)) - V_\omega(0,x_0) \le -2\int_0^{t_k} \mathcal{Q}_\omega(s,x(s),\xi(s))\,ds \\ -\delta\int_0^{t_k}\left(|x(s)|^2 + |\xi(s)|^2\right)ds. \end{aligned} \tag{3.6}$$

Since

$$\liminf_{k\to\infty} V(t_k,x(t_k)) \ge 0, \tag{3.7}$$

we obtain

$$\langle x_0, m^\delta(\omega)x_0\rangle \geq -2\gamma + \delta\int_0^\infty \left(|x(s)|^2 + |\xi(s)|^2\right)ds. \tag{3.8}$$

Hence,

$$\begin{aligned}\int_0^\infty \left(|x(s)|^2 + |\xi(s)|^2\right)ds &\leq \frac{1}{\delta}(\langle x_0, m^\delta(\omega)x_0\rangle + 2\gamma) \\ &\leq \kappa(|x(0)|^2 + \gamma),\end{aligned} \tag{3.9}$$

where κ does not depend on ω, x, and ξ. This completes the proof of the validity of the Absolute stability criterion 2.2.

Now, we prove that if the Absolute stability criterion 2.2 holds, then both the Frequency condition 2.5 and the Nonoscillation condition 2.6 are valid. Assume, for contradiction, that at least one of the Frequency condition 2.5 and the Nonoscillation condition 2.6 does not hold. By [5, Theorem 4.3], condition (F) of that theorem is violated. That is, for each integer $r \geq 1$, there exist $\omega_r \in \Omega$ and a pair $(x_r, \xi_r) \in L^2([0,\infty), \mathbb{R}^n) \times L^2([0,\infty), \mathbb{R}^m)$ with the following properties. First, if ξ_r is substituted on the right-hand side of (2.9) with $\omega = \omega_r$, then $x_r(t)$ is the solution of (2.9) with $\omega = \omega_r$ satisfying $x_r(0) = 0$. Second, we have

$$\int_0^\infty \mathcal{Q}_\omega(s, x_r(s), \xi_r(s))ds \geq -\frac{1}{r}\int_0^\infty \left(|x(s)|^2 + |\xi(s)|^2\right)ds, \tag{3.10}$$

where $\omega = \omega_r$. Clearly, there is no loss of generality in assuming that

$$\int_0^\infty \left(|x(s)|^2 + |\xi(s)|^2\right)ds = 1 \tag{3.11}$$

for each $r \geq 1$. Since

$$\limsup_{t\to\infty}\int_0^t \mathcal{Q}_\omega(s, x_r(s), \xi_r(s))ds \geq -\frac{1}{r}, \tag{3.12}$$

where $\omega = \omega_r$. We have that (x_r, ξ_r) is an admissible pair for ω_r with $\gamma = \gamma_r = 1/r$, where $r \geq 1$. Clearly, if

$$1 = \int_0^\infty \left(|x(s)|^2 + |\xi(s)|^2\right)ds \geq \kappa(|x_r(0)|^2 + \gamma_r), \tag{3.13}$$

then $\kappa \geq r$. We conclude that the Absolute stability criterion 2.2 does not hold. This is a contradiction, so the Absolute stability criterion 2.2 does indeed imply both the Frequency condition 2.5 and the Nonoscillation condition 2.6. This completes the proof of Theorem 3.2. □

Remark 3.3. It is sometimes convenient to relax the condition that $G(\omega) \geq 0$ for all $\omega \in \Omega$ (though not the condition that $R(\omega) < 0$ for all $\omega \in \Omega$). A modified version of Theorem 3.2 is still true. We indicate the necessary changes in the statement and proof of Theorem 3.2 following the outline of Yakubovich's discussion in [23].

First of all, we generalize the definition of minimal stability in the following way. We say that the family of (2.9), together with the family of constraints (2.11), is minimally stable if, for each $\omega \in \Omega$ and each $x_0 \in \mathbb{R}^n$, there are an admissible pair $(x^M(t,\omega,x_0), \xi^M(t,\omega,x_0))$, together with a sequence $(t_k^M(\omega,x_0))$ and numbers $\gamma^M(\omega,x_0)$, such that $x^M(0,\omega,x_0) = 0$ and $x^M(t_k,\omega,x_0) \to 0$ as $k \to \infty$, and $\inf\{\lambda^{-2}\gamma^M(\omega,\lambda x_0)\} = 0$. Next, we modify the statement of Theorem 3.2 by restricting attention to families (2.9) and (2.11) which are minimally stable. Finally, we modify that part of the proof of Theorem 3.2 regarding the positive semidefiniteness of m^δ by using the pairs (x^M, ξ^M) in place of the pairs $(x, \xi \equiv 0)$.

4. Examples. The examples given below are intended to illustrate the strong robustness properties enjoyed by the absolute stability concept.

To begin, let Ω be a compact and translation-invariant subset of $\mathcal{F}_*$. Each $\omega \in \Omega$ defines a quintuple $(A_\omega, B_\omega, G_\omega, g_\omega, R_\omega)$ of bounded and uniformly continuous matrix-valued functions with corresponding control process (2.9) and quadratic form $\mathcal{Q}_\omega$. Suppose that the family of Hamiltonian systems (2.13) satisfies the Frequency condition 2.5 and the Nonoscillation condition 2.6.

It is convenient to embed Ω in a still-larger function space $\mathcal{G}_*$ which contains $\mathcal{F}_*$. For integers $k \geq 1$ and $\ell \geq 1$, let

$$\mathcal{G}_{k\ell} = \{g : \mathbb{R} \longrightarrow \mathcal{M}_{k\ell} \mid g \text{ is bounded and measurable}\}. \tag{4.1}$$

Endow $\mathcal{G}_{k\ell}$ with the usual weak-$*$ topology; this topology may be defined as follows. A sequence $(g_n) \subset \mathcal{G}_{k\ell}$ converges to $g \in \mathcal{G}_{k\ell}$ if, for each $\psi \in L^1(\mathbb{R})$, there holds

$$\int_{-\infty}^{\infty} g_n(s)\psi(s)ds \longrightarrow \int_{-\infty}^{\infty} g(s)\psi(s)ds. \tag{4.2}$$

Each norm-closed ball

$$\left\{g \in \mathcal{G}_{k\ell} \mid \|g\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}} |g(t)| \leq a\right\} \tag{4.3}$$

is weak-$*$ compact ($a > 0$). There is a Bebutov flow $\{\tau_t \mid t \in \mathbb{R}\}$ on $\mathcal{G}_{k\ell}$ defined by

$$\tau_t(g)(s) = g(t+s), \tag{4.4}$$

where $(t,s) \in \mathbb{R}$ and $g \in \mathcal{G}_{k\ell}$.

Next, let

$$\mathcal{G}_* = \mathcal{G}_{nn} \times \mathcal{G}_{nm} \times \mathcal{G}_{nn} \times \mathcal{G}_{nm} \times \mathcal{G}_{mm}. \tag{4.5}$$

Clearly, Ω may be identified with a subset (again called Ω) of $\mathcal{G}_*$. Let N be a number such that, for each $\omega = (\omega_1,\dots,\omega_5) \in \Omega$, we have $\|\omega_i\|_\infty \le N$ for $1 \le i \le 5$. Let

$$\Upsilon = \{y = (y_1,\dots,y_5) \in \mathcal{G}_* \mid \|y_i\|_\infty \le N,\ 1 \le i \le 5\}. \tag{4.6}$$

Then, Υ is a compact, connected, translation-invariant subset of $\mathcal{G}_*$, and $\Omega \subset \Upsilon$.

There is a metric d on Υ, which is compatible with the weak-$*$ topology. For each $\varepsilon > 0$, let

$$\Omega_\varepsilon = \{\hat{\omega} \in \Upsilon \mid d(\hat{\omega},\omega) \le \varepsilon\}. \tag{4.7}$$

Now, by hypothesis, the family of Hamiltonian systems (2.13) has an exponential dichotomy over Ω. Moreover, for each $\omega \in \Omega$, the dichotomy projection P_ω has the property that the Lagrange plane $\lambda_\omega = \mathrm{Im}[P_\omega]$ does not lie on the vertical Maslov cycle $\mathscr{C} \subset \Lambda$. By a basic perturbation theorem of Sacker and Sell [18], there exists $\hat{\varepsilon} > 0$ such that the family of (2.13) with $\hat{\omega} \in \Omega_{\hat{\varepsilon}}$ has an exponential dichotomy over $\Omega_{\hat{\varepsilon}}$. Moreover, the dichotomy projections $\{P_{\hat{\omega}} \mid \hat{\omega} \in \Omega_{\hat{\varepsilon}}\}$ have sufficient continuity properties to ensure that $P_{\hat{\omega}} \notin \mathscr{C}$ for all $\hat{\omega} \in \Omega_{\hat{\varepsilon}}$.

We can thus conclude that if $\hat{\Omega} \subset \Omega_{\hat{\varepsilon}}$ is any weak-$*$ compact and translation-invariant set, then the families (2.9) and (2.11) with $\hat{\omega} \in \hat{\Omega}$ satisfy the Absolute stability criterion 2.2; there is a constant $\kappa \ge 0$ such that, if (x,ξ) is an admissible pair for some $\hat{\omega} \in \hat{\Omega}$, then

$$\int_0^\infty \left(|x(s)|^2 + |\xi(s)|^2\right) ds \le \kappa\left(|x(0)|^2 + \gamma\right). \tag{4.8}$$

For example, if $\varepsilon > 0$ is sufficiently small and if

$$\hat{\Omega} = \{\hat{\omega} = (\hat{\omega}_1,\dots,\hat{\omega}_5) \in \Omega_{\hat{\varepsilon}} \mid \exists \omega = (\omega_1,\dots,\omega_5) \in \Omega \text{ such that } \|\hat{\omega}_i - \omega_i\|_\infty \le \varepsilon\}, \tag{4.9}$$

then the families (2.9) and (2.11) with $\hat{\omega} \in \hat{\Omega}$ satisfy the Absolute stability criterion 2.2. As a very special case, let $\omega_0 = (A,B,G,g,R)$ be a quintuple of T-periodic, bounded, and measurable matrix-valued functions of the appropriate dimensions. Assume that the corresponding system (1.8) satisfies the Frequency condition and the Nonoscillation condition as formulated by Yakubovich in [23]. Then, there is an $\varepsilon > 0$ such that, if A_1, B_1, G_1, g_1, and R_1 are bounded and measurable matrix-valued functions of the appropriate dimensions satisfying $\|A_1\|_\infty \le \varepsilon,\dots,\|R_1\|_\infty \le \varepsilon$, and if $\hat{A} = A + A_1,\dots,\hat{R} = R + R_1$, then the control process

$$\dot{x} = \hat{A}(t)x + \hat{B}(t)\xi, \tag{4.10}$$

together with the integral quadratic constraint corresponding to

$$\mathfrak{Q}(t,x,\xi) = \langle x, \hat{G}(t)x\rangle + \langle x, \hat{g}(t)\xi\rangle + \langle \xi, \hat{R}(t)\xi\rangle, \tag{4.11}$$

is absolutely stable. As an even more special case, the functions $A_1,\dots,R_1$ might be periodic with at least one period incommensurate with T.

The above results actually follow from standard roughness criteria for exponential dichotomies relative to uniform perturbations [4, 12]. However, since the definition of $\Omega_{\hat{\varepsilon}}$ makes reference to the weak-$*$ topology and not to the norm topology on $\mathcal{G}_*$, absolute stability holds also for perturbed families $\hat{\Omega}$, which are not norm-close to Ω. To illustrate this point, let $\mathbb{T}^k$ be the k-dimensional torus with angular variables $(\theta_1,\dots,\theta_k) \bmod 2\pi$. Let $\alpha_1,\dots,\alpha_k$ be real numbers. Write $\theta = (\theta_1,\dots,\theta_k)$ and $\alpha = (\alpha_1,\dots,\alpha_k)$. Let A, B, G, g, and R be matrix-valued functions, of the appropriate sizes, defined and continuous on $\mathbb{T}^k$. For each $\theta \in \mathbb{T}^k$, the functions $t \to A(\theta+\alpha t),\dots,t \to R(\theta+\alpha t)$ are quasiperiodic functions.

Let $H(\cdot)$ be the matrix-valued function on $\mathbb{T}^k$ obtained by substituting $A(\cdot)$, $\dots,R(\cdot)$ on the right-hand side of (1.8):

$$H(\theta) = \begin{bmatrix} G - gR^{-1}g^* & (A^* - BR^{-1}g^*)^* \\ A - BR^{-1}g^* & -BR^{-1}B^* \end{bmatrix}. \tag{4.12}$$

Here, all the entries in the matrix have an argument θ. Suppose that for some fixed frequency vector $\bar{\alpha} = (\bar{\alpha}_1,\dots,\bar{\alpha}_k)$, the family of Hamiltonian equations

$$J\frac{dz}{dt} = H(\theta + \bar{\alpha}t)z \tag{4.13}$$

has an exponential dichotomy over $\mathbb{T}^k$. Further, suppose that for each $\theta \in \mathbb{T}^k$, the projection $\bar{P}_\theta$ does not lie on $\mathscr{C}$. We remark that, if the frequencies $\alpha_1,\dots,\alpha_k$ are independent over $\mathbb{Q}$, then equations (4.13) have an exponential dichotomy over $\mathbb{T}^k$ of the family (4.13) if and only if just one equation of (4.13) admits an exponential dichotomy. However, we allow the frequencies $\alpha_1,\dots,\alpha_k$ to be dependent over $\mathbb{Q}$; hence, we must explicitly assume that equations (4.13) have an exponential dichotomy over all of $\mathbb{T}^k$.

Now, the family (4.13) can be embedded in $\mathcal{G}_*$ in the obvious way: with each $\theta \in \mathbb{T}^k$, associate the quintuple $\omega(t) = (A(\theta+\bar{\alpha}t),\dots,t \to R(\theta+\bar{\alpha}t))$. We now vary the frequency vector $\bar{\alpha}$. Applying the result of Sacker and Sell, we see that there exists $\varepsilon > 0$ so that, if $|\alpha - \bar{\alpha}| \le \varepsilon$, then the family (4.13) with $\theta \in \mathbb{T}^k$ obtained by substituting α for $\bar{\alpha}$ has an exponential dichotomy projection $P_\theta \notin \mathscr{C}$ with $\theta \in \mathbb{T}^k$. Thus, we see that, for all frequency vectors α near $\bar{\alpha}$, the families (1.1) and (1.5) to which (4.13) corresponds satisfy the Absolute stability criterion 2.2.

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