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L-CORRESPONDENCES: THE INCLUSION $L^p(\mu,X) \subset L^q(\upsilon,Y)$

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ABSTRACT. In order to study inclusions of the type $L^p(\mu,X) \subset L^q(\upsilon,Y)$, we introduce the notion of an L-correspondence. After proving some basic theorems, we give characterizations of some types of L-correspondences and offer a conjecture that is similar to an equimeasurability theorem.

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1. INTRODUCTION.

Inclusions of one L^p space in another have been the subject of several previous articles. Most recently, Miamee [2] studied when $L^p(\mu) \subset L^q(\upsilon)$, where μ and υ are (possibly different) measures on (Ω,Σ). As mentioned in Miamee's article, those results extend even to the setting $L^p(\mu,X) \subset L^q(\upsilon,X)$, where X is a Banach space. The purpose of this article is to extend this notion even further, to the setting $L^p(\mu,X) \subset L^q(\upsilon,Y)$, where X and Y are (possibly different) Banach spaces. Of course, the usual meaning of inclusion would prohibit $L^p(\mu,X)$ from being a subset of $L^q(\upsilon,Y)$ if X is not a subset of Y. In order to circumvent this difficulty, we introduce the notion of an L-correspondence. After proving some basic theorems, we characterize some types of L-correspondences and offer a conjecture.

Throughout, (Ω,Σ) will be a measurable space, μ and υ will be non-zero, finite, complete measures on (Ω,Σ), and X and Y will be Banach spaces. The Lebesgue-Bochner spaces are denoted as usual; we define $L(\mu,X,p)$ as the linear space consisting of individual functions (not identified by μ-a.e. equality) whose equivalence classes are in $L^p(\mu,X)$. We also restrict ourselves to the case $1 \le p,q < \infty$.

In [2], Miamee also distinguished between $L^p(\mu) \subset L^q(\upsilon)$ in the sense of equivalence classes and in the sense of individual functions. Miamee's Lemma stated that $L^p(\mu) \subset L^q(\upsilon)$ in the sense of equivalence classes if and only if $\mu << \upsilon$, $\upsilon << \mu$, and $L^p(\mu) \subset L^q(\upsilon)$ in the sense of individual functions. "Inclusion" was then defined as meeting those equivalent conditions. We use this as our starting point in the next section.

2. L-CORRESPONDENCES: A NATURAL EXTENSION OF INCLUSION.

In order to motivate our definition of an inclusion $L^p(\mu,X) \subset L^q(\upsilon,Y)$, consider again the situation $Y = X$, where Miamee's definition applies. If $L^p(\mu,X) \subset L^q(\upsilon,X)$, then the identity mapping $I: L(\mu,X,p) \to L(\upsilon,X,q)$ is defined; also, we have that for all $f,g \in L(\mu,X,p)$, $f = g$ μ-a.e. if and only if $I(f) = I(g)$ υ-a.e. Considering the fact that the identity is a linear injection, we offer the following definition (we use $\overline{f}$ to represent the equivalence class of f in the associated Lebesgue-Bochner space).

To simplify matters, during the sequel whenever we write $T: L(\mu, X, p) \to L(\upsilon, Y, q)$ we mean that T is injective and linear.

DEFINITION. A map $T: L(\mu, X, p) \to L(\upsilon, Y, q)$ is called an L-correspondence if $\hat{T}: L^p(\mu, X) \to L^q(\upsilon, Y)$ defined by $\hat{T}(\overline{f}) = \overline{T(f)}$ is well defined and injective. If, in addition, T maps onto every equivalence class that it maps into, it is called exact.

It is simple to show, given the above definition, that any $T: L(\mu, X, p) \to L(\upsilon, Y, q)$ is an L-correspondence if and only if it has the property that $f = g$ μ-a.e. if and only if $T(f) = T(g)$ υ-a.e. for all $f, g \in L(\mu, X, p)$. That, in a sense, corresponds to Miamee's Lemma. However, the analogy does not hold completely; it is possible to have an L-correspondence with μ not absolutely continuous with respect to υ (see the example at the end of this article).

A look at Miamee's "Main Theorem" suggests that $\hat{T}$ may have to be bounded. To see that this is not the case, let $\Omega = \{\omega\}$ and let $\mu: \Sigma = \{\emptyset, \Omega\} \to \mathbf{R}$ be given by $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. Then $L(\mu, X, p) = X$ and $L(\mu, Y, q) = Y$, and any injective unbounded linear operator from X to Y gives an unbounded L-correspondence. However, a theorem analagous to the other direction of Miamee's theorem does hold, as presented next.

PROPOSITION 1. Suppose $T: L(\mu, X, p) \to L(\upsilon, Y, q)$ satisfies $f = g$ μ-a.e. if $T(f) = T(g)$ υ-a.e. and there is a positive constant C such that $\|\overline{T(f)}\|_{q,\upsilon} \le C\|\overline{f}\|_{p,\mu}$ for all $f \in L(\mu, X, p)$. Then T is an L-correspondence.

PROOF. Suppose $f = g$ μ-a.e.; then $\|\overline{f-g}\|_{p,\mu} = 0$. Thus, $\|\overline{T(f-g)}\|_{q,\upsilon} = 0$ and $T(f) = T(g)$ υ-a.e.

If T is an L-correspondence such that $\hat{T}$ is bounded, we will call T bounded. Note also that if T happens to be continuous in the topology of pointwise convergence, Miamee's closed graph argument shows that $\hat{T}$ is bounded.

We now wish to show that L-correspondences are, in some sense, the same as inclusion in the setting $Y = X$. The sense in which this is true will be given after the next theorem.

THEOREM 2. Suppose $S: X \to Y$ is a linear map and $T: L(\mu, X, p) \to L(\upsilon, Y, q)$ is defined by $T(f) = S \circ f$. We have (i) if S is a continuous injection and $L^p(\mu, X) \subset L^q(\upsilon, X)$, then T is a bounded L-correspondence; (ii) if S is an isomorphism and $T$is an L-correspondence then $L^p(\mu, X) \subset L^q(\upsilon, X)$.

PROOF. For (i), suppose $L^p(\mu, X) \subset L^q(\upsilon, X)$. By Miamee's theorem, there is a positive constant C such that $\|f\|_{q,\upsilon} \le C\|f\|_{p,\mu}$ for all $f \in L(\mu, X, p)$. Let T be as stated. Since $\upsilon << \mu$, it can be seen that $S \circ f$ is measurable by taking limits of simple functions. Also, $\int_\Omega \|T(f)(\omega)\|^q \, d\upsilon(\omega) \le \|S\|^q \int_\Omega \|f(\omega)\|^q \, d\upsilon(\omega) < \infty$, and T is well-defined. It is straightforward to show that T is linear and injective. Thus, the integral inequality just obtained shows that $\hat{T}$ is bounded. Now, suppose $T(f) = T(g)$ υ-a.e. Then $S(f(\omega)) = S(g(\omega))$ υ-a.e., and $f(\omega) = g(\omega)$ υ-a.e. since S is injective. But, $\mu << \upsilon$, and therefore $f = g$ μ-a.e. By Proposition 1, T is a (bounded) L-correspondence.

Suppose the hypotheses of (ii) hold. Then $f = g$ μ-a.e. if and only if $T(f) = T(g)$ υ-a.e. Also, since S is an isomorphism, $T(f) = T(g)$ υ-a.e. if and only if $f = g$ υ-a.e. Let $0 \ne x \in X$. Then $x\chi_E = 0$ μ-a.e. if and only if $x\chi_E = 0$ υ-a.e., and we have both $\mu << \upsilon$ and $\upsilon << \mu$. Thus, given $f \in L(\mu, X, p)$, $\int_\Omega \|f(\omega)\|^q \, d\upsilon(\omega) = \int_\Omega \|S^{-1}(T(f)(\omega))\|^q \, d\upsilon(\omega) \le \|S^{-1}\|^q \int_\Omega \|T(f)(\omega)\|^q \, d\upsilon(\omega) < \infty$, and $f \in L(\upsilon, X, q)$. By Miamee's Lemma, $L^p(\mu, X) \subset L^q(\upsilon, X)$.

COROLLARY 3. $L^p(\mu, X) \subset L^q(\upsilon, X)$ if and only if the identity map $I: L(\mu, X, p) \to L(\upsilon, X, q)$ is an L-correspondence. When I is an L-correspondence, it is both bounded and exact.

It can be shown that if the isomorphism S in Theorem 2 is surjective and $T(f) = S \circ f$ defines an L-correspondence, then T is exact.

3. BASIC CHARACTERIZATION THEOREMS AND A CONJECTURE.

Theorem 2 gives a way to construct some bounded L-correspondences. A natural question to ask is whether or not there are conditions under which a bounded L-correspondence must have been constructed in that manner. A necessary condition can quickly be obtained: Let $x \in X$ and $E \in \Sigma$. Then $T(x\chi_E) = S(x)\chi_E$. The next theorem shows that this is almost sufficient.

THEOREM 4. Let $T: L(\mu, X, p) \to L(\upsilon, Y, q)$ be a bounded L-correspondence such that given $x \in X$ and $E \in \Sigma$, there is some $y \in Y$ such that $T(x\chi_E) = y\chi_E$. Then there exists a bounded linear injection $S: X \to Y$ such that $T(f) = S \circ f$ υ-a.e. for all $f \in L(\mu, X, p)$.

PROOF. Define $S: X \to Y$ by $S(x) = y$ where $T(x\chi_\Omega) = y\chi_\Omega$. Let $E \in \Sigma$. Then $T(x\chi_E) + T(x\chi_{\Omega \setminus E}) = S(x)\chi_\Omega$, and therefore $T(x\chi_E) = S(x)\chi_E$. Let $f = \sum_{i=1}^n x_i\chi_{E_i}$ be a simple function in canonical form. Then we have

$$T(f) = \sum_{i=1}^n T(x_i\chi_{E_i}) = \sum_{i=1}^n S(x_i)\chi_{E_i} = \sum_{i=1}^n S \circ (x_i\chi_{E_i}) = S \circ f. \tag{3.1}$$

We now wish to show that S is a bounded linear injection. A simple calculation shows the linearity of S. For boundedness, let $x \in X$ and note that $\|x\chi_\Omega\|_{p,\mu} = \|x\|\mu(\Omega)^{1/p}$ and $\|S(x)\chi_\Omega\|_{q,\upsilon} = \|S(x)\|\,\upsilon(\Omega)^{1/q}$. However, T is bounded; thus, there is some $M \geq 0$ such that $\|S(x)\chi_\Omega\|_{q,\upsilon} \leq M\|x\chi_\Omega\|_{p,\mu}$. Therefore,

$$\|S(x)\| \leq M\frac{\mu(\Omega)^{1/p}}{\upsilon(\Omega)^{1/q}}\|x\|, \tag{3.2}$$

and S is bounded. Now suppose $S(x) = 0$. Then $T(x\chi_\Omega) = S(x)\chi_\Omega = 0$. Since T is injective, $x = 0$ and S is injective.

Finally, let $f \in L(\mu, X, p)$. Let (f_n) be a sequence of simple functions in $L(\mu, X, p)$ such that $f_n \to f$ in $L^p(\mu, X)$ and $f_n \to f$ μ-a.e. Then $T(f_n) \to T(f)$ in $L^q(\upsilon, Y)$. Choose a subsequence (still denoted by (f_n)) such that $T(f_n) \to T(f)$ pointwise υ-a.e. Note that $\upsilon << \mu$. Thus, there is a υ-null set H off which both $f_n \to f$ pointwise and $T(f_n) \to T(f)$ pointwise. Since S is continuous, $T(f_n) = S \circ f_n \to S \circ f$ pointwise off H. Thus, $T(f) = S \circ f$ υ-a.e.

Next, we show that we cannot guarantee strict equality of $T(f)$ and $S \circ f$ under the conditions of Theorem 4.

PROPOSITION 5. Let T, S be as in Theorem 4 and suppose there is a non-empty μ-null set. Then there is a bounded L-correspondence $T': L(\mu, X, p) \to L(\upsilon, Y, q)$ such that $T'(x\chi_E) = S(x)\chi_E$ for all $x \in X$ and $E \in \Sigma$, $T'(f) = S \circ f$ υ-a.e. for all $f \in L(\mu, X, p)$, and for some $f \in L(\mu, X, p)$, $T'(f) \neq S \circ f$.

PROOF. Let E be a non-empty μ-null set. Let A be a Hamel basis for the subspace of $L(\mu, X, p)$ consisting of all simple functions and let $f \in L(\mu, X, p)$ be a non-simple function. Let B be a Hamel basis of $L(\mu, X, p)$ including A and f. Let $0 \neq x \in X$. Then given $g \in L(\mu, X, p)$, g is expressible as a finite linear combination $\alpha_g f + \cdots$ of elements of B in a unique way. Note that if g is simple, $\alpha_g = 0$. Now define $T': L(\mu, X, p) \to L(\upsilon, Y, q)$ by $T'(g) = T(g) + \alpha_g S(x)\chi_E$. Then T' is linear and $T'(x\chi_E) = S(x)\chi_E$ for all $x \in X$ and $E \in \Sigma$.

To see that T' is injective, suppose $T'(g) = 0$. Then $T(g) = S(-\alpha_g x)\chi_E$. As T is injective, $g = -\alpha_g x\chi_E$. Since g is a simple function, $-\alpha_g = 0$, and $g = 0$.

Finally, recall that $\upsilon << \mu$, and thus $T'(g) = T(g)$ υ-a.e. Consequently, T' is a bounded L-correspondence and $T'(g) = S \circ g$ υ-a.e. for all $g \in L(\mu, X, p)$. However, $T'(f) \neq T(f) = S \circ f$.

The previous theorems dealt with representing L-correspondences by using a continuous linear injection $S: X \to Y$. However, as we are not restricted to using a "natural" embedding for our L-correspondences, we may also choose to rearrange our measure space. As an example, let (Ω, Σ, μ) be the standard Lebesgue measure space (on $[0,1]$), and let $Y = X$ be an arbitrary Banach space. For $f \in L(\mu, X, p)$, define

$$T(f)(t)=\begin{cases} f(2t) & \text{if } t\le \frac{1}{2} \\ 0 & \text{otherwise}\end{cases}, \tag{3.3}$$

for $t\in[0,1]$. Then $T:L(\mu,X,p)\to L(\mu,X,p)$ is a bounded L-correspondence not satisfying the hypotheses or conclusions of Theorem 4. One characteristic that T does still possess is that it sends single-step functions to single-step functions, i.e., given $x\in X$ and $E\in\Sigma$, there exists $y\in Y$ and $H\in\Sigma$ such that $T(x\chi_E)=y\chi_H$. We shall now explore that general setting. The proof of the following Lemma is left to the reader.

LEMMA 6. Let $T:L(\mu,X,p)\to L(\upsilon,Y,q)$ be an L-correspondence that sends single-step functions to single-step functions. Then there is a set function $\psi:\Sigma\to\Sigma$ such that for any $x\in X$ and $E\in\Sigma$, there is some $y\in Y$ such that $T(x\chi_E)=y\chi_{\psi(E)}$. Additionally, if ψ is not a constant function, it is injective and there exists a linear injection $S:X\to Y$ such that $T(x\chi_E)=S(x)\chi_{\psi(E)}$ for all $x\in X$ and $E\in\Sigma$.

It will be shown in the example at the end of this article that the case ψ is constant may occur. Now, suppose ψ is injective. Is it possible that ψ is a (lattice) homomorphism generated by a measurable point mapping φ, in such a way that $T(f)=S\circ f\circ\varphi$ υ-a.e. for all f? Since $\varphi(\Omega)$ may not be Ω, as in the example before Lemma 6, we cannot hope for quite so much. However, we may be able to come close. Suppose singletons are measurable. Let 0 be an object not in Ω, let $\Omega'=\Omega\cup\{0\}$, $\Sigma'=\Sigma\cup\{A\cup\{0\}|A\in\Sigma\}$, and define μ' on Σ' by $\mu'(A)=\mu(A\cap\Omega)$. Define $\varphi:\Omega\to\Omega'$ by

$$\varphi(t)=\begin{cases} \omega & \text{if } t\in\psi(\{\omega\}) \\ 0 & \text{if } t\notin\bigcup_{\omega\in\Omega}\psi(\{\omega\})\end{cases}. \tag{3.4}$$

Finally, for $f\in L(\mu,X,p)$, define $f(0)=0$. We then have the following theorem, the proof of which is similar to that of Theorem 4.

THEOREM 7. Suppose singletons are measurable, $T:L(\mu,X,p)\to L(\upsilon,Y,q)$ is a bounded L-correspondence taking single-step functions to single-step functions, and ψ is injective. If ψ maps onto $\varphi^{-1}(\Sigma)$, then $T(f)=S\circ f\circ\varphi$ υ-a.e. for all $f\in L(\mu,X,p)$.

It is obvious that φ is a measurable point mapping under the hypotheses of Theorem 7; in fact, φ must be measurable in order to obtain the conclusion $T(f)=S\circ f\circ\varphi$ υ-a.e. To see this, let $E\in\Sigma$ such that $\varphi^{-1}(E)$ is not measurable. Then $T(x\chi_E)=S\circ x\chi_E\circ\varphi=S(x)\chi_{\varphi^{-1}(E)}$, which is not measurable, yielding a contradiction. Nevertheless, we offer the conjecture that either it is always the case that ψ maps onto $\varphi^{-1}(\Sigma)$ or that that hypothesis may be removed from the statement of Theorem 7 anyway. This amounts to proving something similar to an equimeasurability theorem in Lebesgue-Bochner spaces (Koldobskiî [1] has obtained some equimeasurability results in that setting).

We close with an example of a bounded L-correspondence in which ψ is constant and μ is not absolutely continuous with respect to υ. Let $\Omega=\mathbb{N}$, $\Sigma=\mathcal{P}(\mathbb{N})$, $\mu(E)=\sum_{n\in E}\frac{1}{2^n}$, and υ be a measure on (Ω,Σ) with a non-empty null set. Define $T:L(\mu,R,p)\to L(\upsilon,\ell^p,q)$ by $T(f)=(\frac{1}{2^n}f(n))_{n=1}^{\infty}\chi_\Omega$. Then $\int_\Omega\|f\|^p\,d\mu=\sum_{n=1}^{\infty}|f(n)|^p\frac{1}{2^n}\ge\sum_{n=1}^{\infty}|\frac{1}{2^n}f(n)|^p$, and T is well-defined. It is quick to see that T is a linear injection. Since $f=g$ μ-a.e. if and only if $f=g$ if and only if $T(f)=T(g)$ υ-a.e., T is an L-correspondence. Since $\|T(f)\|_{q,\upsilon}\le\|f\|_{p,\mu}\,\upsilon(\Omega)^{1/q}$, T is bounded.

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