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PEXIDER FUNCTIONAL EQUATIONS - THEIR FUZZY ANALOGS

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ABSTRACT. In this paper we shall interpret and study the Pexider functional equations in the context of Fuzzy Set Theory. In particular, we shall present a general procedure for obtaining the fuzzy analog of the Pexider functional equations and then solve the resulting equations.

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1. MOTIVATION.

The functional equation

$$I(pq) = I(p) + I(q) \tag{1.1}$$

may be interpreted as giving the amount of information I due to two independent events A and B with probabilities p and q, respectively. The functional equation (1.1) is one of Cauchy equations, and has been dealt with extensively (see Aczél [1-2]). However, it is more often than not that we do not have the exact values of the probabilities p and q because not enough data is available or because p and q partially reflect the decision maker's subjective opinion. As such, the decision maker will not exactly know the amount of information, $I(\cdot)$, generated by the independent event A, B and $A \cap B$. Rather the decision maker has an amount of belief in each of the possible values of such quantities as $I(\cdot)$. The belief is a number between 0 and 1. The idea of assigning beliefs to possible values has been introduced by Zadeh [12-13] under the name of Fuzzy Set Theory. For this reason $I(\cdot)$ is a "fuzzy" number representing the true information given for a specific probability p resulting from some imprecise measurement.

This discussion, along with results obtained by Kreinovich and Deeba [9] for another Cauchy equation

$$m(x+y) = m(x) + m(y) \tag{1.2}$$

which was formulated in the fuzzy setting to address the question:"with what accuracy can we measure masses $m(x)$ if we use an imprecise mass standard?", motivate us to consider the Cauchy functional equations in the setting of Fuzzy Set Theory, (see Deeba et al. [3]). Quantities such as $I(p)$ or $m(x)$ are therefore fuzzy numbers, and values of p and x are positive numbers.

In the fuzzy setting, it is natural to consider assumptions such as

(i) the difference between an expert's opinion and the true probability should not be too large;
(ii) if the amount of information generated by some event of probability pq is $I(pq)$ while the actual amount is u, then we can find two independent events with estimated probabilities p and q, and actual information contents u_1 and u_2 such that $u = u_1 + u_2$.

Using the notation of Kreinovich and Deeba [9], let $P_x(u)$ be the statement that the true value of the information is u and the estimated value is x. The degree of belief in the statement $P_x(u)$ is $N(x)$, where $N(x)$ is a fuzzy number that is a continuous function from the set of real numbers $\mathbb{R}$

to the interval $[0,1]$ such that $\lim_{u\to\infty} N(x)(u) = 0$ and $N(x)(u) = 0$ for $u < 0$. $N(x)(u)$ denotes the belief that the true value is u based on measurement x. Let $t(P_x(u))$ denote the degree of belief in $P_x(u)$. Then the degree of belief in $P_{x+y}(u)$ can be defined as follows:

$$t(P_{x+y}(u)) = \sup_{u_1+u_2=u} (\min\{t(P_x(u_1)), t(P_y(u_2))\}) \tag{1.3}$$

where t denotes the degree of belief in the corresponding statement (see Fuller [6], Hersch and Caramazza [7], Oden [11], and Zimmerman [14]). More generally,

$$t(P_{f(x,y)}(u)) = \sup(\min\{t(P_x(u_1)), t(P_y(u_2))\}) \tag{1.4}$$

where the sup is taken over all u_1 and u_2 for which $f(u_1, u_2) = u$. Formula (1.4) can be viewed as a special case of the "Extension Principle" (see Dubois and Prade [4]).

Assume now that the probabilities p and q for independent events A and B come from distinct sources S_2 and S_3. The resulting estimates of the information for $A \bigcap B$ come from the combination of S_2 and S_3, which we call S_1 and is given by

$$I_1(pq) = I_2(p) + I_3(q) \tag{1.5}$$

where $I_2(p)$ and $I_3(q)$ are the information generated by A and B, and represent the amount of belief in the possible values of A and B. Notice that each $I_i(\cdot)$ is a fuzzy number and (1.5) is a generalization of (1.1). The right hand side of (1.5) is defined as follows: for every $y > 0$

$$[I_2(p) + I_3(q)](y) = \sup(\min\{I_2(p)(y_1) + I_3(q)(y_2)\}) \tag{1.6}$$

where the sup is taken over all y_1 and y_2 such that $y = y_1 + y_2$. Equation (1.5) is the fuzzy analog of one of the Pexider equation.

As noted earlier, we may extend the arguments of Kreinovich and Deeba [9] to consider the fuzzy analog of the Pexider equation

$$f(x + y) = g(x) + h(y)$$

which is a generalization of (1.2). In this setting (1.2) reads:

$$m_1(x + y) = m_2(x) + m_3(y)$$

with x and y representing mass measurements (see Kreinovich [8], and Kreinovich and Deeba [9]), and measurements m_i, $(i = 1, 2, 3)$ in this situation are with respect not to the same standard as Kreinovich and Deeba discussed in [9], but with respect to different imprecise standards. It is also clear that similar arguments could be made for measuring many physical quantities. For example, by considering rates of growth and their imprecise measurements one would arrive at the fuzzy functional equation

$$N_1(x_1 + x_2) = N_2(x_1)N_3(x_2) \tag{1.7}$$

where N_2 and N_3 are estimates of growth at x_1 and x_2 obtained by different sources, and N_1 is the estimate resulting from those two sources at $x_1 + x_2$.

The right hand side of (1.7) would be defined, for any $y > 0$, as:

$$[N_2(x_1)N_3(x_2)](y) = \sup(\min\{N_1(x_1)(y_1), N_2(x_2)(y_2)\})$$

where the sup is taken over all y_1 and y_2 for which $y_1y_2 = y$.

The above discussion leads us to the study of the four Pexider functional equations where the functional values are fuzzy numbers as opposed to real numbers. These equations are:

$$f(x+y) = g(x) + h(y) \tag{1.8}$$

$$f(xy) = g(x) + h(y) \tag{1.9}$$

$$f(x+y) = g(x)h(y) \tag{1.10}$$

$$f(xy) = g(x)h(y) \tag{1.11}$$

We will thus consider the equation

$$A(x * y) = B(x) \Box C(y) \tag{1.12}$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are fuzzy numbers, $*$ is a binary operation on the reals, and $\Box$ is a binary operation on numbers properly extended to fuzzy numbers. Our task is to give the proper formulation of (1.12) and specialize the result to address the four cases of the Pexider equations.

In Section 2 we continue to give background information while in Section 3 we present the necessary lemmas needed in the analysis of (1.12). The four Pexider equations in the fuzzy setting will be addressed in Section 4.

2. PRELIMINARIES.

In this section, we present, for the sake of completeness, more background material needed for this manuscript. For a detailed study, we refer the reader to Dubois and Prade [5], and Zadeh [12].

Let X be any set. A is a fuzzy set of X if A is a function from X into the interval $[0,1]$. The value of $A(x)$ is sometimes referred to as the membership of x in the fuzzy subset A. $\mathbb{R}$ and $\mathbb{R}^+$ denote throughout the set of reals and positive reals, respectively.

Let A be a fuzzy subset of $\mathbb{R}^+$. A is said to be convex if

$$A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\}$$

for all x_1, x_2 in $\mathbb{R}^+$ and $t \in [0,1]$. By an α-level of the fuzzy subset A, denoted by $[A]_\alpha$, we mean

$$[A]_\alpha = \{x \in \mathbb{R}^+ \mid A(x) \geq \alpha\}.$$

It is well-known that the α-levels of a fuzzy subset A determine A, (see Dubois and Prade [5]). It can be easily verified that for any fuzzy A:

A is convex if and only if $[A]_\alpha$ is convex for all $\alpha \in [0,1]$. (C1)

Throughout we shall consider the case of continuous fuzzy sets. In this situation, for any fuzzy subset A we have

A is convex if and only if $[A]_\alpha$ is a closed interval for all $\alpha \in [0,1]$. (C2)

We recall from Section 1 that a fuzzy number N is a continuous, convex fuzzy set such that $\lim_{y\to\infty} N(y) = 0$ and $N(y) = 0$ for all $y < 0$. The fuzzy number N is said to be normalized if there exists a unique y^* such that $N(y^*) = 1$.

Let $\Box$ be a binary operation on $\mathbb{R}$ or some subset of $\mathbb{R}$. We assume that this operation satisfies:

(a) $\Box$ is increasing; that is, if $x_1 < x_2$ and $y_1 < y_2$, then $x_1 \Box y_1 < x_2 \Box y_2$.

(b) $\Box$ is continuous.

Define a function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by

$$g(x,y) = x \Box y.$$

The Extension Principle (see Dubois and Prade [5], or Zadeh [12]) states that g can be extended to fuzzy subsets of $\mathbb{R} \times \mathbb{R}$ as follows:

$$g(A, B)(y) = \sup(\min\{A(y_1), B(y_2)\}), \tag{2.1}$$

where sup is taken over all y_1, y_2 such that $g(y_1, y_2) = y$, and A and B are fuzzy subsets of $\mathbb{R}$. In this manner, we can extend the operation $\Box$ to fuzzy numbers.

Finally, we will need a basic result due to Nguyen [10]. A basic question that we need to address in developing the results of this paper is: if g is as above, when do we have

$$[g(M, N)]_\alpha = g([M]_\alpha, [N]_\alpha), \tag{2.2}$$

that is, when is the α-level of $g(M, N)$ the image by g of the α-levels of M and N for any fuzzy subset M and N of $\mathbb{R}$?

It was shown by Nguyen [10] that (2.2) holds if and only if the sup in (2.1) is assumed. That is, there exist x^* and y^* such that

$$g(M, N)(z) = \min\{M(x^*), N(y^*)\}$$

with $z = g(x^*, y^*)$.

3. FUZZY REPRESENTATION OF FUNCTIONAL EQUATIONS.

In this section we shall prove several lemmas for the fuzzy functional equation

$$A(x * y) = B(x) \Box C(y) \tag{3.1}$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are fuzzy non-negative numbers and x, $y \in \mathbb{R}^+$. The operations $*$ and $\Box$ are as defined in (1.12). We note that the assumptions on the operation $\Box$ make it possible to extend it to an operator over fuzzy numbers. These lemmas are basic in understanding how to manipulate and obtain the solution of equation (3.1).

LEMMA 3.1 If h is a function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^+$ defined by

$$h(x, y) = x \Box y$$

is such that the inverse $h^{-1}(z)$ is bounded, then

$$[A(x * y)]_\alpha = [B(x)]_\alpha \Box [C(y)]_\alpha.$$

PROOF: Let N and M be two fuzzy subsets of $\mathbb{R}^+$. By the Extension Principle,

$$\begin{aligned} h(N, M)(z) &= \sup_{x \Box y = z} (\min\{N(x), M(y)\}) \\ &= \sup_{(x,y) \in h^{-1}(z)} (\min\{N(x), M(y)\}). \end{aligned} \tag{3.2}$$

Since $h^{-1}(z)$ is assumed to be bounded and $h^{-1}(\{z\})$ is closed (note: $\Box$ is continuous), it follows that $h^{-1}(z)$ is compact. This implies that there exist x^* and y^* such that the sup in (3.2) is attained. Hence, by Nguyen's result [10],

$$[h(M, N)]_\alpha = h([N]_\alpha, [M]_\alpha).$$

From this it follows that

$$[N \Box M]_\alpha = [N]_\alpha \Box [M]_\alpha.$$

Upon setting $N = B(x)$ and $M = C(y)$, we obtain

$$[B(x)\Box C(y)]_\alpha = [B(x)]_\alpha \Box [C(y)]_\alpha,$$

or equivalently

$$[A(x * y)]_\alpha = [B(x)]_\alpha \Box [C(y)]_\alpha.$$

Remark 3.1 What makes $h^{-1}(z)$ bounded? This is the case if $S = \{(x,y) \mid x\Box y = z\}$ is bounded. In case $\Box =$ '+', and x and y belong to $\mathbf{R}^+$, then for every z, it is clear that x and y belong to $[0,z]$ and therefore, S is bounded. However, if $\Box =$ '$\times$', one has the boundedness property provided that x and y are bounded away from the origin. We shall choose in the latter case $\mathbf{R}^+_a = \{x \mid x \geq a\}$ for $a \in \mathbf{R}^+$, a fixed and assume that

$$\lim_{x\to\infty} a\Box x = +\infty.$$

In this paper, we shall restrict ourselves to $\mathbf{R}^+$ or $\mathbf{R}^+_1$ depending upon the case under consideration.

It can also be easily shown that, since the fuzzy numbers $A(x)$, $B(x)$ and $C(x)$ are continuous, that

LEMMA 3.2 $[A(x)]_\alpha$, $[B(x)]_\alpha$ and $[C(x)]_\alpha$ are closed sets.

LEMMA 3.3 Let $A(x)$, $B(x)$ and $C(x)$ be any fuzzy numbers. Then their α-levels, $[A(x)]_\alpha$, $[B(x)]_\alpha$ and $[C(x)]_\alpha$ are bounded and closed intervals.

PROOF: From (C1) and Lemma 3.2, it follows that $[A(x)]_\alpha$ is a closed interval. Since $\lim_{u\to\infty} A(x)(u) = 0$, the right end point of this interval is finite.

The proof for $[B(x)]_\alpha$ and $[C(x)]_\alpha$ are similar.

Let $L_A(x)$, $L_B(x)$ and $L_C(x)$ be the left endpoints of $[A(x)]_\alpha$, $[B(x)]_\alpha$ and $[C(x)]_\alpha$, respectively. Similarly, let $R_A(x)$, $R_B(x)$ and $R_C(x)$ be the right endpoints of $[A(x)]_\alpha$, $[B(x)]_\alpha$ and $[C(x)]_\alpha$, respectively.

LEMMA 3.4 The left endpoints $L_i(x)$ $(i = A,B,C)$ and the right endpoints $R_i(x)$ $(i = A,B,C)$ of the intervals $[A(x)]_\alpha$, $[B(x)]_\alpha$ and $[C(x)]_\alpha$ satisfy equations

$$\begin{aligned} L_A(x*y) &= L_B(x)\Box L_C(y) \\ R_A(x*y) &= R_B(x)\Box R_C(y) \end{aligned} \qquad (E)$$

PROOF: If $[A(z)]_\alpha = [L_A(z), R_A(z)]$, $[B(z)]_\alpha = [L_B(z), R_B(z)]$ and $[C(z)]_\alpha = [L_C(z), R_C(z)]$, then $[A(x*y)]_\alpha = [B(x)]_\alpha \Box [C(y)]_\alpha$ in Lemma 3.1 and the assumption that the operation $\Box$ is increasing will imply the equations

$$\begin{aligned} L_A(x*y) &= L_B(x)\Box L_C(y) \\ R_A(x*y) &= R_B(x)\Box R_C(y) \end{aligned}$$

Finally, Lemma 3.4 can be restated as:

LEMMA 3.5 The α-levels, $[A(x)]_\alpha$, $[B(x)]_\alpha$ and $[C(x)]_\alpha$, are all of the form $[S_i(x), T_i(x)]$ $(i = A$, B and $C)$, where $S_i(x)$ and $T_i(x)$ $(i = A$, B and $C)$ are solutions of equations (E).

4. PEXIDER FUNCTIONAL EQUATION-FUZZY ANALOG.

In this section we shall consider the solution of the Pexider equations

$$f(x+y) = g(x) + h(y) \qquad (4.1)$$

$$f(xy) = g(x) + h(y) \tag{4.2}$$

$$f(x+y) = g(x)h(y) \tag{4.3}$$

$$f(xy) = g(x)h(y) \tag{4.4}$$

but in the fuzzy setting. The above four equations will be considered as special cases of equation (3.1) given by

$$A(x * y) = B(x)\Box C(y). \tag{F}$$

Case 1. (F) is the fuzzy analog of (4.1) if $*$ and $\Box$ are replaced by addition '+'. We assume that x and y belong to $\mathbb{R}^+$. In this case (4.1) is given by

$$A(x+y) = B(x) + C(y) \tag{F1}$$

with equations (E) reading

$$\begin{aligned} L_A(x+y) &= L_B(x) + L_C(y) \\ R_A(x+y) &= R_B(x) + R_C(y) \end{aligned} \tag{E1}$$

Since the set $S = \{(x,y) \mid x+y = z\}$ is bounded for every fixed z in $\mathbb{R}^+$, it follows that Lemma 3.3 through Lemma 3.5 hold. The solution to the system (E1) is

$$L_A(x) = L_B(x) + L_C(y), \quad \text{and} \quad R_A(x) = R_B(x) + R_C(x).$$

where

$$\begin{aligned} L_A(x) &= k_1x + b_1 + c_1, & R_A(x) &= k_2x + b_2 + c_2; \\ L_B(x) &= k_1x + b_1, & R_B(x) &= k_2x + b_2; \\ L_C(x) &= k_1x + c_1, & R_C(x) &= k_2x + c_2; \end{aligned} \tag{E1}$$

Thus $[A(x)]_\alpha = [L_A(x), R_A(x)]_\alpha[L_{A,\alpha}(x), R_{A,\alpha}(x)]$, where $L_{A,\alpha}(x) = k_{1\alpha}x + b_{1\alpha} + c_{1\alpha}$, and $R_{A,\alpha}(x) = k_{2\alpha}x + b_{2\alpha} + c_{2\alpha}$. The subscript α is used to show dependency on α.

Now, $y \in [A(x)]_\alpha$ if and only if

$$\begin{aligned} A(x)(y) \geq \alpha &\iff L_{A,\alpha}(x) \leq y \leq R_{A,\alpha}(x) \\ &\iff k_{1\alpha}x + b_{1\alpha} + c_{1\alpha} \leq y \leq k_{2\alpha}x + b_{2\alpha} + c_{2\alpha}; \end{aligned} \tag{4.5}$$

For each x fixed, $x \geq 0$, consider the function

$$h_x(t_1, t_2) = t_1x + t_2.$$

By the Extension Theorem, h_x can be extended to fuzzy subsets N_1 and N_2 of $\mathbb{R} \times \mathbb{R}$ as follows:

$$\begin{aligned} h_x(t_1,t_2)(z) &= \sup_{h_x(t_1,t_2)=z} (\min\{N_1(t_1), N_2(t_2)\}) \\ &= \sup_{t_1x+t_2=z} (\min\{N_1(t_1), N_2(t_2)\}) \end{aligned} \tag{4.6}$$

$h_x^{-1}(z) = \{(t_1,t_2) \mid t_1x + t_2 = z\}$. If $t_1, t_2 > 0$, then $h_x^{-1}(z)$ is bounded for every fixed z. Hence, by Nguyen's result in [10].

$$\begin{aligned} [h_x(N_1,N_2)]_\alpha &= h_x([N_1]_\alpha, [N_2]_\alpha) \\ &= x[N_1]_\alpha + [N_2]_\alpha \\ &= [k_{1\alpha}, k_{2\alpha}]x + [b_{1\alpha} + c_{1\alpha}, b_{2\alpha} + c_{2\alpha}] \end{aligned} \tag{4.7}$$

Here, we pick

$$\begin{cases} N_1(\cdot) = \sup\limits_{0<\alpha\leq 1} \alpha \wedge \chi_{[k_{1\alpha},k_{2\alpha}]}(\cdot) \\ N_2(\cdot) = \sup\limits_{0<\alpha\leq 1} \alpha \wedge \chi_{[b_{1\alpha}+c_{1\alpha},b_{2\alpha}+c_{2\alpha}]}(\cdot) \end{cases} \tag{4.8}$$

Equations (4.5) and (4.7) yield

$$[A(x)]_\alpha = [h_x(N_1, N_2)]_\alpha,$$

or

$$A(x) = h_x(N_1, N_2) = N_1 x + N_2,$$

where N_1 and N_2 defined in terms of $[N_1]_\alpha$ and $[N_2]_\alpha$ in (4.8).

We next investigate $B(x)_\alpha$ and $C(x)_\alpha$. It is known that

$$\begin{aligned} [B(x)]_\alpha &= [k_{1\alpha}x + b_{1\alpha}, k_{2\alpha}x + b_{2\alpha}] \\ &= [k_{1\alpha}, k_{2\alpha}]x + [b_{1\alpha}, b_{2\alpha}] \\ &= [N_1]_\alpha x + [B_0]_\alpha \end{aligned} \tag{4.9}$$

where $B_0(\cdot) = \sup_{0<\alpha\leq 1} \alpha \wedge \chi[b_{1\alpha}, b_{2\alpha}]\cdot$.

Similarly,

$$\begin{aligned} [C(x)]_\alpha &= [k_{1\alpha}x + c_{1\alpha}, k_{2\alpha}x + c_{2\alpha}] \\ &= [k_{1\alpha}, k_{2\alpha}]x + [c_{1\alpha}, c_{2\alpha}] \\ &= [N_1]_\alpha x + [C_0]_\alpha \end{aligned} \tag{4.10}$$

where $C_0(\cdot) = \sup_{0<\alpha\leq 1} \alpha \wedge \chi[c_{1\alpha}, c_{2\alpha}]\cdot$.

Also note that $B_0 + C_0 = N_2$. Symbolically,

$$\begin{aligned} A(x) &= N_1 x + N_2 \\ B(x) &= N_1 x + B_0 \\ C(x) &= N_1 x + C_0 \end{aligned} \tag{4.11}$$

with $B_0 + C_0 = N_2$.

THEOREM 4.1 The solution of the fuzzy Pexider equation (F1) is given by (4.11).

This is analogous to the solution of the "classical" Pexider equation (4.1), where N_1, N_2, B_0 and C_0 are real numbers.

Case 2. (F) is the fuzzy analog of equation (4.2) if $*$ is replaced by multiplication '$\cdot$' and $\square$ is replaced by addition '+'. In this case (4.2) reads:

$$A(xy) = B(x) + C(y) \tag{F2}$$

and equations (E) read

$$\begin{aligned} L_A(xy) &= L_B(x) + L_C(y) \\ R_A(xy) &= R_B(x) + R_C(y) \end{aligned} \tag{E2}$$

As in Case 1, for Lemma 3.1 through 3.5 to hold, we need the set $S = \{(x,y) \mid xy = z\}$ to be bounded. This is the case if x and y belong to the set, say, $\mathbb{R}_1^+ = \{\xi \mid \xi > 1\}$. The solutions to the system (E2) are

$$L_A(x) = k_1 \log(b_1 c_1 x), \quad R_A(x) = k_2 \log(b_2 c_2 x);$$

$$L_B(x) = k_1 \log(b_1 x), \quad R_B(x) = k_2 \log(b_2 x);$$

$$L_C(x) = k_1 \log(c_1 x), \quad R_C(x) = k_2 \log(c_2 x);$$

By introducing a transformation $x = e^u$ and $y = e^v$ for all u, v in $\mathbb{R}^+$ (note that $x > 1$ and $y > 1$,) we can reduce equation (F2) to a new equation:

$$A(e^{u+v}) = B(e^u) + C(e^v). \tag{4.12}$$

Let $\tilde{A}(\xi) = A(e^\xi)$, $\tilde{B}(\xi) = B(e^\xi)$, and $\tilde{C}(\xi) = C(e^\xi)$. Then equation (4.12) becomes

$$\tilde{A}(u+v) = \tilde{B}(u) + \tilde{C}(v).$$

This equation is solved in Case 1:

$$\begin{aligned}\tilde{A}(u) &= N_1 u + N_2\\ \tilde{B}(u) &= N_1 u + B_0\\ \tilde{C}(u) &= N_1 u + C_0\end{aligned}$$

where

$$\begin{aligned}N_1(\cdot) &= \sup_{0<\alpha\leq 1} \alpha \wedge \chi_{[k_{1\alpha},k_{2\alpha}]}(\cdot),\\ B_0(\cdot) &= \sup_{0<\alpha\leq 1} \alpha \wedge \chi_{[k_{1\alpha}\log b_{1\alpha},k_{2\alpha}\log b_{2\alpha}]}(\cdot),\\ C_0(\cdot) &= \sup_{0<\alpha\leq 1} \alpha \wedge \chi_{[k_{1\alpha}\log c_{1\alpha},k_{2\alpha}\log c_{2\alpha}]}(\cdot).\\ N_2(\cdot) &= B_0 + C_0\end{aligned}$$

(These are not the same k's, b's and c's as above.)

Therefore,

$$\begin{aligned}A(x) = \tilde{A}(\log x) &= N_1 \log x + B_0 + C_0\\ B(x) = \tilde{B}(\log x) &= N_1 \log x + B_0\\ C(x) = \tilde{C}(\log x) &= N_1 \log x + C_0\end{aligned} \tag{4.13}$$

THEOREM 4.2 The solution of the fuzzy Pexider equation (F2) is given by (4.13).

These results are analogous to the "classical" solution of the Pexider equation (4.2) with the proper interpretation of N_1, N_2, B_0 and C_0.

Case 3. Equation (F) is the fuzzy analog of equation (4.3) if $*$ is replaced by '+' and '□' is replaced by '$\cdot$'. For this case, (F) reads

$$A(x+y) = B(x)C(y) \tag{F3}$$

and equations (E) read as:

$$\begin{aligned}L_A(x+y) &= L_B(x)L_C(y)\\ R_A(x+y) &= R_B(x)R_C(y)\end{aligned} \tag{E3}$$

For Lemma 3.1 through 3.5 to hold, the set $S = \{(x,y) \mid xy = z\}$ must be bounded. This is the case since we restrict ourselves to the set $\mathbb{R}_1^+ = \{\nu \mid \nu > 1\}$. We note that we do not know if all solutions of (E3) over $\mathbb{R}_1^+$ are the restrictions of solutions of (E3) over $\mathbb{R}^+$ and the question of uniqueness to the best of our knowledge is open.

The solution that will be exhibited for (F3) will be valid for $x > 1$ and in here we will assume that $A(x) = 0$ for $x \leq 1$.

The solutions to the system (E3) are

$$L_A(x) = b_1c_1e^{k_1x}, \quad R_A(x) = b_2c_2e^{k_2x};$$

$$L_B(x) = b_1e^{k_1x}, \quad R_B(x) = b_2e^{k_2x};$$

$$L_C(x) = c_1e^{k_1x}, \quad R_C(x) = c_2e^{k_2x};$$

Thus $[A(x)]_\alpha = [b_{1\alpha}c_{1\alpha}e^{k_{1\alpha}x}, b_{2\alpha}c_{2\alpha}e^{k_{2\alpha}x}]$. Now $y \in [A(x)]_\alpha$ if and only if

$$A(x)(y) \geq \alpha \iff b_{1\alpha}c_{1\alpha}e^{k_{1\alpha}x} \leq y \leq b_{2\alpha}c_{2\alpha}e^{k_{2\alpha}x} \tag{4.14}$$

Now consider the function $h_x(t_1,t_2) = t_2e^{t_1x}$. Let $p_x(t_1,t_2) = \log h_x(t_1,t_2) = \log t_2 + t_1x = \tilde{t_2} + t_1x$. (or, $h_x(t_1,t_2) = e^{p_x(t_1,t_2)}$.) As in Case 1, we have

$$\begin{aligned} [p_x(N_1,N_2)]_\alpha &= p_x([N_1]_\alpha,[N_2]_\alpha) \\ &= [N_1]_\alpha + [N_2]_\alpha \\ &= [k_{1\alpha},k_{2\alpha}]x + [\log(b_{1\alpha}c_{1\alpha}),\log(b_{2\alpha}c_{2\alpha})] \end{aligned} \tag{4.15}$$

where

$$\begin{cases} N_1(\cdot) &= \sup_{0<\alpha\leq 1} \alpha \wedge \chi_{[k_{1\alpha},k_{2\alpha}]}(\cdot) \\ N_2(\cdot) &= \sup_{0<\alpha\leq 1} \alpha \wedge \chi_{[\log(b_{1\alpha}c_{1\alpha}),\log(b_{2\alpha}c_{2\alpha})]}(\cdot) \end{cases} \tag{4.16}$$

By the relation between h_x and p_x we conclude

$$\begin{aligned} [h_x(N_1,N_2)]_\alpha &= [e^{p_x(N_1,N_2)}]_\alpha \\ &= [b_{1\alpha}c_{1\alpha}e^{k_{1\alpha}x}, b_{2\alpha}c_{2\alpha}e^{k_{2\alpha}x}] \end{aligned}$$

So, $A(x) = h_x(N_1,N_2)$, where N_1 and N_2 defined in terms of $[N_1]_\alpha$ and $[N_2]_\alpha$ in (4.16), or

$$A(x) = N_2e^{N_1x}. \tag{4.17}$$

Similarly, we can find B(x) and C(x).

$$B(x) = B_0e^{N_1x} \tag{4.18}$$

where $B_0(\cdot) = \sup_{0<\alpha\leq 1} \alpha \wedge \chi_{[b_{1\alpha}, b_{2\alpha}]}(\cdot)$, and

$$C(x) = C_0e^{N_1x} \tag{4.19}$$

where $C_0(\cdot) = \sup_{0<\alpha\leq 1} \alpha \wedge \chi_{[c_{1\alpha}, c_{2\alpha}]}(\cdot)$.

Also note that $B_0C_0 = N_2$.

THEOREM 4.3 The solution of the fuzzy Pexider equation (F3) is given by (4.17), (4.18) and (4.19).

They are analogous to the solution of the "classical" Pexider equation (4.3) with N_1, N_2, B_0 and C_0 being all real numbers.

Case 4. (F) is the fuzzy analog of equation (4.4) if both $*$ and $\square$ are replaced by '$\cdot$'. For this case equation (F)

$$A(xy) = B(x)C(y) \tag{F4}$$

and equations (E) read as:

$$\begin{aligned} L_A(xy) &= L_B(x)L_C(y) \\ R_A(xy) &= R_B(x)R_C(y) \end{aligned} \tag{E4}$$

Like in Case 3, we restrict ourselves for all x and y in $\mathbb{R}_1^+$. $\mathbb{R}_1^+$ forms a semi-group under multiplication and it generates the group $(\mathbb{R}^+, \cdot)$. Therefore, all solutions of equation (F4) over $\mathbb{R}_1^+$ are the restriction of the solutions over $\mathbb{R}^+$ (see Aczél [1-2]).

Like in Case 2, by introducing a transformation $x = e^u$ and $y = e^v$ for all u, v in $\mathbb{R}^+$ (note that $x > 1$ and $y > 1$, we can reduce equation (4.4) to a new equation:

$$A(e^{u+v}) = B(e^u)C(e^v). \tag{4.20}$$

Let $\tilde{A}(\xi) = A(e^\xi)$, $\tilde{B}(\xi) = B(e^\xi)$, and $\tilde{C}(\xi) = C(e^\xi)$. Then equation (4.20) becomes

$$\tilde{A}(u+v) = \tilde{B}(u)\tilde{C}(v).$$

This equation is solved in Case 3:

$$\begin{aligned}\tilde{A}(u) &= N_2 e^{N_1 u}\\ \tilde{B}(u) &= B_0 e^{N_1 u}\\ \tilde{C}(u) &= C_0 e^{N_1 u}\end{aligned}$$

Note that $B_0 C_0 = N_2$.

Therefore,

$$\begin{aligned}A(x) = \tilde{A}(\log x) &= N_2 e^{N_1 \log x} &= N_2 x^{N_1}\\ B(x) = \tilde{B}(\log x) &= B_0 e^{N_1 \log x} &= B_0 x^{N_1}\\ C(x) = \tilde{C}(\log x) &= C_0 e^{N_1 \log x} &= C_0 x^{N_1}\end{aligned} \tag{4.21}$$

THEOREM 4.4 The solution of the fuzzy Pexider equation (F4) is given by (4.21).

These results are analogous to the "classical" solutions of equation (4.4), where N_1, N_2, B_0 and C_0 are real numbers.

Remark. In this manuscript, we attempt to formulate and solve an important class of functional equations in the setting of Fuzzy Set Theory. This may generate the motivation to study such equations from this perspective given that they arise in many settings, (e.g. information theory, economics, etc.) where measurements may be imprecise.

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