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THE REGULAR OPEN-OPEN TOPOLOGY FOR FUNCTION SPACES

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ABSTRACT. The regular open-open topology, T_{roo}, is introduced, its properties for spaces of continuous functions are discussed, and T_{roo} is compared to T_{oo}, the open-open topology. It is then shown that T_{roo} on $H(X)$, the collection of all self-homeomorphisms on a topological space, (X,T), is equivalent to the topology induced on $H(X)$ by a specific quasi-uniformity on X, when X is a semi-regular space.

KEY WORDS AND PHRASES. Compact-open topology, admissible topology, open-open topology quasi-uniformity, regular open set, semi-regular space, topological group.
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1. INTRODUCTION.

A set-set topology is one which is defined as follows: Let (X,T) and (Y,T^*) be topological spaces. Let $\mathbf{U}$ and $\mathbf{V}$ be collections of subsets of X and Y, respectively. Let $F \subset Y^X$, the collection of all functions from X into Y. Define, for $U \in \mathbf{U}$ and $V \in \mathbf{V}$, $(U,V) = \{f \in F : f(U) \subset V\}$. Let $S(\mathbf{U},\mathbf{V}) = \{(U,V) : U \in \mathbf{U} \text{ and } V \in \mathbf{V}\}$. If $S(\mathbf{U},\mathbf{V})$ is a subbasis for a topology $T(\mathbf{U},\mathbf{V})$ on F then $T(\mathbf{U},\mathbf{V})$ is called a set-set topology.

Some of the most commonly discussed set-set topologies are the compact-open topology, T_{co}, which was introduced in 1945 by R. Fox [1], and the point-open topology, T_p. For T_{co}, $\mathbf{U}$ is the collection of all compact subsets of X and $\mathbf{V} = T^*$, the collection of all open subsets of Y, while for T_p, $\mathbf{U}$ is the collection of all singletons in X and $\mathbf{V} = T^*$.

In section 2 of this paper, we shall introduce and discuss the regular open-open topology for function spaces. It will be shown which of the desirable properties T_{roo} possesses. In section 3, Pervin and almost-Pervin spaces are explained.

The fact that T_{roo}, on $H(X)$, is actually equivalent to the regular-Pervin topology of quasi-uniform convergence will be discussed in section 4 along with the topic of quasi-uniform convergence. The advantage of the regular open-open topology is the set-set notation which provides us with simple notation and, hence, our proofs are more concise than those using the cumbersome notation of the quasi-uniformity.

We assume a basic knowledge of quasi-uniform spaces. An introduction to quasi-uniform spaces may be found in Fletcher and Lindgren's [2] or in Murdeshwar and Naimpally's book [3].

Throughout this paper we shall assume (X,T) and (Y,T^*) are topological spaces.

2. THE REGULAR OPEN-OPEN TOPOLOGY.

A subset, W, of X is called a regular open set provided $W = Int_X(Cl_X(W))$. If we let $\mathbf{U}$ be the collection of all regular open sets in X and $\mathbf{V} = T^*$, then $S_{roo} = S(\mathbf{U},\mathbf{V})$ is the subbasis for a topology, T_{roo}, on any $F \subset Y^X$, which is called the regular open-open topology.

A topological space, X, is called semi-regular provided that for each $U \in X$ and each $x \in U$ there exists a regular open set, V, in X, such that $x \in V \subset U$. One can easily show that if (X,T) is a semi-regular space then $T_{roo} \subset T_{oo}$, the open-open topology (Porter, [4]) which has as a subbasis the set $S_{oo} = \{(U,V) : U \in T \text{ and } V \in T^*\}$.

We now examine some of the properties of function spaces the regular open-open topology possesses. The first two theorems also hold for the open-open topology even even when X is not semi-regular. The proofs of these two theorems are straightforward and are left to the reader.

THEOREM 1. Let (X,T) be a semi-regular space and $F \subset C(X,Y)$. If (Y,T^*) is T_i for i = 0, 1, 2, then (F,T_{roo}) is T_i for i = 0, 1, 2.

A topology, T', on $F \subset Y^X$ is called an admissible (Arens [5]) topology for F provided the evaluation map, E: $(F,T') \times (X,T) \rightarrow (Y,T^*)$, defined by $E(f,x) = f(x)$, is continuous.

THEOREM 2. If $F \subset C(X,Y)$ and X is semi-regular, then T_{roo} is admissible for F.

Arens also has shown that if T' is admissible for $F \subset C(X,Y)$, then T' is finer than T_{co}. From this fact and Theorem 2, it follows, as it does for T_{oo}, that $T_{co} \subset T_{roo}$ when X is semi-regular.

THEOREM 3. The sets of the form (U,V) where both U and V are regular open sets in X form a subbasis for $(H(X),T_{roo})$.

PROOF. Let (U,V) be a subbasic open set in $(H(X),T_{roo})$. i.e., U is a regular open set and O is an open set, not necessarily regular. Let $f \in (U,O)$. Then $f(U) \subset O$, so $f \in (U,f(U)) \subset (U,O)$ and $f(U)$ is a regular open set.

Let $(G,\circ)$ be a group such that (G,T) is a topological space, then (G,T) is a topological group provided the following two maps are continuous. (1) $m : G \times G \rightarrow G$ defined by $m(g_1,g_2) = g_1 \circ g_2$ and $\Phi : G \rightarrow G$ defined by $\Phi(g) = g^{-1}$. If only the first map is continuous, then we call (G,T) a quasi-topological group (Murdeshwar and Naimpally [3]).

Note that $H(X)$ with the binary operation $\circ$, composition of functions, and identity element e, is a group. It is not difficult to show that if (X,T) is a topological space and G is a subgroup of $H(X)$ then (G,T_{oo}) is a quasi-topological group. However, (G,T_{oo}) is not always a topological group (Porter, [4]) since Φ is not always continuous although m is always continuous. But we discover the following about the regular open-open topology.

THEOREM 4. Let X be a semi-regular space and let G be a subgroup of $H(X)$. Then (G, T_{roo}) is a topological group.

PROOF. Let X be a semi-regular space and let G be a subgroup of $H(X)$. Let (U, V) be a subbasic open set in T_{roo} such that both U and V are regular open sets. Let $(f, g) \in m^{-1}((U, V))$. Then, $f \circ g(U) \subset V$ and $g(U) \subset f^{-1}(V)$. So, $(f, g) \in (g(U), V) \times (U, g(U)) \in T_{roo} \times T_{roo}$. But $(g(U), V) \times (U, g(U)) \subset m^{-1}((U, V))$. Thus, m is continuous.

Note that the inverse map $\Phi : G \to G$ is bijective and that $\Phi^{-1} = \Phi$. Thus, in order to show that Φ is continuous, it suffices to show that Φ is an open map. To this end, let (O, U) be a subbasic open set in T_{roo} where O and U are both regular open sets. Clearly, $\Phi((O, U)) = ((X \setminus U), (X \setminus O))$ since we are dealing with homeomorphisms. Note that if C, K are regular closed sets then $Int_X C$, $Int_X K$ are regular open sets. Thus, since $(X \setminus O)$, $(X \setminus U)$ are regular closed sets, $Int_X(X \setminus U), Int_X(X \setminus O)$ are regular open sets. Again, since G is a set of homeomorphisms, $(X \setminus U, X \setminus O) = (Int_X(X \setminus U), Int_X(X \setminus O))$ but this is in T_{roo}. Therefore, $\Phi(O, U)$ is an open set in T_{roo}. So, Φ is open and we are done.

3. PERVIN AND ALMOST-PERVIN SPACES.

A topological space, (X, T), is called a Pervin space (Fletcher [4]) provided that for each finite collection, $\mathcal{A}$, of open sets in X, there exists some $h \in H(X)$ such that $h \neq e$ and $h(U) \subset U$ for all $U \in \mathcal{A}$. A topological space, (X, T) is called almost-Pervin provided that for each finite collection,$\mathcal{A}$, of regular open sets, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O) \subset O$ for all $O \in \mathcal{A}$.

Topologies are rarely interesting if they are the trivial or discrete topology. We have previously shown (Porter, [4]) that $(H(X), T_{oo})$ is not discrete if and only if (X, T) is a Pervin space. The situation for T_{roo} is similar.

THEOREM 5. $(H(X), T_{roo})$ is not discrete if and only if (X, T) is almost-Pervin.

PROOF. First, assume that (X, T) is an almost-Pervin space. Let W be a basic open set in T_{roo} which contains e; i.e. $W = \bigcap_{i=1}^{n}(O_i, U_i)$ where $O_i \subset U_i$ for each $i = 1, 2, 3, ..., n$ and O_i and U_i are regular open sets in X. $\{O_i : i = 1, 2, 3, ..., n\}$ is a finite collection of regular open sets in X, and X is an almost-Pervin space, hence, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O_i) \subset O_i \subset U_i$. So, $h \in W$ and $h \neq e$. Therefore, $(H(X), T_{roo})$ is not a discrete space.

Now assume that $(H(X), T_{roo})$ is not discrete. Let V be a finite collection of regular open sets in X. Let $O = \bigcap_{U \in V} (U, U)$. Then, O is a basic open set in $(H(X), T_{roo})$ which is not a discrete space. Hence, there exists $h \in O$ with $h \neq e$. So, (X, T) is almost-Pervin.

The above proof, along with the few needed definitions involving T_{roo}, is an example of the simplification that the definition of T_{roo} offers over the quasi-uniform definition and notation.

4. QUASI-UNIFORM CONVERGENCE.

Recall that if Q is a quasi-uniformity on X, then the topology, T_Q, on X, which has as its neighborhood base at x, $B_x = \{U[x] : U \in Q\}$, is called the topology induced by Q. The ordered triple (X, Q, T_Q) is called a quasi-uniform space. A topological space, (X, T) is quasi-uniformizable provided there exists a quasi-uniformity, Q, such that $T_Q = T$. In 1962, Pervin [7] proved that every topological space is quasi-uniformizable by giving the following construction.

Let (X, T) be a topological space. For each $O \in T$, define the set $S_O = (O \times O) \cup ((X \setminus O) \times X)$. Let $S = \{S_O : O \in T\}$. Then S is a subbasis for a quasi-uniformity, P, for X, called the Pervin quasi-uniformity and, as is easily shown, $T_P = T$.

If we use the same basic structure as above but change the subbasis to $S = \{S_O : O$ is a regular open set $\}$ then the quasi-uniformity induced will be called the regular-Pervin quasi-uniformity, RP.

If (X, Q) is a quasi-uniform space then Q induces a topology on $H(X)$ called the topology of quasi-uniform convergence w.r.t. Q, as follows: For each set $U \in Q$, let us define $W(U) = \{(f, g) \in H(X) \times H(X) : (f(x), g(x)) \in U \text{ for all } x \in X\}$. Then, $B(Q) = \{W(U) : U \in Q\}$ is a basis for Q^*, the quasi-uniformity of quasi-uniform convergence w.r.t. Q (Naimpally [8]). Let T_{Q^*} denote the topology on $H(X)$ induced by Q^*. T_{Q^*} is called the topology of quasi-uniform convergence w.r.t. Q^*. If P is the Pervin quasi-uniformity on X, T_{P^*} is the Pervin topology of quasi-uniform convergence and if RP is the regular-Pervin quasi-uniformity on X, then T_{RP} is called the regular-Pervin topology of quasi-uniform convergence, T_{RP^*}.

It has been shown that the open-open topology is equivalent to the Pervin topology of quasi-uniform convergence (Porter, [4]). It is also true that the regular open-open topology is equivalent to the regular-Pervin topology of quasi-uniform convergence. The method of two proofs are exactly the same and we leave this one for the reader.

THEOREM 6. Let (X, T) be a topological space and let G be a subgroup of $H(X)$. Then, $T_{roo} = T_{RP^*}$ on G.

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