Internat. J. Math. & Math. Sci.
VOL. 19 NO. 2 (1996) 411-414



# ON THE BASIS OF THE DIRECT PRODUCT OF PATHS AND WHEELS

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(Received February 24, 1994 and in final form March 22, 1995)

**ABSTRACT.** The basis number, $b(G)$, of a graph $G$ is defined to be the least integer $k$ such that $G$ has a $k$-fold basis for its cycle space In this paper we determine the basis number of the direct product of paths and wheels. It is proved that $P_2 \wedge W_n$is planar, and $b(P_m \wedge W_n) = 3$, for all $m \geq 3$ and $n \geq 4$.

**KEY WORDS AND PHRASES.** Basis number, cycle space, paths, and wheels.
**1991 AMS SUBJECT CLASSIFICATION CODE.** 05C99.

## 1. INTRODUCTION.

Throughout this paper, we consider only finite, undirected, simple graphs. Our notations and terminology will be standard except as indicated For undefined terms, see [3].

Let $G$ be a graph, and let $e_1, ..., e_q$ be an ordering of its edges. Then any subset $H$ of edges in $G$ corresponds to a (0,1)-vector $(a_1, ..., a_q)$ in the usual way, with $a_i = 1$ if $e_i \in H$ and $a_i = 0$ if $e_i \notin H$. These vectors form a $q$-dimensional vector space, denoted by $(\mathbf{Z}_2)^q$ over the field of two elements $\mathbf{Z}_2$.

The vectors in $(\mathbf{Z}_2)^q$ which corresponds to the cycles in $G$ generate a subspace called the cycle space of $G$, denoted by $C(G)$. We shall say, however, that the cycles themselves, rather than the vectors corresponding to the cycles, generate $C(G)$. It is well known that (see [3], p. 39)

$$\dim C(G) = \gamma(G) = q - p + k\,, \tag{1.1}$$

where $q$ is the number of edges, $p$ is the number of vertices, $k$ is the number of connected components, and $\gamma(G)$ is the cyclomatic number of $G$. A basis for $C(G)$ is called $k$-fold, if each edge of $G$ occurs in at most $k$ of the cycles in the basis. The basis number of $G$ (denoted by $b(G)$) is the smallest integer $k$ such that $C(G)$ has a $k$-fold basis. The fold of an edge $e$ in a basis $B$ for $C(G)$ is defined to be the number of cycles in $B$ containing $e$, and denoted by $f_B(e)$

The direct product [5] (or conjunction [3]) of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph denoted by $G_1 \wedge G_2$ with vertex set $V_1 \times V_2$, in which $(v_1, u_1)$ is joined to $(v_2, u_2)$ whenever $v_1v_2 \in E_1$ and $u_1u_2 \in E_2$. It is clear that $d_{G_1 \wedge G_2}(v_i, u_j) = d_{G_1}(v_i)\, d_{G_2}(u_j)$, where $d_H(v)$ is the degree of vertex $v$ in the graph $H$. Thus the number of edges in $G_1 \wedge G_2$ is $2|E_1|\,|E_2|$.

Let $P_m$ denote a path with $m$-vertices, and let $W_n$ denote a wheel with $n$ vertices.

The first important result about the basis number was given by MacLane in 1937 (see [4]), when he proved that a graph $G$ is planar if and only if $b(G) \leq 2$ In 1981, Schmeichel [6] proved that $b(k_n) = 3$ for $n \geq 5$, and for $m, n \geq 5$, $b(k_{m,n}) = 4$. In 1982 Banks and Schmeichel [2] proved that $b(Q_n) = 4$, for $n \geq 7$, where $Q_n$ is the $n$-cube In 1989, Ali [1] proved that $b(C_m \wedge P_n) \leq 2$, and for all $m, n \geq 3$, $b(C_m \wedge C_n) = 3$, where $C_m$ is a cycle with $m$ edges. Next we restate Theorem 1 of [2]:

**THEOREM 1.2.** For any connected graph $G$,

$$\sum_{v \in V(G)} \left\lceil \frac{b(G)\, d(v)}{2} \right\rceil \geq (\text{girth}\, G) \dim(C(G)) ,$$

where $d(v)$ denotes the degree of a vertex $v$.

The purpose of this paper is to determine the basis number of $P_m \wedge W_n$. In fact it is proved that $b(P_m \wedge W_n) = 3$, for all $m \geq 3$. It is also proved that $P_2 \wedge W_n$ is planar.

## 2. MAIN RESULTS

In what follows let $\{1, 2, ..., m\}$ be the vertices of $P_m$ and let $\{1, 2, ..., n\}$ be the vertices of $W_n$, with the vertex 1 of $W_n$ of degree $n-1$, and all other vertices of degree 3.

**LEMMA 2.1.** If $G = P_2 \wedge W_n$, then $G$ is connected.

**PROOF.** This is clear since $W_n$ has an odd cycle, namely a 3-cycle (see [3], p. 25). QED

**COROLLARY 2.2.** If $G = P_2 \wedge W_n$, then $\dim C(G) = 2n - 3$.

**PROOF.** Just apply (1.1) and Lemma 2.1. QED

**THEOREM 2.3.** If $G = P_2 \wedge W_n$, then $b(G) = 2$ and hence $G$ is planar.

**PROOF.** Consider the following sets of 4-cycles in $G$:

$$\begin{aligned}
E_1 &= \{(1,1)(2,i+1)(1,i+2)(2,i+3)(1,1) : i = 1, 2, 3, ..., n-3\} \\
E_2 &= \{(2,1)(1,i+1)(2,i+2)(1,i+3)(2,1) : i = 1, 2, 3, ..., n-3\} \\
E_3 &= \{(1,1)(2,n-1)(1,n)(2,2)(1,1)\} \\
E_4 &= \{(1,1)(2,n)(1,2)(2,3)(1,1)\} \\
E_5 &= \{(2,1)(1,n-1)(2,n)(1,2)(2,1)\}
\end{aligned}$$

Let $B = \bigcup_{j=1}^{5} E_j$, then $|B| = 2n - 3 = \dim C(G)$. Next we show that $B$ is an independent set of cycles in $C(G)$.

It is clear that $E_1$ consists of $n-3$ independent cycles, in fact if $C$ is a cycle in $E_1$, then $C$ contains the edge $(1, i+2)(2, i+3)$, which is not an edge of any other cycle in $E_1$, hence $C$ cannot be written as a linear combination of the rest of the cycles in $E_1$. A similar argument shows that $E_2$ consists of $n-3$ independent cycles, and clearly each of $E_3$, $E_4$ and $E_5$ consists of exactly one cycle; thus the cycles in each $E_j (j = 1, ..., 5)$ are independent.

Each cycle of $E_2$ contains the dege $(2,1)(2,i+1)$ which is not in $E_1$, hence $E_1 \cup E_2$ is an independent set of cycles. The cycle $E_3$ contains the dege $(1,n)(2,2)$, which is not in $E_1 \cup E_2$, hence $E_1 \cup E_2 \cup E_3$ is an independent set of cycles. The cycle $E_4$ contains the dege $(2,n)(1,2)$, which is not in $E_1 \cup E_2 \cup E_3$, hence $E_1 \cup E_2 \cup E_3 \cup E_4$ is an independent set of cycles. Finally it is clear that the cycle $E_5$ cannot be written as a linear combination of the cycles in $\bigcup_{j=1}^{4} E_j$. Hence $B = \bigcup_{j=1}^{5} E_j$ is an independent set of cycles in $G$, and, since $|B| = \dim C(G)$, $B$ is a basis for $C(G)$.

Next, we show that $B$ is a 2-fold basis of $C(G)$. Notice that if $e$ is an edge of $E = E_1 \cup E_3 \cup E_4$ of the form $\{(1,1)(2,i+3) : i = 1, ..., n-3\}$ then $f_E(e) = 2$, and if $e_¿$ is an edge of $E$, which is not of the given form, then $f_E(e_¿) = 1$. Moreover, if $e$ is an edge of $E_¿ = E_2 \cup E_5$ of the form $\{(2,1)(1,i+1) : i = 1, ..., n-2\}$, then $f_{E_¿}(e) \leq 2$, and if $e_¿$ is an edge of $E_¿$, which is not of the given form then $f_{E_¿}(e_¿) = 1$, now clearly the edges of the above two forms are disjoint, hence $f_B(e) \leq 2$ for any $e \in G$; thus $b(G) \leq 2$. Now $b(G) > 1$ because each cycle must have at least 3 edges, which is more than the number of edges in $G$. Thus $b(G) = 2$, and hence $G$ is planar.

**REMARK 2.4.** If $G = P_m \wedge W_n$, then for all $m \geq 3$, $n \geq 4$, we have:

$$\dim C(G) = 3m - 4(m+n) + 5.$$

**THEOREM 2.5.** If $G = P_m \wedge W_n$, then for all $m \geq 3$, $b(G) \geq 3$, and hence $G$ is nonplanar.

**PROOF.** If $b(G) \leq 2$, then by Theorem 1 2, we have

$$\sum_{v\in V(G)} d(v) \geq \sum_{v\in V(G)} \left\lceil \frac{b(G)\,d(v)}{2} \right\rceil \geq (\text{girth}\,G)\dim(C(G)) ,$$

where, $d(v)$ is the degree of the vertex $v$, hence

$$2|E(G)| \geq 4[|E(G)| - nm + 1] , \quad (\text{girth}\,G = 4)$$

i e,

$$0 \geq 2|E(G)| - 4nm + 4 ,$$

Now if we evaluate and divide the inequality by four we get:

$$0 \geq mn - 2m - 2n + 3 = (m-2)(n-2) - 1 ,$$

and since $n \geq 4$, we have

$$1 \geq (m-1)(n-2) \geq 2(m-2) .$$

Hence $m \leq 2.5 < 3$, thus we conclude that if $m \geq 3$, then $b(G) \geq 3$, hence $G$ is non planar. QED

**THEOREM 2.6.** If $G = P_m \wedge W_n$, then for all $m \geq 3$, $b(G) = 3$.

**PROOF.** The plan here is to give an independent set of cycles $B$ in $C(G)$, such that $|B| =$ dim $C(G)$, and to show that $B$ is a 3-fold basis for $C(G)$ To this end consider the following sets of 4-cycles in $C(G)$ for $k = 1, ..., m-1$, let

$$\begin{aligned} E_k &= \{(k,1)(k+1,i+1)(k,i+2)(k+1,i+3)(k,1) : i = 1,...,n-3\} , \\ E_{k¿} &= \{(k+1,1)(k,i+1)(k+1,i+2)(k,i+3)(k+1,1) : i = 1,...,n-3\} , \\ A_k &= \{(k,1)(k+1,n-1)(k,n)(k+1,2)(k,1)\} , \\ A_{k¿} &= \{(k,1)(k+1,n)(k,2)(k+1,3)(k,1)\} , \text{ and} \\ A_{k¿¿} &= \{(k+1,1)(k,n-1)(k+1,n)(k,2)(k+1,1)\} . \end{aligned}$$

And for $k = 1, ..., m-2$, let

$$D_k = \{(k+1,1)(k+2,i+1)(k+1,i+2)(k,i+1)(k+1,1) : i = 1,...,n-2\} ,$$

and

$$D_{k¿} = \{(k+1,1)(k+2,n)(k+1,n-1)(k,n)(k+1,1)\} .$$

Let

$$F_k = E_k \cup E_{k¿} \cup A_k \cup A_{k¿} \cup A_{k¿¿}\,(k = 1, ..., m-1) .$$

$F = \bigcup_{k=1}^{m-1} F_k$, $H_k = D_k \cup D_{k¿}\,(k = 1, ..., m-2)$, $H = \bigcup_{k=1}^{m-2} H_k$, and let $B = F \cup H$. Then

$$|B| = |F| + |G| = (m-1)(2n-3) + (m-2)(n-1) = 3m - 4n - 4m + 5 = \dim C(G) .$$

For each $k = 1, ..., m-1$, notice that $F_k$ is just a copy of the cycle basis of $P_2 \wedge W_n$ (with $\{k, k+1\}$ as vertices of $P_2$), hence the cycles in each $F_k$ are independent, and since $F_\ell$ is just a copy of the cycle basis of $b_2 \wedge W_n$ (with $\{\ell, \ell+1\}$ as vertices of $P_2$), then it follows that If $k \neq \ell$ in $\{1, ..., m-1\}$, then the cycles in $F_k$ are edge disjoint from the cycles in $F_\ell$, hence $F$ is an independent set of cycles.

Consider $H_k$, for each $k = 1, ..., m-2$, it is clear that the cycles in $H_k$ are edge disjoint, hence $H_k$ is an independent set of cycles. Moreover, if $k \neq \ell$ in $\{1, ..., m-2\}$, then the cycles in $H_k$ are edge disjoint from the cycles in $H_\ell$, hence $H = \bigcup_{k=1}^{m-2} H_k$ is an independent set of cycles Now if $C$ is any 4-cycle in $H$, then $C$ belongs to $H_k$ for some $k$, and clearly $C$ consists of two edges in $F_k$ and two edges in

$F_{k+1}$, hence $C$ cannot be written as a linear combination of cycles in $F$, hence $B = F \cup H$ is an independent set of cycles with $|B| = \dim C(G)$ Thus $B$ is a basis for $C(G)$.

It remains to show that $B$ is a 3-fold basis for $C(G)$, but this is clear since if $e$ is an edge of $G$, then it follows from the result when $m = 2$ that $f_F(e) \leq 2$, and $f_H(e) \leq 1$, hence $f_B(e) \leq 3$ (i e., $b(G) \leq 3$). Now combining this with Theorem 2 5, we see that $B$ is a 3-fold basis for $C(G)$ QED

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