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ON EXTENSION OF PAIRWISE Θ-CONTINUOUS MAPS

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ABSTRACT. The aim of the paper is to find suitable conditions so as to ultimately establish the existence and uniqueness of the extension of a pairwise θ-continuous map onto an arbitrary extension-space of a bitopological space.

KEY WORDS AND PHRASES. Bitopological extension, pairwise θ-continuity, pairwise θ-proper, pairwise *-free.
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1. INTRODUCTION.

The problem of extending continuous maps on a topological space X to a given extension X* of X has been dealt with extensively by many mathematicians. For example it is known (see [2]) that a continuous map $f : X \to Y$ with Y a compact Hausdorff space can be extended by a continuous map onto an extension X* of X iff for each pair of closed disjoint subsets A and B of Y, the closures of $f^{-1}(A)$ and $f^{-1}(B)$ in X* are disjoint. One of the different interesting generalizations of this result arises when continuity and compactness are replaced by θ-continuity and quasi-H-closedness (QHC) respectively under a suitably changed condition (see Rudolf [5]).

Bitopological versions of QHC spaces and θ-continuous functions have been introduced by Mukherjee [4], and Bose and Sinha [1] respectively. It is our purpose here to further generalize the above extension theorem by Rudolf [5]. For this we suitably modify and redefine the appliances used by Rudolf to ultimately establish the existence and uniqueness of the extension of a pairwise θ-continuous map onto an arbitrary extension of a bitopological space under certain conditions.

By spaces X and Y we shall mean bitopological spaces (see Kelly [3])(X,Q_1,Q_2) and (Y,P_1,P_2) respectively. For any $A \subset X$, Q_i-intA and Q_i-clA will respectively stand for the interior and closure of A in (X,Q_i), where i=1,2. A set A is called an ij-regularly open set (Singal and Arya [6]) if $A=Q_i\text{-int } Q_j\text{-cl}A$, and complement of such a set is called ij-regularly closed where (and also in future discussion) i,j=1,2 and $i \neq j$. A space X is called pairwise Hausdorff (Kelly [3]) if for $x,y \in X$ with $x \neq y$, there exist $U \in Q_1$ and $V \in Q_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

DEFINITION 1. (see Bose and Sinha [1]) A function (or map) $f : (X,Q_1,Q_2) \rightarrow (Y,P_1,P_2)$ is called ij-θ-continuous if for each $x \in X$ and each P_1-open neighbourhood (henceforth nbd., for short) U of f(x), there is a Q_i-open nbd V of x with $f(Q_j\text{-cl}V) \subset P_j\text{-cl}U$. f is called pairwise θ-continuous if it is 12- as well as 21-θ-continuous.

DEFINITION 2. (see Singal and Arya [6]) A subset A of a space (X,Q_1,Q_2) is said to be pairwise dense if every non-empty subset of X which is the intersection of a Q_1-open set and a Q_2-open set, has non-void intersection with A.

2. MAIN THEOREM AND ASSOCIATED RESULTS.

DEFINITION 3. A space (X^*,Q_1^*,Q_2^*) is said to be an extension of a space (X,Q_1,Q_2) if $Q_1^*/X=Q_1, Q_2^*/X=Q_2$ and X is pairwise dense in X^*.

For an extension (X^*,Q_1^*,Q_2^*) of (X,Q_1,Q_2), a map $f : (X,Q_1,Q_2) \rightarrow (Y,P_1,P_2)$ and a point x of X^* (of Y) let N_x^{*i} (resp. N_x^i) the family of all Q_i^*-open (P_i-open) nbds of x in X^* (resp. in Y), for i=1,2. For $x \in X^*$, $N^i(f,N_x^{*i})$ shall denote the P_i-open filter on Y generated by the family $\{f(U_x \cap X) : U_x \in N_x^{*i}\}$ (i=1,2).

DEFINITION 4. A map $f : X \rightarrow Y$ is ij-θ-proper if for each $x \in X^*-X$, $N^j(f,N_x^{*j})$ has non-void P_i-adherence, where (X^*,Q_1^*,Q_2^*) is an extension of (X,Q_1,Q_2). The map f is called pairwise θ-proper if for each $x \in X^*-X$, $[\cap\{P_1\text{-cl}U : U \in N^2(f,N_x^{*2})\}] \cap [\cap\{P_2\text{-cl}U : U \in N^1(f,N_x^{*1})\}] \neq \emptyset$.

THEOREM 1. Let (X^*,Q_1^*,Q_2^*) be an extension of a space (X,Q_1,Q_2) and $f^* : (X^*,Q_1^*,Q_2^*) \rightarrow (Y,P_1,P_2)$ be an ij-θ-continuous extension of an ij-θ-continuous map $f : X \rightarrow Y$ on X^*. Then $f^*(x) \in \cap\{P_i\text{-cl}U : U \in N^j(f,N_x^{*j})\}$, for each $x \in X^*-X$.

PROOF. Let $y = f^*(x) \notin P_i\text{-cl}U$, for some $U \in N^j(f,N_x^{*j})$. We consider the P_i-open nbd $U_y=Y-P_i\text{-cl}U$. By ij-θ-continuity of f^*, there exists a Q_i^*-open nbd U_x of x such that $f^*(Q_j^*\text{-cl}U_x) \subset P_j\text{-cl}U_y \subset Y-U$ (since $U \in P_j$), and hence $f^*(U_x \cap X)=f(U_x \cap X) \subset Y-U$. Since $U \in N^j(f,N_x^{*j})$, U contains a set of the form $f(U_x' \cap X)$, for some $U_x' \in N_x^{*j}$. Now, $f(U_x \cap X) \cap f(U_x' \cap X) \subset (Y-U) \cap U = \emptyset$ implies that $f(U_x \cap U_x' \cap X)=\emptyset$, a contradiction because X is pairwise dense in X^*.

LEMMA 1. Let (X^*,Q_1^*,Q_2^*) be an extension of a space (X,Q_1,Q_2) and let $f : (X,Q_1,Q_2) \rightarrow (Y,P_1,P_2)$ be an arbitrary map. Then for each $x \in X^*$ and each $y \in Y$, $y \in \cap\{P_i\text{-cl}U : U \in N^j(f,N_x^{*j})\}$ iff $x \in \cap\{Q_j^*\text{-cl}\ f^{-1}(P_j\text{-cl}U_y) : U_y \in N_y^i\}$.

PROOF. Let $y \in \cap\{P_i\text{-cl}U : U \in N^j(f,N_x^{*j})\}$. Then for each P_i-open nbd U_y of y and each $U \in N^j(f,N_x^{*j})$ we have $U_y \cap U \neq \emptyset$, i.e., $P_j\text{-cl}U_y \cap U \neq \emptyset$ which gives $P_j\text{-cl}U_y \cap f(U_x \cap X) \neq \emptyset$ for each $U_y \in N_y^i$ and each $U_x \in N_x^{*j}$. For otherwise, $P_j\text{-cl}U_y \cap f(U_x \cap X) \neq \emptyset$ for some $U_y \in N_y^i$ and some $U_x \in N_x^{*j}$. Then $V=Y-P_j\text{-cl}U_y \supset f(U_x \cap X)$ and hence $V \in N^j(f,N_x^{*j})$ for which $V \cap U_y=\emptyset$, a contradiction. Now, $f^{-1}(P_j\text{-cl}U_y) \cap (U_x \cap X) \neq \emptyset$ and hence $f^{-1}(P_j\text{-cl}U_y) \cap U_x \neq \emptyset$, for each $U_x \in N_x^{*j}$ and each $U_y \in N_y^i$. Thus $x \in \cap\{Q_j^*\text{-cl}\ f^{-1}(P_j\text{-cl}U_y) : U_y \in N_y^i\}$. Reversing the argument we get the reverse implication.

DEFINITION 5. Let (X^*,Q_1^*,Q_2^*) be an extension of a space (X,Q_1,Q_2). A map $f : X \rightarrow Y$ is called ij-*-free if for each $x \in X^*-X$, each $y \in Y$ and each ij-regularly closed set A in Y with $y \notin A$, there exists a P_i-open nbd U_y of y with $x \notin Q_j^*\text{-cl}\ f^{-1}(P_j\text{-cl}U_y) \cap Q_i^*\text{-cl}\ f^{-1}(A)$. The map f is called pairwise *-free if it is 12- as well as 21-*-free. The map f is called ij-*-proper if f is ij-θ-proper and ij-*-free; f will be called pairwise *-proper if it is pairwise θ-proper and pairwise *-free.

THEOREM 2. Let (X^*,Q_1^*,Q_2^*) be an extension of (X,Q_1,Q_2). Then for each pairwise

*-proper map f from X to a pairwise Hausdorff space Y, the set $[\cap\{P_1\text{-cl}U : U\in N^2(f,N_x{*}^2)\}]\cap[\cap\{P_2\text{-cl}V : V\in N^1(f,N_x{*}^1)\}]$ is a singleton for each $x\in X^*-X$.

PROOF. Let $x\in X^*-X$. Since f is pairwise θ-proper, suppose that $y\in[\cap\{P_1\text{-cl}U : U\in N^2(f,N_x{*}^2)\}]\cap[\cap\{P_2\text{-cl}V : V\in N^1(f,N_x{*}^1)\}]$. By Lemma 1, $x\in[\cap\{Q_2{*}\text{-cl } f^{-1}(P_2\text{-cl}U_y) : U_y\in N_y^1\}]\cap[\cap\{Q_1{*}\text{-cl } f^{-1}(P_1\text{-cl}U_y) : U_y\in N_y^2\}]$. We consider a point $y'\in Y$ such that $y'\neq y$. Since Y is pairwise Hausdorff, there exists a P_2-open nbd $V_{y'}$ of y' such that $y\notin P_j\text{-cl}V_{y'}$. Now, f being 12-*-free there exists a P_j-open nbd U_y of y such that $x\notin Q_2{*}\text{-cl } f^{-1}(P_2\text{-cl}U_y)\cap Q_1{*}\text{-cl } f^{-1}(P_1\text{-cl}V_{y'})$. Since $x\in\cap\{Q_2{*}\text{-cl } f^{-1}(P_2\text{-cl}U_y) : U_y\in N_y^1\}$, $x\notin Q_1{*}\text{-cl } f^{-1}(P_1\text{-cl}V_{y'})$ and thus $y'\notin\cap\{P_2\text{-cl}V : V\in N^1(f,N_x{*}^1)\}$ (by Lemma 1).

LEMMA 2. For an ij-θ-continuous map $f:X\rightarrow Y$ and $U\in P_j$, $f(Q_i\text{-cl } f^{-1}(U))\subset P_i\text{-cl}U$.

We are now in a position to prove the main theorem of this paper as follows :

THEOREM 3. Let $(X^*,Q_1{*},Q_2{*})$ be an extension of (X,Q_1,Q_2). Then each pairwise θ-continuous, pairwise *-proper function $f : X\rightarrow Y$ possesses a pairwise θ-continuous extension $f^*:X^*\rightarrow Y$. The extension is unique if Y is pairwise Hausdorff.

PROOF. For $x\in X$ we take $f^*(x)=f(x)$, and for each $x\in X^*-X$ we choose and fix a point of $[\cap\{P_1\text{-cl}U:U\in N^2(f,N_x{*}^2)\}]\cap[\cap\{P_2\text{-cl}V:V\in N^1(f,N_x{*}^1)\}]$ and define it to be $f^*(x)$; the latter choice is possible since f is pairwise θ-proper.

We first prove that for each P_j-open set U of Y,

$$f^*((X^*-X)\cap Q_i{*}\text{-cl } f^{-1}(P_i\text{-cl}U))\subset P_i\text{-cl}U \tag{2.1}$$

If not, then for some $x\in X^*-X$, there exist a P_j-open set U in Y with $x\in(X^*-X)\cap Q_1{*}\text{-cl}f^{-1}(P_i\text{-cl}U)$ but $f^*(x)$ (= y, say) $\notin P_i\text{-cl}U$. Since f is ij-*-free, there exists a P_i-open nbd V of y such that $x\notin Q_j{*}\text{-cl } f^{-1}(P_j\text{-cl}V)\cap Q_i{*}\text{-cl}f^{-1}(P_i\text{-cl}U)$. Now since $y = f^*(x)\in\cap\{P_j\text{-cl}U : U\in N^j(f,N_x{*}^j)\}$, Lemma 1 gives $x\in Q_j{*}\text{-cl } f^{-1}(P_j\text{-cl}V)$ which implies $x\notin Q_i{*}\text{-cl } f^{-1}(P_i\text{-cl}U)$, contradicting the choice of x. This proves (2.1).

Now to prove the pairwise θ-continuity of f*, we first consider $x\in X$. Suppose $f^*(x)$ (= f(x)) = y and let U_y be an arbitrary P_j-open nbd of y. By ji-θ-continuity of f there exists a Q_j-open nbd U_x of x such that $f(Q_i\text{-cl}U_x)\subset P_i\text{-cl}U_y$, i.e.,

$$Q_i\text{-cl}U_x\subset f^{-1}(P_i\text{-cl}U_y) \tag{2.2}$$

Define $U_x{*} =\cup\{U\in Q_j{*} : U\cap X = U_x\}$ which is a $Q_j{*}$-open nbd of x. Then using the pairwise denseness of X we have

$$Q_i{*}\text{-cl}U_x{*} = Q_i\text{-cl}(U_x{*}\cap X) = Q_i{*}\text{-cl}U_x\subset Q_i{*}\text{-cl}f^{-1}(P_i\text{-cl}U_y) \tag{2.3}$$

Again, $f^*(Q_i{*}\text{-cl}U_x{*}) = f^*((X^*-X)\cap Q_i{*}\text{-cl}U_x{*})\cup f^*(X\cap Q_i{*}\text{-cl}U_x{*}) = f^*((X^*-X)\cap Q_i{*}\text{-cl}U_x{*})\cup f(Q_i\text{-cl}U_x)\subset P_i\text{-cl}U_y$ (by virtue of (2.1), (2,2) and (2.3)).

Next we consider $x\in X^*-X$, and let U_y be a ji-regularly open set containing $f^*(x)$ (= y, say). Hence $y\notin Y-U_y$, where $Y-U_y$ is a ji-regularly closed set. Since $y\in\cap\{P_j\text{-cl}V : V\in N^i(f,N_x{*}^i)\}$, Lemma 1 gives $x\in\cap\{Q_i{*}\text{-cl } f^{-1}(P_i\text{-cl}U) : U\in N_y{}^j\}$. Then by ji-*-freeness of f, $x\notin Q_j{*}\text{-cl } f^{-1}(Y-U_y)$. Then $U_x=X^*-Q_j{*}\text{-cl}f^{-1}(Y-U_y)$ is a $Q_j{*}$-open nbd of x. But $Y=P_i\text{-cl}U_y\cup(Y-U_y)$. Thus $X=f^{-1}(P_i\text{-cl}U_y)\cup f^{-1}(Y-U_y)$. Then $X^* = Q_i{*}\text{-cl } f^{-1}(P_i\text{-cl}U_y)\cup Q_j{*}\text{-cl } f^{-1}(Y-U_y)$. If not, let $x_o\in X^*-X$ but $x_o\notin$ R.H.S. Let V be any $Q_i{*}$-open nbd of x_o. Since $x_o\in V\cap[X^*-Q_j{*}\text{-cl } f^{-1}(Y-U_y)]$ (note that $x_o\notin$ R.H.S.) and X is pairwise dense in X*, $V\cap[X^*-Q_j{*}\text{-cl } f^{-1}(Y-U_y)]\cap X\neq\emptyset$ which gives $V\cap f^{-1}(P_i\text{-cl}U_y)\neq\emptyset$. Hence $x_o\in Q_i{*}\text{-cl } f^{-1}(P_i\text{-cl}U_y)$, a contradiction. Now since $U_x\cap Q_j{*}\text{-cl } f^{-1}(Y-U_y) = \emptyset$, $U_x\subset Q_i{*}\text{cl } f^{-1}(P_i\text{-cl}U_y)$, i.e.,

$$Q_i{*}\text{-cl}U_x\subset Q_i{*}\text{-cl } f^{-1}(P_i\text{-cl}U_y) \tag{2.4}$$

Again, $U_x \cap X = X - Q_j^*\text{-cl } f^{-1}(Y-U_y) \subset X - f^{-1}(Y-U_y) = f^{-1}(U_y)$. Thus

$$Q_i\text{-cl}(U_x \cap X) \subset Q_i\text{-cl } f^{-1}(U_y) \tag{2.5}$$

Now, $f^*(Q_i^*\text{-cl}U_x) = f^*((X^*-X) \cap Q_i^*\text{-cl}U_x) \cup f^*(X \cap Q_i^*\text{-cl}U_x) = f^*((X^*-X) \cap Q_i^*\text{-cl}U_x) \cup f(Q_i\text{-cl}(U_x \cap X))$ (as X is pairwise dense in X^*) $\subset P_i\text{-cl}U_y$ (by (2.1), (2.4), (2.5) and Lemma 2, noting that f is ij-θ-continuous). If U_y' be any P_j-open nbd od $f^*(x)$, then $U_y = P_j\text{-int } P_i\text{-cl}U_y'$ is a ji-regularly open set containing y. Thus by what we have obtained so far, there is a Q_j^*-open nbd U_x of x with $f^*(Q_i^*\text{-cl}U_x) \subset P_i\text{-cl}U_y = P_i\text{-cl}U_y'$. Hence f^* is ji-θ-continuous at each point of X^*-X. Thus we infer that $f^* : X^* \to Y$ is ji-θ-continuous. The ij-θ-continuity of f^* can similarly be dealt with. The uniqueness of the extension f^* of f follows from Theorems 1 and 2.

REMARK 1. Putting $Q_1 = Q_2$ and $P_1 = P_2$ in the above theorem, we get Theorem 3.1 of Rudolf [5]. If X and Y are topological spaces, then the θ-properness of a map $f : X \to Y$ is ensured by the QHC property of Y (see [5] for details). In bitopological setting, the definition of pairwise QHC property of Y (cf. [4]) implies that $\cap\{P_i\text{-cl}U : U \in N^j(f, N_x^{*j})\} \neq \emptyset$, for i,j=1,2 (i≠j). But it is not necessary that $[\cap\{P_1\text{-cl}U : U \in N^2(f, N_x^{*2})\}] \cap [\cap\{P_2\text{-cl}U : U \in N^1(f, N_x^{*1})\}] \neq \emptyset$. Hence in our case, the role of pairwise θ-properness of f in Theorem 3 cannot be replaced, in general, by pairwise quasi H-closedness of (Y, P_1, P_2). Nevertheless, taking $Q_1=Q_2$ and $P_1=P_2$ we see that every *-free θ-continuous map from a topological space X to any H-closed topological space Y can be extended uniquely over any extension space X^* of X.

EXAMPLE 1. Let $X^*=Y=R$ (=the set of reals), $Q_1^*=P_1$=the usual topology on R and $Q_2^*=P_2$=the lower limit topology on R. If X = the set of rationals and $Q_i=Q_i^*/X$, for i=1 and 2, then clearly (X^*, Q_1^*, Q_2^*) is an extension of (X, Q_1, Q_2) and also, the map $f : (X, Q_1, Q_2) \to (Y, P_1, P_2)$, defined by $f(x)=x$ ($x \in X$), is pairwise θ-continuous and pairwise *-proper. Since (Y, P_1, P_2) is pairwise Hausdorff, f has a unique pairwise θ-continuous extension over X^*, by Theorem 3.

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