HOMOMORPHISM AND SIGMA POLYNOMIALS

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ABSTRACT. By establishing a connection between the sigma polynomial
and the homomorphism polynomial, many of the proofs for computing the
sigma polynmial are simplified, the homomorphism polynomial can be
identified for several new classes of graphs, and progress can be made
on identifying homomorphism polynomials.

1.   INTRODUCTION.

Throughout this paper, a graph will refer to a simple graph, and
for a given graph G, let G_L be the graph obtained by inserting an edge
between two non-adjacent vertices, let G_R be the graph obtained by
obtained by contracting the edge inserted in G_L. Thus $G = G_L + G_R$ is
the standard decomposition of G used for obtaining the chromatic
polynomial of G. Further, let G_P be the insertion of a pseudo-edge
between two non-adjacent vertices (pictorially, a pseudo-edge is a
dashed edge between the two vertices: we will interpet this to mean
that the vertices are still non-adjacent, but we have considered this
pair of vertices already), in which case G_R denotes the graph obtained
by contracting the pseudo-edge. A graph is pseudo-complete if every
pair of vertices is either adjacent or pseudo-adjacent. We denote a
pseudo-complete graph by $K_{p(m)}$, where p is the number of vertices and
m is the number of pseudo-edges.

A homomorphism from G onto a graph G' is a mapping ϕ from V(G)
onto V(G') such that if vertices u and v are adjacent in G, then $\phi(u)$
and $\phi(v)$ are adjacent in G'. Bari, [1], showed that the set of all
homomorphic images of G, h(G), can be found by using the recursion

$$h(G) = h(G_P) \cup h(G_R), \tag{1.1}$$

which terminates when each graph is pseudo-complete. We find the set
of homomorphic images of G by ignoring the pseudo-edges in these
graphs.

For a $(p, \binom{p}{2} - m)$ graph, this recursive process yields the
homomorphism polynomial

$$h(G; x,y) = \sum_{K_{p(m)} \in H} x^p y^m, \qquad (1.2)$$

where H is the set of pseudo-complete graphs obtained in the final
step of the recursive process. As discussed by Bari, [1], the
homomorphism polynomial has several interesting properties, including
the fact that it yields the chromatic number $\chi(G)$ as the minimum
degree of the terms of $h(G; x,0)$ and that the chromatic polynomial of
G, $P(G; \lambda)$, can be obtained from $h(G; x, 1)$ by replacing the x^n with
the falling factorial $\lambda^{(n)} = \lambda(\lambda-1)...(\lambda-n+1)$. For example,

$$h(C_4; x,y) = x^4 y^2 + 2x^3 y + x^2 \qquad (1.3)$$

and

$$P(C_4; \lambda) = \lambda^{(4)} + 2\lambda^{(3)} + \lambda^{(2)}. \qquad (1.4)$$

Korfhage, [5], introduced the sigma polynomial of G and defined
it as follows. For all p, $\sigma(K_p; s) = 1$. If G is not complete, suppose
that $\sigma(G_L; s)$ is a polynomial of degree m and $\sigma(G_R; s)$ is a polynomial
of degree n, then $\sigma(G; s)$ is a polynomial of degree max$\{n, m+1\}$,
defined by

$$\sigma(G) = \sigma(G_L; s) \cdot s^{n-m+1} + \sigma(G_R; s), \quad \text{if } n \geq m+1 \qquad (1.5)$$

$$\sigma(G) = \sigma(G_L; s) + \sigma(G_R; s) \cdot s^{m-n+1}, \quad \text{if } n < m+1.$$

It was introduced as a way to investigate the chromatic polynomial,
and can be obtained from the $P(G, \lambda)$ in the following manner. If we
write the chromatic polynomial of the (p,q)-graph G in falling
factorial form,

$$P(G; \lambda) = \sum_{i=0}^{p} a_i \lambda^{(p-i)}, \qquad (1.6)$$

then

$$\sigma(G; s) = \sum_{i=0}^{k} a_i s^{k-i}, \qquad (1.7)$$

where $k = p - \chi(G)$. Thus, as an example, we have

$$P(C_4; \lambda) = \lambda^{(4)} + 2\lambda^{(3)} + \lambda^{(2)} \qquad (1.8)$$

and

$$\sigma(C_4; s) = s^2 + 2s + 1. \qquad (1.9)$$

As our examples illustrate, there is a connection between the
homomorphism polynomial and the sigma polynomial. By utilizing their
relationship to the chromatic polynomial we obtain the following
theorem.

THEOREM 1. For any graph G, $h(G; s,1) \cdot s^{-\chi} = \sigma(G; s)$.

PROOF. Since $P(G; \lambda) = h(G; x,1)$ with x^n replaced with $\lambda^{(n)}$, and $\sigma(G; s) = P(G; \lambda)/\lambda^{(\chi)}$ with $\lambda^{(n)}$ replaced with s^n, the result follows by direct substitution of s for x in $h(G; x,1)x^{-\chi}$.

By utilizing this theorem, we can obtain many of the computational formulas for $\sigma(G; s)$ from those given for $h(G; x,y)$. We will do this in section 2. Further, the work done on characterizing graphs with certain sigma polynomials will further the classification process for homomorphism polynomials. We will investigate these processes in sections 3 and 4.

2. REDUCTION THEOREMS.

A wide variety of reduction and characterization theorems concerning homomorphisms appear in Girse, [4]. We provide two examples here to illustrate their range, and their relationship to results on the sigma polynomial. The corollaries given here are found originally in Korfhage's work, [5], but follow directly from the theorems by applying Theorem 1.

THEOREM 2. If $G = G_1 + G_2$ (the join), then
$$h(G; x,y) = h(G_1; x,y) \cdot h(G_2; x,y). \qquad (2.1)$$

COROLLARY 2.1. For any graph G, $\sigma(G+v; s) = \sigma(G; s)$, where $G+v$ is the join of the vertex v and G.

THEOREM 3. If $G = K_{p_1} \cup K_{p_2}$, where $p_1 \geq p_2$, then

$$h(G; x,y) = \sum_{k=0}^{p_2} k! \binom{p_1}{k} \binom{p_2}{k} x^{p_1 + p_2 - k} y^{(p_1-k)(p_2-k)}. \qquad (2.2)$$

COROLLARY 3.1. $\sigma(K_{p_1} \cup K_{p_2}; s) = \sum_{i=0}^{p_2} \binom{p_2}{i} p_1^{(p_2-i)} s^i$, $p_2 \leq p_1$. (2.3)

However, not all of the characterization theorems concerning sigma polynomials are derivable from corresponding results on homomorphism polynomials. For example, if we let H be a graph with distinct vertices u and v, let w be a vertex not in H, let G_1 be $H \cup w$ with the edge uw, and let G_2 be $H \cup v$ with the edge vw, then we have the following theorem and corollary from Korfhage, [5].

THEOREM 4. $\sigma(G_1; s) = \sigma(G_2; s)$.

COROLLARY 4.1. All trees on $p+1$ vertices have the same sigma polynomial.

More general versions of these do not hold concerning homomorphism polynomials; specifically, empirical evidence suggests that every tree has a distinct homomorphism polynomial.

3. SPECIFIC CLASSES.

The sigma polynomial for several classes of graphs have been identified, and the techniques used to do this can be generalized to identify the homomorphism polynomials of these classes. Our first results in this direction are modifications of similar results in Korfhage, [5].

THEOREM 5. Let G_k be a $(2k-2)$ regular graph on $p = 2k$ vertices.
Then

$$h(G_k; x,y) = (x + x^2 y)h(G_{k-1}; x,y) = \sum_{i=\chi}^{p} \binom{p}{i-\chi} x^i y^{i-\chi} . \qquad (3.1)$$

PROOF. We have

$$h(G_k; x,y) = h(G_{k_p}; x,y) + h(G_{k_R}; x,y). \qquad (3.2)$$

But G_{k_R} is a vertex joined to G_{k-1} and G_{k_p} is $\overline{K}_2$ joined to G_{k-1}, so
that

$$h(G_{k_R}; x,y) = (x)h(G_{k-1}; x,y) \qquad (3.3)$$

and

$$h(G_{k_p}; x,y) = (x^2 y)h(G_{k-1}; x,y). \qquad (3.4)$$

Thus, the recursion follows, and the polynomial follows from the fact
that $h(G_2; x,y) = x^2 y + x$.

THEOREM 6. Let G be the graph on $n+2$ vertices, with $n+1$ vertices
of degree n and one vertex of degree 2. If $n > 1$,

$$h(G; x,y) = x^{n+2} y^n + \left[(n-1)y + y^{n-1} \right] x^{n+1} + (n-1)x^n. \qquad (3.5)$$

PROOF. Let v be the vertex of degree 2, with neighbors u and w.
Since u, w, and all other vertices not adjacent to v are of degree n,
u and w must not be adjacent. Thus there are two types of elementary
homomorphisms possible: identifying u and w, or identifying v to
another vertex. Thus the result follows from direct computation.

4. QUADRATIC FORMS.

In this section, we will use the fact that if $h(x,y)$ is the
homomorphism polynomial of some graph, then $h(s,1)s^{-\chi}$ is the sigma
polynomial of the same graph, and the fact that a graph G has a sigma
polynomial of the form

$$s^2 + ms + C \qquad (4.1)$$

if and only if the homomorphism polynomial of G is of the form

$$x^p y^m + P(y)x^{p-1} + Cx^{p-2}, \qquad (4.2)$$

where $P(y)$ is a polynomial in y with coefficients summing to m. The
latter follows from the fact that G has a quadratic sigma polynomial
if and only if G has chromatic number $p-2$, which is true if and only
if the smallest term of the homomorphism polynomial of G is of the
form Cx^{p-2}.

We obtain the following theorems by generalizing similar results
concerning sigma polynomials.

From Theorem 5 in Xu, [7], we can get the following bounds on the
coefficients of $h(G; x,y)$.

THEOREM 7. Let G be a (p,q) graph with homomorphism polynomial

$$h(G;\ x,y) = \sum_{i=\chi}^{p} P_i(y)x^i \text{ and let } m = \binom{p}{2} - q \ .$$

Then, for $i = \chi,\ldots,p$,

$$P_i(1) \leq \min\left\{ \binom{m}{i+m-p},\ \left\{\begin{matrix}p\\p-i\end{matrix}\right\} \right\}, \qquad (4.3)$$

where $\left\{\begin{matrix}p\\p-i\end{matrix}\right\}$ is a Stirling number of the second kind.

We introduce the following graphs, first defined by Frucht and Giudici in [3], to obtain a characterization of certain homomorphism polynomials. The graph T(a,b,c,d) has two distinguished vertices t_1 and t_2, which are adjacent to each other. There are a vertices adjacent to both t_1 and t_2; b vertices adjacent to t_1 only; c vertices adjacent to t_2 only; and d isolated vertices. Without loss of generality, we can let $b \geq c$, and to avoid trivial graphs, assume $a + c > 0$. The graph U(a,b,c,d) is obtained from T(a,b,c,d) by deleting the edge between t_1 and t_2.

From work by Frucht and Giudici, [3], and Li and Whitehead, [6], concerning these graphs and quadratic sigma polynomials, we obtain the following theorem, with the specific polynomials found by direct computation.

THEOREM 8. A graph G has a homomorphism polynomial of the form

$$x^p y^m + P(y)x^{p-1} + Cx^{p-2},$$

where P(1) = m and C is constant, if and only if the complement of G is either $C_5 \cup dK_1$, T(a,b,c,d), or U(a,b,c,d) different from U(1,0,0,d) where a + c > 0. In fact,

$$h(\overline{C_5 \cup dK_1}) = x^{d+5}y^5 + 5x^{d+4}y^2 + 5x^{d+3}, \qquad (4.4)$$

$$h(\overline{T(a,b,c,d)}) = x^{p+2}y^{2a+b+c+1}$$

$$+ x^{p+1}(y^a + by^{a+c} + ay^{a+c} + (a+c)y^{a+b})$$

$$+ x^p(a+b)(a+c), \qquad (4.5)$$

and

$$h(\overline{U(a,b,c,d)}) = x^{p+2}y^{2a+b+c} + x^{p+1}(by^{a+c} + ay^c + ay^b + cy^{a+b})$$

$$+ x^p(b(a+c) + a(a+c-1)) \qquad (4.6)$$

where p = a + b + c.

Other characterizations of quadratic sigma polynomials found by Dhurandhur, [2], Korfhage. [5], and Xu, [7], can be generalized for homomorphism polynomials. For example, we can obtain the following theorem.

THEOREM 9. The polynomial $x^p y^m + P(y)x^{p-1} + Cx^{p-2}$, where P(1) = m, is a homomorphism polynomial for some graph G iff

$m - 2 \leq C \leq m^2/4$, and there exists an integer $b \leq m/2$ such that $b(m - 1 - b) \leq C \leq b(m - b)$.

Results similar to these can be generated for homomorphism polynomials of the form

$$x^p y^m + P_{p-1}(y) x^{p-1} + P_{p-2}(y) x^{p-2} + Cx^{p-3} \qquad (4.7)$$

by generalizing the related work by Dhurandhur, [2].

REFERENCES

[1]   BARI R., Homomorphism polynomials of graphs, <u>Journal</u> <u>of</u> <u>Combinatorics, Information</u> <u>&</u> <u>Systems</u> <u>Sciences</u>, v7 #1 p56-64 (1982).

[2]   DHURANDHUR M., Characterization of quadratic and cubic σ-polynomials, <u>Journal</u> <u>of</u> <u>Combinatorial</u> <u>Theory</u> <u>Series</u> <u>B</u>, v37 p210-220 (1984).

[3]   FRUCHT R. and GIUDICI R., Some chromatically unique graphs with seven points, <u>Ars</u> <u>Combinatorica</u>, v16A p161-172 (1983).

[4]   GIRSE R., Homomorphism polynomials of certain graphs, <u>Journal</u> <u>of</u> <u>Combinatorics, Information</u> <u>&</u> <u>Systems</u> <u>Sciences</u>, v10 p62-68 (1985).

[5]   KORFHAGE R., σ-polynomials and graph coloring, <u>Journal</u> <u>of</u> <u>Combinatorial</u> <u>Theory</u> <u>Series</u> <u>B</u>, v24 p137-153 (1977).

[6]   LI N. and WHITEHEAD E., Characterizations of graphs having quadratic σ-polynomials, unpublished.

[7]   XU S., On σ-polynomials, <u>Discrete</u> <u>Mathematics</u>, v69 p189-194 (1988).