# PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND ORDER INTEGRO-ORDINARY DIFFERENTIAL EQUATIONS WITH GENERAL KERNEL AND CARATHÉODORY NONLINEARITIES*

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**ABSTRACT.** We study the existence of solutions for the periodic boundary value problem for some second order integro-differential equations with a general kernel. Also we develop the monotone method to approximate the extremal solutions of the problem.
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## 1. INTRODUCTION

The purpose of this paper is to study the following periodic boundary value problem for a second order nonlinear integro-ordinary differential equation

$$-u''(t) = f(t, u(t), Ku(t)), u(0) = u(2\pi), u'(0) = u'(2\pi) \tag{1.1}$$

where $f : I \times \mathbf{R}^2 \to \mathbf{R}$ is a Carathéodory function, $K$ is an integral operator in $L^2(I)$ with kernel $k \in L^2(J), J = I \times I$. Related to (1.1) we consider the linear problem

$$-u''(t) + Mu(t) + N[Ku](t) = h(t), u(0) = u(2\pi), u'(0) = u'(2\pi) \tag{1.2}$$

where $M, N \in \mathbf{R}$, and $h \in L^2(I)$.

By a solution $u$ of (1.1) we mean a function $u \in H^2(I)$ such that the function $t \in I \to f(t, u(t), [Ku](t))$ is a function in $L^2(I)$ satisfying the equation for a.e. $t \in I$, and $u(0) = u(2\pi), u'(0) = u'(2\pi)$.

Problem (1.1) is considered in [7] with $f$ continuous and $K$ a Volterra integral operator with positive kernel. The authors developed the monotone iterative method for (1.1) based on a comparison result. As it is pointed out in [6], the method of [7] is not applicable to the general situation. Erbe and Guo studied problem (1.1) with $f$ continuous, and $k$ continuous and positive. They first considered the linear problem (1.2) and gave an estimate on $\|k\|_\infty$. We note that in [6], $Ku = NTu + N_1Su$ with $N, N_1$ real numbers, $T$ an integral operator of Volterra type, and $S$ an integral operator of Fredholm type. Problem (1.1) is studied in [8] under the assumption that $f(t, u, v)$ is continuous and increasing in $v$, and in [7] for $f$ continuous and $K$ of Volterra type. Following the ideas of [6] we study (1.1) in the general case, i.e., $f$ is a Carathéodory function and $k$ is an $L^2$ kernel. Also, we do not require $k$ to have constant sign on $J$. For the linear problem (1.2) to have a unique solution, we give an estimate on $\|k\|_2$ that improves the estimation given in [6], and our estimate is the best possible in the sense that if equality is attained, then the existence-uniqueness result for (1.2) is not valid anymore.

## 2. PBVP FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

We recall here, for convenience of the reader, some results for the following periodic boundary value problem (PBVP) for a linear second order ordinary differential equation. The problem

$$-u''(t) + Mu(t) = h(t); \quad u(0) = u(2\pi); \quad u'(0) = u'(2\pi) \tag{2.1}$$

with $M = m^2, m > 0$ and $h \in L^2(I)$, has a unique solution given by the expression

$$u(t) = \int_0^{2\pi} G(t,s)h(s)ds \,,$$

the Green function $G$ is given by

$$G(t,s) = \begin{cases} \dfrac{1}{2m(e^{2m\pi} - 1)}[e^{m(t-s)} + e^{m(2\pi - t + s)}]; & 0 \leq s \leq t \leq 2\pi \\[2mm] \dfrac{1}{2m(e^{2m\pi} - 1)}[e^{m(s-t)} + e^{m(2\pi - s + t)}]; & 0 \leq t \leq s \leq 2\pi \end{cases}$$

Moreover, $G$ is continuous on $J = I \times I$,

$$\int_0^{2\pi} G(t,s)ds = \frac{1}{M}, \quad \min\{G(t,s):(t,s) \in J\} = \frac{e^{m\pi}}{m(e^{2m\pi} - 1)} \equiv a,$$

and $\max\{G(t,s):(t,s) \in J\} = \frac{1+e^{2m\pi}}{2m(e^{2m\pi}-1)} \equiv b$.

Since $G(t,s) \geq a > 0$ for every $(t,s) \in J$ we obtain the following maximum principles:

$$h \geq 0 \quad \text{a.e. on} \quad I \quad \text{implies} \quad u \geq 0 \text{ on} \quad I\,, \tag{2.3}$$

$$h \leq 0 \quad \text{a.e. on} \quad I \quad \text{implies} \quad u \leq 0 \text{ on} \quad I \tag{2.4}$$

Obviously, $h = 0$ implies $u = 0$, but if $h > 0$ on a set of positive measure of $I$, then $u(t) \geq a \int_0^{2\pi} h(s)ds > 0$ for every $t \in I$.

## 3. LINEAR INTEGRO-DIFFERENTIAL EQUATIONS

We now consider the integro-differential problem (1.2) with $M > 0$, $N \in \mathbf{R}, [Ku](t) = \int_0^{2\pi} k(t,s)u(s)ds, k \in L^2(J)$ and $h \in L^2(I)$.

Note that $K$ is an integral operator, and it could be either of Volterra or Fredholm type. Even $K$ can be of mixed type as in [6] where

$$Ku = NTu + N_1 Su \tag{3.1}$$

with $N, N_1$ real numbers, $T$ an integral operator of Volterra type with kernel $k_0$, and $S$ an integral operator of Fredholm type with kernel $k_1$. Thus, $k = N \cdot k_0 + N_1 \cdot k_1$.

In what follows, $\| \ \|_p$ denotes the usual norm in $L^p$, $1 \leq p \leq \infty$.

According to the results of section 2, we have that $u$ is a solution of (1.2) if and only if

$$u(t) = \int_0^{2\pi} G(t,s)[h(s) - N[Ku](s)]ds \,. \tag{3.2}$$

Using Fubini's theorem it is easy to see that for any $t \in I$

$$-N \int_0^{2\pi} G(t,s)[Ku](s)ds = \int_0^{2\pi} \tau(t,s)u(s)ds \,,$$

with

$$\tau(t,s) = -N \int_0^{2\pi} G(t,r)k(r,s)ds \tag{3.3}$$

Therefore, equation (3.2) is equivalent to the following abstract equation

$$u(t) = w(t) + [Tu](t) \tag{3.4}$$

where

$$w(t) = \int_0^{2\pi} G(t,s)h(s)\,ds$$

and

$$[Tu](t) = \int_0^{2\pi} \tau(t,s)u(s)\,ds \ .$$

In consequence, $u$ is a solution of the linear problem (1.2) if and only if $u$ is a fixed point of the operator $T_w : L^2(I) \to L^2(I)$, $T_w(u) = w + Tu$.

**THEOREM 3.1.** *Suppose that*

$$\|\tau\|_2 < 1 \ , \tag{3.5}$$

*then (1.2) has a unique solution* $u = \lim_{n \to \infty} u_n$, $u_0 \in L^2(I)$, $u_{n+1} = T_w(u_n)$, $n \geq 0$. *Moreover, the solution is given by the following relations*

$$u(t) = \int_0^{2\pi} [G(t,s) + H(t,s)]h(s)\,ds \ , \tag{3.6}$$

$$H(t,s) = \int_0^{2\pi} L(t,r)G(r,s)\,dr \ , \tag{3.7}$$

$$L = \sum_{n=1}^{\infty} \tau_n \ , \quad \tau_1 = \tau \ , \quad \tau_n(t,s) = \int_0^{2\pi} \tau_{n-1}(t,r)\tau(r,s)\,dr \ , \quad n \geq 2 \ . \tag{3.8}$$

**PROOF.** Note that $\|T_w(u) - T_w(v)\|_2 \leq \|\tau\|_2 \cdot \|u - v\|_2$ for any $u, v \in L^2(I)$. By the contraction principle of Banach we have that $T_w$ has a unique fixed point which is the solution of (1.2). Now, we choose $u_0 = w$. Using again Fubini's theorem, it is easy to see that

$$u_n(t) = w(t) + \int_0^{2\pi} H_n(t,s)h(s)\,ds \ ,$$

$$H_n(t,s) = \int_0^{2\pi} L_n(t,r)G(r,s)\,dr \ ,$$

$$L_n = \sum_{i=1}^{n} \tau_i \ .$$

We have that $\|\tau_i\|_2 \leq \|\tau\|_2^i$, and taking into account (3.5) we see that the series $\sum_{i=1}^{\infty} \tau_i$ is convergent in $L^2(J)$. We now deduce that $\{H_n\} \to H$ in $L^2(J)$ and the validity of formulas (3.6), (3.7), (3.8). ∎

Taking into account (3.3) we have that $\|\tau\|_2 \leq \|k\|_2 \|G\|_2$. In consequence, $\|k\|_2 \|G\|_2 < 1$ implies that the linear problem (1.2) has a unique solution. It is possible to give different estimates for $k$ that imply that (3.5) holds. For instance, if there exists $c > 0$ such that

$$\left| \int_0^{2\pi} k(t,s)ds \right| < c \quad \text{for every} \quad s \in I \tag{3.9}$$

then $|\tau(t,s)| \leq bc$ for every $(t,s) \in J$ and $\|\tau\|_2 < 2\pi bc$. Therefore, $c < \frac{1}{2\pi b}$ implies that the linear problem (1.2) is uniquely solvable.

In the case that $k \in L^\infty(J)$ then $|\tau(t,s)| \leq \frac{|k|_\infty}{M}$ for a.e. $(t,s) \in J$. Thus, $\|\tau\|_2 < \frac{2\pi|k|_\infty}{M}$ and, in this situation, $\|k\|_\infty < \frac{M}{2\pi}$ implies that $\|\tau\|_2 < 1$.

Note that in the case that $K$ is given by (3.1), Erbe and Guo gave the estimate (see formula (4) in [6])

$$N_0 \| K_0 \|_\infty + N_1 \| k_1 \|_\infty < \frac{M}{2\pi}$$

which obviously implies (3.5).

A natural question is if Theorem 3.1 remains valid in the critical case $\| \tau \|_2 = 1$. The following example shows that in such a case the linear problem (1.2) may have either no solution or an infinite number of solutions, thus showing that the estimate (3.5) is as sharp as possible.

**EXAMPLE.** Take $k = c \cdot G, c \in \mathbf{R}$ such that $\| k \|_2 = 1$. The integral operator $T$ associated to $\tau$ is compact and selfadjoint. Thus, $+1$ or $-1$ is an eigenvalue of $T$. Suppose that $+1$ is an eigenvalue and choose $u \neq 0$ with $Tu = u$. Thus, by the Fredholm alternative theorem, the equation $u = w + Tu$ has either no solution or an infinite number of solutions.

## 4. MAXIMUM PRINCIPLE

We are now interested in obtaining a similar result to (2.4) for the linear integro-differential problem (1.2). Using the representation (3.6) for the solution of (1.2), we see that it is equivalent to show that $G + H \geq 0$ a.e. on $J$. Since $G(t,s) \geq a$ for any $(t,s) \in J$, we can affirm that $G + H \geq 0$ a.e. on $J$ if, for instance, $\| H \|_\infty \leq a$. We first give an estimate for $\| H \|_\infty$.

**THEOREM 4.1.** *Suppose that* $k \in L^\infty(J)$ *and*

$$\| k \|_\infty < \frac{M}{2\pi}. \tag{4.1}$$

*Then, (3.5) holds and*

$$\| H \|_\infty \leq \frac{\| k \|_\infty}{M(M - 2\pi \| k \|_\infty)}. \tag{4.2}$$

**PROOF.** From the relation (3.3) we deduce that $\tau \in L^\infty(J)$, and

$$\| \tau \|_\infty \leq \frac{\| k \|_\infty}{M} = d < \frac{1}{2\pi}.$$

On the other hand, $\tau_n \in L^\infty(J)$ for any $n \in \mathbf{N}$, and $\| \tau \|_\infty \leq d(2\pi d)^{n-1}$. This implies that the series $\sum_{n=1}^\infty \tau_n$ is convergent in $L^\infty(J)$. Therefore,

$$\| L \|_\infty = \left\| \sum_{n=1}^\infty \tau_n \right\|_\infty \leq \frac{d}{1 - 2\pi d}$$

and

$$\| H \| \leq \frac{\| L \|_\infty}{M} \leq \frac{d}{M(1 - 2\pi d)}$$

which is precisely estimate (4.2). ∎

We note that the right hand side of (4.2) tends to 0 when $\| k \|_\infty$ tends to 0. Thus, we obtain the following maximum principle for the linear equation (1.2).

**THEOREM 4.2.** *Assume that* $k \in L^\infty(J)$ *and*

$$\| k \|_\infty < \frac{Mme^{m\pi}}{e^{2\pi m} - 1 + 2\pi me^{\pi m}} = r. \tag{4.3}$$

*Then, we have that (3.5) holds and* $G + H \geq 0$ *a.e. on* $J$.

**PROOF.** We first note that the following inequality holds true:

$$\frac{me^{m\pi}}{e^{2\pi m} - 1 + 2\pi m e^{\pi m}} < \frac{1}{2\pi}. \tag{4.4}$$

Hence, $\|k\|_\infty < \frac{M}{2\pi}$. Now, using (4.2) we have for a.e. $(t,s) \in J$ that

$$G(t,s) + H(t,s) \geq a - \frac{\|k\|_\infty}{M(M - 2\pi\|k\|_\infty)}.$$

Combining (4.2) and (4.3) we obtain that $\|H\|_\infty \leq a$.

Therefore, we can write that $G(t,s) + H(t,s) \geq a - \|H\|_\infty > +0$ for a.e. $(t,s) \in J$, completing the proof of the theorem. ∎

As a consequence, we obtain that if inequality (4.3) holds then

$$h \geq 0 (\leq 0) \text{ a.e. on } I \text{ implies } u \geq 0 (\leq 0) \text{ a.e. on } J . \tag{4.5}$$

We now consider the case when the kernel has constant sign on $J$. If $k \leq 0$ a.e. on $J$, then $\tau \geq 0$ a.e. on $J$ and we have that $H \geq 0$ a.e. on $J$. Then, trivially the maximum principle (4.5) holds.

If $k \geq 0$ a.e. on $J$, then $\tau \leq 0$ a.e. on $J$ and the previous reasoning is not valid. However, we have that $(-1)^n\tau_n \geq 0$ a.e. on $J$, $n \geq 1$ and this is useful to prove the following result.

**THEOREM 4.3.** *Suppose that $k \in L^\infty(J)$ is such that $k \geq 0$ a.e. on $J$ and*

$$\|k\|_\infty \leq \frac{m}{8\pi^2 e^{m\pi}}\left[\sqrt{(e^{2\pi m} - 1)^2 + 16\pi^2 M e^{2\pi m}} - (e^{2\pi m} - 1)\right] = r_+. \tag{4.6}$$

*Then, (3.5) holds and $G + H \geq 0$ a.e. on $J$.*

**PROOF.** We first note that $r_+ < \frac{M}{2\pi}$ and then $\|k\|_\infty < \frac{M}{2\pi}$. On the other hand,

$$L \geq \sum_{\substack{n=1 \\ n \text{ odd}}}^\infty \tau_n$$

and

$$\left\|\sum_{\substack{n=1 \\ n \text{ odd}}}^\infty \tau_n\right\|_\infty \leq \sum_{\substack{n=1 \\ n \text{ odd}}}^\infty \|\tau_n\|_\infty \leq \sum_{\substack{n=1 \\ n \text{ odd}}}^\infty d(2\pi d)^{2n-2} = \frac{d}{1 - (2\pi d)^2} = \frac{M\|k\|_\infty}{M^2 - 4\pi^2\|k\|_\infty^2}.$$

Hence,

$$L(t,s) \geq \frac{-M\|k\|_\infty}{M^2 - 4\pi^2\|k\|_\infty^2}$$

for a.e. $(t,s) \in J$,

$$H(t,s) \geq \frac{-M\|k\|_\infty}{M^2 - 4\pi^2\|k\|_\infty^2}$$

for $(t,s) \in J$. In consequence,

$$G + H \geq a - \frac{-M\|k\|_\infty}{M^2 - 4\pi^2\|k\|_\infty^2} = a_1$$

on $J$. Now, $a_1 \geq 0$ if and only if

$$4\pi^2 e^{m\pi}\|k\|_\infty^2 + m(e^{2\pi m} - 1)\|k\|_\infty - M^2 e^{m\pi} \leq 0,$$

and this is true for $\|k\|_\infty \in [0, r_+]$. ∎

Note that estimate (4.6) improves (4.3) since $r < r_+$.

## 5.  MONOTONE ITERATIVE METHOD

We now consider the nonlinear equation (1.1). We recall that $f$ is a Carathéodory function if $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in I$, $f(\cdot, u, v)$ is measurable for any $u, v \in \mathbf{R}$, and for any $R > 0$ there exists $\sigma = \sigma_R \in L^2(I)$ such that $|f(t, u, v)| \leq \sigma(t)$ for a.e. $t \in I$ for any $u, v \in \mathbf{R}$ with $\max(|u|, |v|) \leq R$. Define $H^2_{2\pi}(I) = \{ u \in H^2(I) : u(0) = u(2\pi), u'(0) = u'(2\pi) \}$. By a solution $u$ of (1.1) we mean a function $u \in H^2_{2\pi}(I)$ such that $t \in I \rightarrow [F(u)](t) = f(t, u(t), [Ku](t))$ is a function of $L^2(I)$ satisfying the equation for a.e. $t \in I$.

We say that $\alpha \in H^2_{2\pi}(I)$ is a lower solution for (1.1) if $F\alpha \in L^2(I)$, and

$$-\alpha''(t) \leq f(t, \alpha(t), [K\alpha](t)) \quad \text{for a.e.} \quad t \in I. \tag{5.1}$$

Similarly, we define an upper solution as a function $\beta \in H^2_{2\pi}(I)$ such that $F\beta \in L^2(I)$, and

$$-\beta''(t) \leq f(t, \beta(t), [K\beta](t)) \quad \text{for a.e.} \quad t \in I. \tag{5.2}$$

In $u \in L^2(I)$, in general, $Fu$ is not a function of $L^2(I)$. The condition that $F$ maps $L^2(I)$ into $L^2(I)$ is equivalent [1] to the existence of $b \in L^2(I)$, $a \in \mathbf{R}$ such that $|f(t, u, v)| \leq b(t) + a(|u| + |v|)$ for a.e. $t \in I$ and every $u, v \in \mathbf{R}$.

Now suppose that condition (3.9) is verified. Obviously, (3.9) is satisfied if, for instance, $k \in L^\infty(J)$. However, $k(t, s) = s^{-1/2}$ is a kernel that satisfies (3.9) but does not belong to $L^\infty(J)$. Then for any $u \in L^\infty(I)$ we have that

$$\left| [Ku](t) \right| = \left| \int_0^{2\pi} k(t, s) u(s) ds \right| < c \|u\|_\infty,$$

and $Ku \in L^\infty(I)$. In consequence, $|f(t, u, [Ku](t))| \leq \sigma_R(t)$ for a.e. $t \in I$, where $R = \max(\|u\|_\infty, c\|u\|_\infty)$, and $Fu \in L^2(I)$.

Thus, if condition (3.9) holds and $\alpha \in H^2_{2\pi}(I)$, then $K\alpha \in L^\infty(I)$ and $F\alpha \in L^2(I)$.

If $\alpha, \beta$ are lower and upper solutions of (1.1) respectively, we shall assume that

$$\alpha \leq \beta \quad \text{on} \quad I. \tag{5.3}$$

Thus, $k \geq 0 (\leq 0)$ implies that $K\alpha \leq (\geq)K\beta$ a.e. on $I$.

In the case that $k \geq 0$ a.e. on $J$, we introduce the following condition: there exist $M > 0, N > 0$ such that

$$f(t, u, v) - f(t, w, x) \geq -M(u - w) - N(v - x) \tag{5.4}$$

for a.e. $t \in I$, $\alpha(t) \leq w \leq u \leq \beta(t)$, and $[K\alpha](t) \leq x \leq v \leq [K\beta](t)$.

If $k \leq 0$ a.e. on $J$, we shall use the condition: there exist $M > 0, N > 0$ such that

$$f(t, u, v) - f(t, w, x) \geq -M(u - w) - N(x - v). \tag{5.5}$$

for a.e. $t \in I$, $\alpha(t) \leq w \leq u \leq \beta(t)$, and $[K\beta](t) \leq v \leq x \leq [K\alpha](t)$.

**THEOREM 5.1.** *Assume that $k \in L^\infty(J)$, $k \geq 0$ a.e. on $J$, and $\|k\|_\infty \leq \frac{r}{|N|}$. In addition, suppose that there exist $\alpha, \beta \in H^2_{2\pi}(I)$ lower and upper solutions of (1.1) respectively such that (5.3) and (5.4) hold. Then, there exists monotone sequences $\{\alpha_n\} \uparrow \phi$, and $\{\beta_n\} \downarrow \psi$ uniformly on $I$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Here $\phi$ and $\psi$ are the minimal and maximal solutions of (1.1) respectively on $[\alpha, \beta]$. Moreover, these sequences verify $\alpha_0 \leq \ldots \leq \alpha_n \leq \ldots \leq \beta_n \leq \ldots \leq \beta_0$.*

**PROOF.** For $\eta \in [\alpha, \beta]$, let us consider the following linear periodic boundary value problem

$$-u'' + Mu + N[Ku] = h_\eta(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \tag{5.6}$$

where $h(t) = h_\eta(t) = f(t, \eta(t), [K\eta](t)) + M\eta + N[K\eta](t)$.

This linear problem has a unique solution $u = A\eta$ in view of Theorem 3.1 since $\|N \cdot k\|_\infty \leq r$. Moreover, in this case $G + H \geq 0$ a.e. on $J$.

The operator $A$ is well defined from $[\alpha, \beta]$ to $[\alpha, \beta]$ and $A$ is increasing.

Indeed, let $\eta \in [\alpha, \beta]$ and define $v = u - \alpha$. Thus, using (5.4), we obtain

$$-v'' + Mv + NKv = f(t, \eta, K\eta) + M\eta + NK\eta - (-\alpha'' + M\alpha + NK\alpha) \geq$$

$$f(t, \eta, K\eta) + M\eta + NK\eta - f(t, \alpha, K\alpha) - M\alpha - NK\alpha \geq 0 .$$

Hence, by virtue of Theorem 4.3 we deduce that $v \geq 0$ on $I$. Similarly, one can show that $u \leq \beta$ on $I$.

To show the monotonicity of $A$, let $\eta_i \in [\alpha, \beta]$, $u_i = A\eta_i$, $i = 1, 2$, and $w = u_1 - u_2$. Hence,

$$-w'' + Mw + NKw = f(t, \eta_1, K\eta_1) + M\eta_1 + NK\eta_1 - f(t, \eta_2, K\eta_2) - M\eta_2 - NK\eta_2 \geq 0$$

and then, $w \geq 0$.

We now define $\alpha_0 = \alpha$, $\alpha_{n+1} = A\alpha_n$, $n \geq 0$. By the properties of the operator $A$, the sequence $\{\alpha_n\}$ is increasing and uniformly bounded on $I$. Then, $\{\alpha_n\} \uparrow \phi$ pointwise on $I$. Writing the integral representation for $A\alpha_n$ and using standard arguments we obtain that $\phi$ is actually a solution of (1.1). Analogously, defining $\beta_0 = \beta$, $\beta_{n+1} = A\beta_n$, $n \geq 0$, $\{\beta_n\} \downarrow \psi$, where $\psi$ is solution of (1.1).

To show that $\phi$ and $\psi$ are the minimal and maximal solutions of (1.1) in $[\alpha, \beta]$, let $u \in [\alpha, \beta]$ be a solution of (1.1). Then, $Au = u$, and using the properties of the operator $A$ we have that $\alpha_n \leq u \leq \beta_n$ for every $n \in N$. Passing to the limit when $n \to \infty$ we obtain that $\phi \leq u \leq \psi$.  ∎

If $k \leq 0$ a.e. on $J$, then we can use directly that $G + H \geq 0$ a.e. on $J$ to obtain the following result.

**THEOREM 5.2.** *Suppose that $k \in L^\infty(J)$, $k \leq 0$ a.e. on $J$, and $\|k\|_\infty \leq \frac{M}{2\pi \cdot |N|}$. In addition, assume that there exist $\alpha, \beta \in H^2_{2\pi}(I)$ lower and upper solutions of (1.1) respectively such that (5.3) and (5.5) hold. Then, there exist monotone sequences $\{\alpha_n\} \uparrow \phi$, and $\{\beta_n\} \downarrow \psi$ uniformly on $I$ with*

$$\alpha = \alpha_0 \leq \ldots \leq \alpha_n \leq \ldots \leq \beta_n \leq \ldots \leq \beta_0 = \beta ,$$

*and $\phi, \psi$ are the minimal and maximal solutions of (1.1) respectively on $[\alpha, \beta]$.*

**PROOF.** For $\eta \in [\alpha, \beta]$, let us consider the linear problem (5.6). In this case, $K\alpha \geq u = K\eta \geq K\beta$ on $I$. As in the proof of Theorem 5.1 we have that (5.6) has a unique solution $u = A\eta$. The operator $A$ is well defined from $[\alpha, \beta]$ to $[\alpha, \beta]$, and it is increasing. As in the proof of Theorem 5.1 we construct monotone sequences $\{\alpha_n\} \uparrow \phi$ and $\{\beta_n\} \downarrow \psi$, where $\phi$ and $\psi$ are the minimal and maximal solutions of (1.1) respectively between $\alpha$ and $\beta$.  ∎

It may occur that

$$K\alpha \leq K\beta \quad \text{a.e. on} \quad I \tag{5.7}$$

even if $\alpha \leq \beta$ and $k$ changes sign on $J$ since $[K\alpha](t)$ depends on the value of $k(t, s)\alpha(s)$, $0 \leq s \leq 2\pi$. (For example, take $\varepsilon > 0$, $k(t, s) = 1$ for $0 \leq s \leq 2\pi - \varepsilon$, $k(t, s) = -1$ for $2\pi - \varepsilon < s \leq 2\pi$, and $\alpha = 1$, $\beta = 2$.) In such a situation, we can apply the previous results to obtain.

**THEOREM 5.3.** *Suppose that $k \in L^\infty(J)$, $N \cdot \|k\|_\infty < r$, and (5.3), (5.4), (5.7) hold. Then, there exist monotone sequences $\{\alpha_n\} \uparrow \phi$, and $\{\beta_n\} \downarrow \psi$ uniformly on $I$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ and $\alpha_0 \leq \ldots \leq \alpha_n \leq \ldots \leq \beta_n \leq \ldots \leq \beta_0$. Here $\phi$ and $\psi$ are the minimal and maximal solutions of (1.1) respectively on $[\alpha, \beta]$.*

If $K\alpha \geq K\beta$ a.e. on $J$ (even if $k$ has no constant sign on $J$), then we have an analogous result using (5.5).

In the general case ($k$ has no constant sign on $J$) we are able to deal with a linear integral perturbation of the ordinary differential equation $-u'' = f(t,u)$, being such a perturbation of the type $N \cdot Ku, N \in \mathbf{R}$. In concrete, we consider the problem

$$-u'' = f(t,u) + N \cdot Ku, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \tag{5.8}$$

We now require the following condition: there exists $M > 0$ such that

$$f(t,u) - f(t,v) \geq -M(u - v) \tag{5.9}$$

for a.e. $t \in I, \alpha(t) \leq v \leq u \leq \beta(t)$. In this case, for $\eta \in [\alpha, \beta]$, the linear problem (5.6) reads

$$-u'' + Mu = f(t,\eta) + M\eta + NK\eta .$$

**THEOREM 5.4.** *Consider the nonlinear problem (5.8) were $f$ satisfies condition (5.9). Suppose that $k \in L^\infty(I)$ is such that $|N| \, \|k\|_\infty < r$, and that there exist $\alpha, \beta \in H_2^{2\pi}(I)$ lower and upper solution of (5.8) respectively with $\alpha \leq \beta$ on $I$. Then, there exists monotone sequences $\{\alpha_n\} \uparrow \phi$, and $\{\beta_n\} \downarrow \psi$ uniformly on $I$ with $\alpha = \alpha_0 \leq \ldots \leq \alpha_n \leq \ldots \leq \beta_n \leq \ldots \leq \beta_0 = \beta$. Here $\phi$ and $\psi$ are the minimal and maximal solutions of (5.8) respectively on $[\alpha, \beta]$.*

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