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TOPOLOGICAL PROPERTIES AND MATRIX TRANSFORMATIONS OF CERTAIN ORDERED GENERALIZED SEQUENCE SPACES

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ABSTRACT. In this note, we carry out investigations related to the mixed impact of ordering and topological structure of a locally convex solid Riesz space (X, τ) and a scalar valued sequence space λ, on the vector valued sequence space $\lambda(X)$ which is formed and topologized with the help of λ and X, and vice versa. Besides, we also characterize o-matrix transformations from $c(X)$, $\ell^{\infty}(X)$ to themselves, $cs(X)$ to $c(X)$ and derive necessary conditions for a matrix of linear operators to transform $\ell^1(X)$ into a simple ordered vector valued sequence space $\Lambda(X)$.

KEY WORDS AND PHRASES. Riesz spaces, order complete, ideal, diagonal property; positive, sequentially order continuous and o-precompact linear operators; locally convex solid Riesz spaces, Riesz seminorms, Lebesgue, pre-Lebesgue, Fatau property; ordered vector valued sequence spaces, matrix transformations.
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1. INTRODUCTION.

In the direction of generalizing the Köthe theory of perfect sequence spaces [13], [16], the concept of a vector valued sequence space (VVSS) or a generalized sequence space defined with the help of a scalar valued sequence space (SVSS) was introduced by Pietsch [21], who also used these spaces in the study of absolutely summing operators and nuclear spaces, cf. [20], [22]. Moreover, the topological properties of VVSS which are defined by using the seminorms generating the topology of a locally convex space and a SVSS and have been found useful in the study of λ-nuclearity, λ-bases, operators of λ-type, absolutely λ-summing operators, [4], [5], (see also [14] and [15] for bases theory and its applications related to nuclearity etc. in locally convex spaces) and completeness of the space via their mixed structure [9], are sufficiently explored now [4], [8], [18], [24]. However, if the underlying locally convex space and the SVSS have additional structure of ordering, it is natural to inquire the mixed impact of the topology and the order structure on the VVSS. This study when X has an order structure, was taken up by Walsh [27] for certain type of VVSS, who applied these spaces to study K-absolutely summing and majorizing operators introduced for Banach lattices in [25]. In the present paper, we carry out investigations related to this mixed impact of ordering and topological structure of the locally convex solid Riesz spaces and the SVSS on the VVSS and vice-versa. Besides, we also continue our study of o-matrix transformations initiated in [10] for particular ordered VVSS introduced in [11].

2. PREREQUISITES.

In this section, we mention the salient features of Riesz spaces, locally convex solid Riesz spaces, scalar and vector valued sequence spaces required for our present work. However for unexplained terms in these theories, the reader is urged to look into [1], [17], [19], [26].

Throughout the sequel, X denotes a Riesz space ordered by the cone K. The notations $x_\alpha \uparrow$ (resp. $x_\alpha \downarrow$) are used for an increasing (resp. decreasing) net $\{x_\alpha\}$ in X and $x_\alpha \uparrow x$ (resp. $x_\alpha \downarrow x$) provided $x_\alpha \uparrow$ and $\sup x_\alpha = x$ (resp. $x_\alpha \downarrow$ and $\inf x_\alpha = x$).

A net $\{x_\alpha\}$ in X <u>order converges</u> to X, written as $x_\alpha \xrightarrow{(o)} x$, if there exists a net $\{y_\alpha\}$ in X with $y_\alpha \downarrow \theta$ and $|x_\alpha - x| \leq y_\alpha$ for each α; and it converges <u>relatively uniformly</u> to x, if there exists $u \in K$ such that for any $\epsilon > o$, there exists $\alpha_o \in \Lambda$ satisfying the condition $|x_\alpha - x| \leq \epsilon u$, $\alpha \geq \alpha_o$, where u is known as the <u>regulator of convergence</u>. A sequence $\{x_n\}$ in X is said to be <u>order-Cauchy</u> (respectively, relatively <u>uniformly Cauchy</u>) if there exists some sequence $y_n \downarrow 0$ in X such that $|x_m - x_k| \leq y_n$ for all $m,k \geq n$ (resp. for any $\epsilon > o$, there exists $k \in N$ such that $|x_n - x_m| \leq \epsilon u$ for $n,m \geq k$); accordingly, X is <u>order-Cauchy complete</u> [[3],[12]] (resp. <u>uniformly complete</u>) if every order-Cauchy (resp. relatively uniformly Cauchy) sequence is order convergent (resp. relatively uniformly convergent) in X.

Relating the above concepts of convergence and completeness, we have the following well known result contained in [1], [19], [3].

PROPOSITION 2.1. (i) In a σ-order complete Riesz space X with the diagonal property, the order, and uniform convergence coincide for sequences; (ii) every order complete Riesz space is uniformly complete; and (iii) every σ-order complete Riesz space is order-Cauchy complete.

We now equip X with a locally convex topology τ generated by the family $\mathcal{D}$ of Riesz seminorms p's, i.e., $p(x) \leq p(y)$ for $|x| \leq |y|$ in X, so that (X, τ) is a locally convex solid (l.c.s.) Riesz space. We refer [1] for various terms and results on the theory of locally convex solid Riesz spaces. However, we say a Riesz seminorm p on X is,
(i) <u>σ-Lebesgue</u> if $p(x_n) \to o$ for any sequence $x_n \downarrow \theta$ in X; (ii) <u>Lebesgue</u> if $p(x_\alpha) \to o$ for any net $\{x_\alpha\}$ in X which decreases to θ in X;

(iii) pre-Lebesgue if for any disjoint order bounded sequence $\{x_n\}$ in X, $p(x_n) \to o$; and (iv) σ-Fatou (resp. Fatou) if $p(x_n) \uparrow p(x)$ (resp. $p(x_\alpha) \uparrow p(x)$) whenever $\theta \le x_n \uparrow x$ (resp. $\theta \le x_\alpha \uparrow x$).

We denote the family of all linear operators from a Riesz space X to another Riesz space Y, by $\mathcal{L}(X,Y)$ and the subspace of $\mathcal{L}(X,Y)$ consisting of all order bounded operators by $\mathcal{L}^b(X,Y)$. The subspaces of $\mathcal{L}^b(X,Y)$ containing order continuous and sequentially order continuous operators are respectively denoted by $\mathcal{L}^c(X,Y)$ and $\mathcal{L}^{so}(X,Y)$. For $Y \equiv \mathbb{R}$, the set of real numbers, we write $X^b \equiv \mathcal{L}^b(X,\mathbb{R})$, $X^c \equiv \mathcal{L}^c(X,\mathbb{R})$ and $X^{so} \equiv \mathcal{L}^{so}(X,\mathbb{R})$.

Concerning these spaces, we have

PROPOSITION 2.2. If Y is an order complete Riesz space, then $\mathcal{L}^b(X,Y)$ is an order complete Riesz space ordered by the cone $\tilde{K} = \{T \in \mathcal{L}^b(X,Y): T(x) \ge \theta, \forall x \in K\}$ where for T in $\mathcal{L}^b(X,Y)$ and $x \in X$, $|T|(|x|) = \sup\{|T(y)| : |y| \le |x|\}$. Further, $\mathcal{L}^{so}(X,Y)$ and $\mathcal{L}^c(X,Y)$ are bands in $\mathcal{L}^b(X,Y)$.

For monotone nets of operators in $\mathcal{L}^b(X,Y)$, we have [26]

THEOREM 2.3. {Let $T_\alpha : \alpha \in \Lambda\}$ be an increasing or decreasing net of operators in $\mathcal{L}^b(X,Y)$ such that $T_\alpha x \xrightarrow{(o)} Tx$, $x \in X$, for some $T \in \mathcal{L}(X,Y)$. Then $T \in \mathcal{L}^b(X,Y)$ and $T_\alpha \xrightarrow{(o)} T$ in $\mathcal{L}^b(X,Y)$.

In case Y is also equipped with a l.c.s. topology, we have another subclass of $\mathcal{L}^b(X,Y)$ as given in [7].

DEFINITION 2.4. An operator T in $\mathcal{L}^b(X,Y)$, where X is a Riesz space and (Y,τ) is an order complete l.c.s. Riesz space, is said to be o-precompact if it maps order bounded subsets of X into precompact subsets of Y. The class of all o-precompact operators from X to Y is denoted by $\mathcal{L}_{op}(X,Y)$.

PROPOSITION 2.5. The space $\mathcal{L}_{op}(X,Y)$ forms a band in $\mathcal{L}^b(X,Y)$, where X is a Riesz space and (Y,τ) is an order complete l.c.s. Riesz space with Lebesgue property.

NOTE. The above result is due to P.G. Dodds and D.H. Fremlin [6] for the special case of Banach lattices; however, in its present form it is to be found in [2] (with a simplified proof due to A.R. Schep).

We follow [18], [23] for the fundamentals of vector valued sequence spaces (VVSS). For the sake of convenience, let us recall [23] the vector spaces $\Omega(X)$ and $\Phi(X)$ consisting of all sequences and finitely non-zero sequences from a vector space X. A VVSS $\Lambda(X)$ is a subspace of $\Omega(X)$ containing $\Phi(X)$. Corresponding to a dual pair $\langle X,Y\rangle$ of vector spaces defined over the some field, the generalized Köthe dual $\Lambda^x(Y)$ of $\Lambda(X)$ is the vector space given by

$$[\Lambda(X)]^x \equiv \Lambda^x(Y) = \{\{y_i\} : y_i \in Y,\ i \ge 1 \text{ and } \sum_{i\ge1} |\langle x_i,y_i\rangle| < \infty,\ \forall\ \{x_i\} \in \Lambda(X)\}.$$

The generalized Köthe dual of $\Lambda^{x}(Y)$ is denoted by $\Lambda^{xx}(X)$ and so on. Members of $\Omega(X)$ are denoted by $\bar{x}$, $\bar{y}$, etc., i.e., $\bar{x} = \{x_i\}$, $\bar{y} = \{y_i\}$, $x_i, y_i \in X$, $\forall\ i \geq 1$. For $x \in X$ and $i \in N$, the set of natural numbers, we write

$$\delta_i^x = \{\theta, \theta, \ldots\ldots\ x, \theta, \theta, \ldots\ldots\} . \quad \text{i-th co-ordinate}$$

Observe that $\Phi(X)$ is spanned by $\{\delta_i^x : x \in X,\ i \in N\}$.

Let us also recall [18] that a VVSS $\Lambda(X)$ is <u>normal</u> if $\{\alpha_i x_i\} \in \Lambda(X)$ where $\{x_i\} \in \Lambda(X)$ and $\{\alpha_i\} \subseteq \mathbb{R}$ with $|\alpha_i| \leq 1$, $\forall\ i \geq 1$ and $\Lambda(X)$ equipped with a locally convex topology $\mathcal{T}$, is <u>simple</u> if for each $\mathcal{T}$-bounded set A, there exists an element $\bar{x}$ in $\Lambda(X)$ such that for each $\bar{y} = \{y_i\} \in A$, $y_i = \alpha_i x_i$ for some sequence $\{\alpha_i\}$ of scalars with $|\alpha_i| \leq 1$, $\forall\ i \geq 1$, i.e., A is contained in the normal hull of the set $\{\bar{x}\}$.

In addition, if X is also a Riesz space, the VVSS $\Lambda(X)$ is an ordered vector space relative to the co-ordinate wise ordering [10]. Indeed we have [10].

PROPOSITION 2.6. If X is an order complete Riesz space, then $\Omega(X)$ and $\Phi(X)$ are also order complete Riesz spaces and a Riesz subspace $\Lambda(X)$ of $\Omega(X)$ is order complete if and only if it is an ideal in $\Omega(X)$. Let us note that in the particular case when $X = \mathbb{R}$, $\Lambda(X)$ is normal if and only if it is a solid subspace of $\Omega(X)$, or equivalently, an ideal in $\Omega(X)$.

PROPOSITION 2.7. Let X be an order complete Riesz space such that $\Lambda(X)$ is an ideal in $\Omega(X)$. Then a linear functional $\bar{f}$ on $\Lambda(X)$ is in $[\Lambda(X)]^{so}$ if and only if there is a unique $\bar{z}$ in $\Lambda^{x}(X^{so})$, $\bar{z} = \{z_n\}$, $z_n \in X^{so}$ for $n \geq 1$ such that

$$\bar{f}(\bar{x}) = \langle \bar{x}, \bar{z} \rangle = \sum_{n=1}^{\infty} \langle x_n, z_n \rangle, \quad \forall\ \bar{x} = \{x_n\} \in \Lambda(X).$$

As examples of ordered vector valued sequence spaces (OVVSS), we recall the following spaces from [11], which are needed in the sequel and are defined over an order complete Riesz space X such that $\langle X, X^{so} \rangle$ forms a dual pair.

$$\ell^1(X) = \{\{x_n\} : x_n \in X,\ n \geq 1 \text{ and } \{\sum_{i=1}^{n} |x_i|\} \text{ order converges in } X\},$$

$$\ell^\infty(X) = \{\{x_n\} : x_n \in X,\ n \geq 1 \text{ and } \sup_n \{|x_n|\} \text{ exists in } X\}, \tag{2.8}$$

$$c(X) = \{\{x_n\} : x_n \in X,\ n \geq 1 \text{ and } \{x_n\} \text{ order converges in } X\},$$

$$c_o(X) = \{\{x_n\} : x_n \in X,\ n \geq 1 \text{ and } x_n \xrightarrow{(o)} \theta \text{ in } X\}.$$

One can easily verify that $\ell^1(X)$, $\ell^\infty(X)$ and $c_o(X)$ are ideals of $\Omega(X)$ and $c(X)$ is a Riesz subspace of $\Omega(X)$, which is not an ideal.

We also define

$$cs(X) = \{\{x_n\} : x_n \in X,\ n \geq 1 \text{ and } \{\sum_{i=1}^{n} x_i\} \text{ order converges in } X\}. \quad (2.9)$$

Note that the space cs(X) is an ordered vector subspace of $\Omega(X)$, which is not a Riesz space.

In particular, when $X \equiv \mathbb{R}$, the VVSS $\Lambda(X)$ written as λ in the sequel, is known as a scalar valued sequence space (SVSS).

Using a normal SVSS λ and a locally convex space (X,τ), another type of a VVSS introduced in [4] is given as

$$\lambda(X) = \{\bar{x} = \{x_i\}: x_i \in X,\ i \geq 1 \text{ and } \{p(x_i)\} \in \lambda,\ \forall\ p \in \mathcal{D}\} \quad (2.10)$$

If X is a locally convex space and λ is equipped with a normal $\mathfrak{S}$-topology T_λ compatible with the dual pair $\langle\lambda, \lambda^x\rangle$ and generated by the family $\{p_S : S \in \mathfrak{S}\}$ of seminorms, where for $\bar{\alpha} = \{\alpha_i\} \in \lambda$,

$$p_S(\bar{\alpha}) = \sup_{\{\beta_i\} \in S} \{\sum_{i \geq 1} |\alpha_i|\ |\beta_i|\} \quad (2.11)$$

and $\mathfrak{S}$ is a family of normal hulls of balanced convex $\sigma(\lambda^x,\lambda)$-bounded subsets of λ^x, covering λ^x, then the topology $T_{\lambda(X)}$ on $\lambda(X)$ generated by the family of seminorms $\{P_s \equiv p_s \circ p : S \in \mathfrak{S},\ p \in \mathcal{D}\}$ defined as

$$P_S(\bar{x}) = (p_S \circ p)(\bar{x}) = p_S(\{p(x_i)\}) \quad (2.12)$$

for $\bar{x} \in \lambda(X)$, is a Hausdorff locally convex topology; cf. [4], p.130.

3. TOPOLOGICAL PROPERTIES OF $\lambda(X)$.

Recalling the space $\lambda(X)$ and the topology $T_{\lambda(X)}$ defined as in (2.10) and (2.12), corresponding to a normal SVSS λ and a locally convex space X, we prove in this section the impact of topologies of λ and X on $T_{\lambda(X)}$ and vice-versa. To begin with, we have

PROPOSITION 3.1. If (X,τ) is a l.c.s. Riesz space generated by the family $\mathcal{D}$ of Riesz seminorms, then $\lambda(X)$ is an ideal in $\Omega(X)$ and $T_{\lambda(X)}$ is a locally convex solid topology on $\lambda(X)$.

PROOF. Straightforward.

A partial converse of the above result is contained in

PROPOSITION 3.2. Let X be a Riesz space equipped with a locally convex topology τ generated by the family $\mathcal{D}$ of seminorms, and $(\lambda(X), T_{\lambda(X)})$ be a locally convex solid Riesz space. Then τ is a locally convex solid topology on X.

PROOF. Consider $p \in \mathcal{D}$, $x,y \in X$ with $|x| \leq |y|$ and a member S of $\mathfrak{S}$ such that $M = \sup\{|\beta_i|: \{\beta_i\} \in S\} > 0$ for some $i \in$ N. Then

$$P_S(\delta_i^x) = (p_S \circ p)(\delta_i^x) \leq (p_S \circ p)(\delta_i^y) = P_S(\delta_i^y)$$

implies that $p(x) \leq p(y)$. Thus each $p \in \mathcal{D}$, is a Riesz seminorm.

From now onwards <u>in this section</u>, we assume that the pair (X,τ)

stands for a Hausdorff locally convex solid Riesz space, where the topology τ is generated by the family $\mathscr{D}$ of Riesz seminorms and λ is equipped with the normal topology $\eta(\lambda, \lambda^{x})$ which is the same as the absolute weak topology $|\sigma|(\lambda,\lambda^{x})$, cf. [1]; and which is an $\mathfrak{S}$-topology where $\mathfrak{S}$ corresponds to the normal balanced convex hulls of singleton sets in λ^{x}. In this case, we write the seminorms defined in (2.12) as

$$P_{\bar\beta}(\bar x) = \sum_{i=1}^{\infty} |\beta_i|\, p(x_i), \quad \{\beta_i\} \in \lambda^{x}, \tag{3.3}$$

where P corresponds to $p \in \mathscr{D}$, and $\mathscr{D}_{\lambda(x)}$ for the family $\{P_{\bar\beta} : p \in \mathscr{D}, \bar\beta \in \lambda^{x}\}$ of Riesz seminorms.

The interrelationship of various properties of Riesz seminorms, for the families $\mathscr{D}$ and $\mathscr{D}_{\lambda(x)}$ are exhibited in

PROPOSITION 3.4. A seminorm $p \in \mathscr{D}$ satisfies σ-Lebesgue property in X if and only if the corresponding seminorms $\{P_{\bar\beta} : \bar\beta \in \lambda^{x}\}$ satisfy the same in $\lambda(X)$.

PROOF. For proving the necessity, consider $\bar\beta = \{\beta_i\} \in \lambda^{x}$ and a sequence $\{\bar x^{n}\}$ in $\lambda(X)$ such that $\bar x^{n} \downarrow \bar\theta$ in $\lambda(X)$. Then $p(x_i^{n}) \to o$, for each $i \geq 1$ and $\sum_{i=1}^{\infty} |\beta_i|\, p(x_i^{n}) < \infty$, $\forall\, n \geq 1$. Hence for an arbitrary fixed $\varepsilon > o$, there exists $i_o \in N$ such that

$$\sum_{i=i_o}^{\infty} |\beta_i|\, p(x_i^{1}) < \varepsilon.$$

Consequently, for each $n \geq 1$

$$P_{\bar\beta}(\bar x^{n}) \leq \sum_{i=1}^{i_o} p(x_i^{n})\, |\beta_i| + \varepsilon .$$

As $\sum_{i=1}^{i_o} p(x_i^{n})\, |\beta_i| \to o$ as $n \to \infty$, $P_{\bar\beta}(\bar x^{n}) \to o$.

For proving the converse, consider a sequence $\{x_n\}$ in X such that $x_n \downarrow \theta$ in X and an element $\bar\beta$ of λ^{x} such that $i \in N$ corresponds to the non-zero co-ordinate of $\bar\beta$. Then

$$|\beta_i|\, p(x_n) = P_{\bar\beta}(\delta_i^{x_n}) \to o \quad \text{as} \quad n \to \infty ,$$

for $\delta_i^{x_n} \downarrow \bar\theta$ in $\lambda(X)$. Thus $p(x_n) \to o$.

PROPOSITION 3.5. A member p of $\mathscr{D}$ satisfies the Lebesgue property if and only if each $P_{\bar\beta}$ for $\bar\beta \in \lambda^{x}$, satisfies the same in $\lambda(X)$.

PROOF. In order to prove that $P_{\bar\beta}$ for $\bar\beta \in \lambda^{x}$, satisfies the Lebesgue property if p does, consider a net $\{\bar x^{\alpha} : \alpha \in \Lambda\}$ in $\lambda(X)$ with $\bar x^{\alpha} \downarrow \bar\theta$ in $\lambda(X)$. Then $p(x_i^{\alpha}) \to o$, $\forall\, i \geq 1$ and for an arbitrary fixed $\varepsilon > o$ as well as for fixed $\alpha_o \in \Lambda$, we can find $i_o \in N$ such that

$$\sum_{i=i_o}^{\infty} |\beta_i|\ p(x_i^{\alpha_o}) < \varepsilon$$

$$\Rightarrow \quad P_{\bar\beta}(\bar{x}^{\alpha}) = \sum_{i=1}^{\infty} |\beta_i|\ p(x_i^{\alpha}) \leq \sum_{i=1}^{i_o} |\beta_i|\ p(x_i^{\alpha}) + \varepsilon \ . \quad \forall\ \alpha \geq \alpha_o \ .$$

As $\sum_{i=1}^{i_o} |\beta_i|\ p(x_i^{\alpha}) \xrightarrow[\alpha]{} o$, $P_{\bar\beta}(\bar{x}^{\alpha}) \to o$.

Replacing sequence by net in the proof of the converse part of preceding result, we may infer the Lebesgue property of p from that of $P_{\bar\beta}$.

PROPOSITION 3.6. A seminorm p in $\mathcal{D}$ possesses pre-Lebesgue property if and only if each $P_{\bar\beta}$ for $\bar\beta \in \lambda^{X}$, possesses the same.

PROOF. For proving the necessity, consider an order bounded disjoint sequence $\{\bar{x}^n\}$ in $\lambda(X)$. Then $|\bar{x}^n| \leq \bar{x}$, for every $n \geq 1$ and for some $\bar{x} = \{x_i\} \in \lambda(X)$, and so $\{x_i^n\}$ is an order bounded disjoint sequence in X for each $i \in N$. Hence $p(x_i^n) \to o$ as $n \to \infty$ for each $i \in N$. Now for any $\bar\beta \in \lambda^X$ and $\varepsilon > o$, choose $i_o \in N$ such that

$$\sum_{i=i_o}^{\infty} |\beta_i|\ p(x_i) < \varepsilon.$$

We now proceed as in the proof of necessary part of Proposition 3.4 in order to get the result.

For sufficiency, observe that for each $i \geq 1$, $\{\delta_i^{x_n}\}$ is an order bounded disjoint sequence in $\lambda(X)$ for any disjoint order bounded sequence $\{x_n\}$ in X. Now use the pre-Lebesgue property of $P_{\bar\beta}$ for deriving the result.

PROPOSITION 3.7. p in $\mathcal{D}$ is a σ-Fatou seminorm if and only if $P_{\bar\beta}$ is so.

PROOF. Let p satisfy the σ-Fatou property. For proving the σ-Fatou property of $P_{\bar\beta}$ where $\bar\beta \in \Lambda^X$, consider $\bar\theta \leq \bar{x}^n \uparrow \bar{x}$ in $\lambda(X)$. Then $\theta \leq x_i^n \uparrow x_i$, $\forall\ i \geq 1$ and so $p(x_i^n) \uparrow_n p(x_i)$ for each $i \in N$. Consequently, $\forall\ m \geq 1$

$$\sum_{i=1}^{m} |\beta_i|\ p(x_i^n) \uparrow_n \sum_{i=1}^{m} |\beta_i|\ p(x_i) \ .$$

If a is any upper bound of the sequence $P_{\bar\beta}(\bar{x}^n)$, then

$$\sum_{i=1}^{m} |\beta_i|\ p(x_i) \leq a \ , \quad \forall \quad m \geq 1 \ .$$

Hence $P_{\bar\beta}(\bar{x}) \leq a$, i.e. $P_{\bar\beta}(\bar{x}^n) \uparrow_n P_{\bar\beta}(\bar{x})$.

The proof of the converse is analogous to the earlier proofs of converse parts and so omitted.

PROPOSITION 3.8. p in $\mathcal{D}$ is a Fatou seminorm if and only if each $P_{\bar\beta}$ is so for $\bar\beta \in \lambda^X$.

PROOF. Analogous to the proof of the preceding result and so omitted.

Concerning the spaces X and $\lambda(X)$, the preceding propositions immediately lead to

THEOREM 3.9. (X,τ) satisfies σ-Lebesgue (resp. Lebesgue, pre-Lebesgue, σ-Fatou or Fatou) property if and only if $(\lambda(X), T_{\lambda(X)})$ does so.

PROOF. Straightforward.

4. o-MATRIX TRANSFORMATIONS.

This section which is divided into two subsections incorporates results on o-matrix transformations from one OVVSS to another OVVSS defined corresponding to Riesz spaces. Whereas the first subsection deals with the o-precompactness of the o-matrix transformation, the second subsection includes characterizations of such transformations on particular OVVSS in terms of component linear operators. Before we pass on to these results, let us recall [10].

DEFINITION 4.1. Let X and Y be two order complete Riesz spaces such that $\Lambda(X)$ and $\mu(Y)$ are ideals in $\Omega(X)$ and $\Omega(Y)$ respectively. A linear map Z from $\Lambda(X)$ to $\mu(Y)$ is said to be an <u>o-matrix transformation</u> if there exists a matrix $[Z_{ij}]$ of linear maps from X to Y such that for every $\bar{x} = \{x_i\}$ in $\Lambda(X)$ and each $i \in N$, the sequence $\{\sum_{j=1}^{k} Z_{ij}(x_j)\}$ order converges to some $y_i \in Y$ and $Z(\bar{x}) = \bar{y}$, where $\bar{y} = \{y_i\}$; in such a case we write $Z \equiv [Z_{ij}]$. The transpose $Z^{\perp}$ of a matrix $Z \equiv [Z_{ij}]$ of linear maps $Z_{ij} : X \to Y$, $i,j \geq 1$ is defined as the transpose of the matrix of adjoint maps, i.e., $Z^{\perp} \equiv [Z^{*}_{ji}]$.

PROPOSITION 4.2. Let $Z \equiv [Z_{ij}]$ be an o-matrix transformations from $\Lambda(X)$ to $\mu(Y)$. The the following statements hold:

(i) For $x \in X$ and $i,j \in N$: $Z_{ij}(x) = (Z(\delta^{x}_{j}))_i$, the i^{th} co-ordinate of $Z(\delta^{x}_{j})$;

(ii) Z is positive if and only if the Z_{ij}'s are positive for each $i,j \in N$.

PROPOSITION 4.3. Let X and Y be order complete Riesz spaces such that $\langle X,X^{so}\rangle$ and $\langle Y,Y^{so}\rangle$ form dual pairs. Assume that the ideals $\Lambda(X)$ and $\mu(Y)$ in $\Omega(X)$ and $\Omega(Y)$ are in duality with $\Lambda^{x}(X^{so})$ and $\mu^{x}(Y^{so})$ respectively. If Z is an order bounded, sequentially order continuous linear map from $\Lambda(X)$ to $\mu(Y)$, given by the matrix $[Z_{ij}]$ of linear maps Z_{ij} from X to Y for $i,j \geq 1$, then the adjoint Z^{*} of Z is an order bounded, sequentially order continuous o-matrix transformation from $\mu^{x}(Y^{so})$ to $\Lambda^{x}(X^{so})$ such that $Z^{*} \equiv [Z^{*}_{ji}]$.

o-PRECOMPACTNESS OF o-MATRIX TRANSFORMATIONS. In order to consider the o-precompactness of the o-matrix transformation, we consider the range space $\mu(Y)$ as $\lambda(Y)$ defined as in (2.10) corresponding to a normal SVSS λ and a l.c.s. Riesz space Y where Y is equipped with the Lebesgue topology so that the topology $T_{\lambda(Y)}$ on $\lambda(Y)$ is generated by seminorms $\{Q_{\bar{\beta}} : \bar{\beta} \in \lambda^{x}, q \in \mathcal{D}_Y\}$, where $Q_{\bar{\beta}}$ is defined as in (3.3).

With these underlying assumptions, we prove

PROPOSITION 4.4. A positive o-matrix transformation $Z \equiv [Z_{ij}]$ from $\Lambda(X)$ to $\lambda(Y)$ is o-precompact if and only if each Z_{ij} is o-precompact.

PROOF. For necessity, use the fact that $Z_{ij}(x) = (Z(\delta_j^x))_i$, for each $i,j \geq 1$ and $x \in X$. In order to prove the converse part, for $i,j \in N$, define linear operators $Z_j^i : \Lambda(X) \to \lambda(Y)$ by

$$Z_j^i(\bar{x}) = \delta_i^{Z_{ij}(x_j)}, \quad \bar{x} = \{x_j\} \in \Lambda(X).$$

Since Z_{ij}'s are o-precompact operators from X to Y, Z_j^i's are also o-precompact; indeed, for $\bar{\beta} \in \lambda^X$ with $\beta_i \neq o$, (the case when $\beta_i = o$, trivially follows) and any ϵ-neighbourhood $U_{\bar{\beta}} = \{\bar{y} \in \lambda(Y): Q_{\bar{\beta}}(\bar{y}) < \epsilon\}$ of $\bar{\theta}$ in $\lambda(Y)$,

$$Z_j^i[-\bar{x},\bar{x}] \subset \delta_i^F + U_{\bar{\beta}},$$

where $\delta_i^F = \{\delta_i^f : f \in F\}$, F being the finite subset of Y obtained corresponding to the neighbourhood $U = \{y \in Y : q(y) < \epsilon/|\beta_i|\}$. Now for $i,n \in N$, define $Z_i^{(n)} : \Lambda(X) \to \lambda(Y)$ by

$$Z_i^{(n)} = \sum_{j=1}^{n} Z_j^i .$$

These linear operators are clearly o-precompact and for $\bar{x} \in \Lambda(X)$, $Z_i^{(n)}(\bar{x}) \xrightarrow{(o)} Z_i(\bar{x})$ as $n \to \infty$, where $Z_i : \Lambda(X) \to \lambda(Y)$ is defined by

$$Z_i(\bar{x}) = \delta_i^{(Z(\bar{x}))_i}, \quad \forall \ i \geq 1 ;$$

cf. Definition 4.1. As $(\lambda(Y),T_{\lambda(y)})$ is a Lebesgue space by Proposition 3.5, $Z_i^{(n)}(\bar{x})$ converges to $Z_i(\bar{x})$ in $T_{\lambda(y)}$.

In order to show the o-precompactness of each Z_i, fix $i \in N$, an interval $[-\bar{x},\bar{x}]$ for $\bar{x} \geq \bar{\theta}$ in $\Lambda(X)$ and a neighbourhood $\bar{U}$ of $\bar{\theta}$ in $\lambda(Y)$. Then we can find another neighbourhood $\bar{V}$ of $\bar{\theta}$ such that $\bar{V} + \bar{V} \subset \bar{U}$. Choose $n_o \in N$ such that

$$Z_i^{(n)}(\bar{x}) - Z_i(\bar{x}) \in \bar{V}, \quad \forall \ n \geq n_o ;$$

and a finite set $\bar{A}$ in $\lambda(Y)$ with

$$Z_i^{(n_o)} [-\bar{x},\bar{x}] \subset \bar{A} + \bar{V} .$$

Hence

$$Z_i[-\bar{x},\bar{x}] \subset \bar{A} + \bar{U}$$

and so Z_i is o-precompact. Consequently, $Z^{(n)}: \Lambda(X) \to \lambda(Y)$, where $Z^n = \sum_{i=1}^{n} Z_i$, $\forall\ n \geq 1$, are also o-precompact operators. As $Z^{(n)}(\bar{x}) \xrightarrow{(o)} Z(\bar{x})$ in $\lambda(Y)$, $\forall\ \bar{x} \in \Lambda(X)$, cf. [10], Proposition 3.5(i) and the topology of $\lambda(Y)$ is Lebesgue, $Z^{(n)}(\bar{x}) \to Z(\bar{x})$ in $T_{\lambda(Y)}$. Now proceeding as in the preceding paragraph, we infer the o-precompactness of Z. This completes the proof.

As an order bounded, sequentially order continuous o-matrix transformation $Z = [Z_{ij}]$ can be written as $Z \equiv [Z_{ij}^{+}] - [Z_{ij}^{-}]$, cf. Propositions 4.2.(v), (vi) and 4.3 of [10], the above result immediately leads to

THEOREM 4.5. Let $Z \equiv [Z_{ij}]$ be an order bounded, sequentially order continuous o-matrix transformation from $\Lambda(X)$ to $\lambda(Y)$. Then Z is o-precompact if and only if Z_{ij} is o-precompact for each $i,j \geq 1$.

CERTAIN o-MATRIX TRANSFORMATIONS. In this subsection, we characterize o-matrix transformations from $c(X)$, $\ell^{\infty}(X)$ to themselves, $cs(X)$ to $c(X)$ and derive necessary conditions for a matrix of linear operators to transform $\ell^{1}(X)$ into a simple OVVSS $\Lambda(X)$. We begin with the following general result:

PROPOSITION 4.6. Let X be a Riesz space, Y an order complete Riesz space and $\{T_n\}$ a sequence of positive linear operators from X to Y. Then a necessary and sufficient condition for the sequence $\{\sum_{i=1}^{n} T_i(x_i)\}$ to order converge in Y whenever $\{x_n\}$ order converges in X is that $\{\sum_{i=1}^{n} T_i\}$ order converges in $\mathcal{L}^{b}(X',Y)$.

PROOF. It suffices to prove the sufficient condition as necessity is immediate from Theorem 2.3.

For an order converging sequence $\{x_n\}$ in X with $|x_n| \leq x$, $\forall\, n \geq 1$, write

$$t_n = \sum_{i=1}^{n} T_i(x_i), \quad n \in N.$$

Then for $n \geq m$,

$$|t_n - t_m| = |\sum_{i=m}^{n} T_i(x_i)| \leq \sum_{i=m}^{n} T_i(x).$$

Since $\{\sum_{i=m}^{n} T_i(x)\}$, being order convergent, is order-Cauchy in Y, $\{t_n\}$ is order-Cauchy and so order converges in Y by Proposition 2.1.(iii).

Making use of the above result, we prove

PROPOSITION 4.7. Let X be an order complete Riesz space with diagonal property and Z_{ij}'s, positive linear operators from X into itself. Then $Z \equiv [Z_{ij}]$ is an o-matrix transformation from $c(X)$ to $c(X)$ so that the transformed sequence order converges to the same limit if and only if

(i) $\sum_{j=1}^{n} Z_{ij} \xrightarrow{(o)} Z_i$ as $n \to \infty$, for some positive linear operator Z_i, $\forall\; i \geq 1$;

(ii) $Z_i(x) \xrightarrow{(o)} x$ as $i \to \infty$ $\forall\; x \in x$; and

(iii) $Z_{ij}(x) \xrightarrow{(o)} \theta$ as $i \to \infty$, $\forall\; x \in X$ and for each $j \in N$.

PROOF. We first derive the necessary conditions (i), (ii) and (iii).

(i) Since $\{\sum_{j=1}^{n} Z_{ij}(x_j)\}$ order converges in X, for each $i \in N$ and for each $\bar{x} = \{x_j\} \in c(X)$, (i) follows from Proposition 4.6.

(ii) For $x \in X$, consider the constant sequence $\{x_n\}$, $x_n = x$, $\forall\, n \geq 1$ in $c(X)$. Then $Z_i(x) \xrightarrow{(o)} x$ by the hypothesis.

(iii) This follows from the hypothesis, equalities $Z_{ij}(x) = (Z(\delta_j^x))_i$, $\forall\, i,j \in N$ and $(\delta_j^x)_i \xrightarrow{(o)} \theta$ as $i \to \infty$ in X, for each $j \geq 1$.

For proving the sufficiency, let us consider a sequence $\bar{x} = \{x_j\} \in c(X)$ with $x_j \xrightarrow{(o)} x$ in X. Then by Proposition 4.6, $\sum_{j=1}^{n} Z_{ij}(x_j) \xrightarrow{(o)} t_i$, say, in X for each $i \geq 1$. Hence by Proposition 2.1(i), for any $\varepsilon > o$, we can find $j_o \in N$ such that

$$|x_j - x| \leq \varepsilon u, \quad \forall\ j \geq j_o ,$$

where u is the regulator of convergence of the sequence $\{x_n\}$. Let $v \in X$ be such that $Z_i(u) \leq v$, $\forall\, i \geq 1$. Then

$$\begin{aligned}|t_i - x| &\leq \Big|\text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j) - \text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x)\Big| + |Z_i(x) - x| \\ &\leq \Big|\sum_{j=1}^{j_o} Z_{ij}(x_j - x)\Big| + \Big|\sum_{j=j_o}^{\infty} Z_{ij}(x_j - x)\Big| + |Z_i(x) - x| \\ &\leq \sum_{j=1}^{j_o} Z_{ij}(|x_j - x|) + \varepsilon v + |Z_i(x) - x| ,\end{aligned}$$

where $\sum_{j=j_o}^{\infty} Z_{ij}(x_j - x) = \text{o-}\lim_{n\to\infty} \sum_{j=j_o}^{n} Z_{ij}(x_j - x)$. Consequently, by (ii), (iii) and Proposition 2.1(i), $t_i \xrightarrow{(o)} x$ in X. This completes the proof.

NOTE. Observe that the restriction of Z on the space $c_o(X)$ has the range contained in $c_o(X)$ under the conditions (i), (ii) and (iii).

For the space $\ell^\infty(X)$, we have the following characterization of o-matrix transformation.

THEOREM 4.8. Let X be an order complete Riesz space. Then a matrix $Z \equiv [Z_{ij}]$ of positive linear operators Z_{ij} from X to X, transforms $\ell^\infty(X)$ into itself if and only if

(i) for each i in N, $\sum_{j=1}^{n} Z_{ij} \xrightarrow{(o)} Z_i$ as $n \to \infty$, for some positive linear operator Z_i on X; and

(ii) for each $x \in X$, $\{Z_i(x): i \geq 1\}$ is an order bounded subset of X.

PROOF. It suffices to prove the sufficiency of the conditions (i) and (ii) as necessity is immediate from the definition of o-matrix transformation Z from $\ell^\infty(X)$ into itself. Indeed, if $\bar{x} = \{x_j\} \in \ell^\infty(X)$, then the sequence $\{\sum_{j=1}^{n} Z_{ij}(x_j): n \geq 1\}$ is order-Cauchy by (i) and so order converges for every $i \geq 1$ by Proposition 2.1(iii). Let

$$t_i = \text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j) , \qquad i \geq 1.$$

Since $|x_j| \leq x$, for each $j \geq 1$ and for some $x \in X$, $\{t_i\} \in \ell^\infty(X)$ by (ii).

In case of the matrices of linear operators transforming the space cs(X) to c(X), we have the following two results dealing separately the sufficient and necessary conditions:

THEOREM 4.9. Let X be an order complete Riesz space with diagonal property and $Z \equiv [Z_{ij}]$, a matrix of positive sequentially order continuous linear operators satisfying the conditions

(i) for each $i \geq 1$, $\text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} |Z_{ij} - Z_{ij+1}| = Z_i$, for some sequentially order continuous linear operator Z_i on X;

(ii) $\{Z_i\}$ is an order bounded sequence in $\mathcal{L}^{so}(X) \equiv \mathcal{L}^{so}(X,X)$; and

(iii) for each $j \geq 1$, $Z_{ij} \xrightarrow{(o)} I$ as $i \to \infty$ in $\mathcal{L}^{so}(X)$ where I is the identity map on X.

Then Z is an o-matrix transformation from cs(X) to c(X) so that the transformed sequence order converges to the sum of the original sequence.

PROOF. We first prove the result for those elements $\bar{x} = \{x_j\} \in$ cs(X), which have the zero sum; indeed, if $\text{o-}\lim_{n\to\infty} \sum_{i=1}^{n} x_j = \theta$, for $\bar{x} = \{x_j\} \in$ cs(X), we show that $t_i = \text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} Z_{ij}(x_j)$, for each $i \in N$, exists and $t_i \xrightarrow{(o)} \theta$ in X.

Let us write $s_m = \sum_{j=1}^{m} x_j$, $m \in N$. Then $s_m \xrightarrow{(o)} \theta$ in X and so for given $\varepsilon > o$, we can find $j_o \in N$ such that $|s_j| \leq \varepsilon u$, $\forall\, j \geq j_o$, where u is the regulator of convergence. Let $A, B \in \mathcal{L}^{so}(X)$ be such that $|Z_i| \leq A$ and $|Z_{i1}| \leq B$, $\forall\, i \geq 1$. Then using the equalities

$$\sum_{j=1}^{m} Z_{ij}(x_j) = \sum_{j=1}^{m-1} (Z_{ij}-Z_{ij+1})(s_j) + Z_{im}(s_m):\ \forall\, i \geq 1,\ m \geq 1; \qquad (*)$$

and

$$\sum_{j=m}^{m+p} Z_{ij}(x_j) = \sum_{j=m}^{m+p-1} (Z_{ij}-Z_{ij+1})(s_j) + Z_{im+p}(s_{m+p}) - Z_{im}(s_{m-1}), \qquad (**)$$
$$\forall\ i,m,p \in N,$$

we infer that

$$\left|\sum_{j=m}^{m+p} Z_{ij}(x_j)\right| \leq \varepsilon\,(3A+2B)u;\quad \forall\ m \geq j_o\ ,$$

and

$$t_i = \text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} (Z_{ij} - Z_{ij+1})(s_j)\ ,\quad \forall\ i \geq 1\ . \qquad (+)$$

Also, from (+) and the condition (iii), we deduce that $t_i \xrightarrow{(o)} \theta$. Thus $Z(\bar{x}) \in c(X)$.

If $\bar{x} \in$ cs(X) is such that $\sum_{j=1}^{\infty} x_j = x \neq \theta$, then the sequence $\{y_j\}$ defined by $y_1 = x_1 - x$, $y_j = x_j$, for $j \geq 2$, is in cs(X) with zero sum. As

$$\sum_{j=1}^{m} Z_{ij}(x_j) = \sum_{j=1}^{m} Z_{ij}(y_j) + Z_{i1}(x) , \quad \forall \quad m \geq 1 ;$$

and $Z_{ij}(x) \xrightarrow{(o)} x$ in X, the result follows from the preceding paragraph.

THEOREM 4.10. Let X be an order complete Riesz space and $Z \equiv [Z_{ij}]$, a matrix of positive and sequentially order continuous linear operators from X into itself such that $Z_{ij} \geq Z_{ij+1}$, for each i,j in N. Assume that Z transforms cs(X) to c(X) so that the transformed sequence order converges to the sum of the original sequence. Then

(i) for each $j \geq 1$ and $x \in X$, $Z_{ij}(x) \xrightarrow{(o)} x$ as $i \to \infty$;

(ii) for each $i \geq 1$, $Z_{in} \xrightarrow{(o)} Z_i$ as $n \to \infty$, where $Z_i \in \mathcal{L}^{so}(X)$; and

(iii) for each $x \in X$, $\{Z_i(x)\}$ is an order bounded sequence of X.

PROOF. (i) This is immediate from the equalities

$$Z_{ij}(x) = (Z(\delta_j^x))_i , \quad \forall \quad i,j \quad N \quad \text{and} \quad x \in X.$$

(ii) As $\theta \leq Z_{in}(x) \uparrow_n$ in X, for each x in K and $i \in N$ and X is order complete, $Z_{in}(x) \uparrow v_i$, for some $v_i \in X$ and for each $i \in N$. Hence, if we define linear operators Z_i's on X as

$$Z_i(x) = \text{o-}\lim_{n\to\infty} Z_{in}(x) ,$$

for $x \in X$, then in view of Proposition 2.2 and Theorem 2.3, $\{Z_i : i \geq 1\} \subset \mathcal{L}^{so}(X)$.

(iii) for each $x \in K$, observe that

$$\theta \leq Z_i(x) \leq Z_{i1}(x) , \quad \forall \quad i \geq 1 ,$$

since $\theta \leq Z_{in}(x) \leq Z_{i1}(x)$, for each i,n in N and $Z_{in}(x) \xrightarrow{(o)} Z_i(x)$ as $n \to \infty$, in X by (ii). Hence (iii) follows from (i).

Restricting further the linear operators Z_{ij} in the hypothesis of the above theorem, we have

THEOREM 4.11. Let $Z \equiv [Z_{ij}]$ and X be as in Theorem 4.10. Assume further that $Z_{ij} \geq Z_{i+j,j}$, $\forall$ i,j N. Then $A \equiv [A_{ij}]$, where $A_{ij} = Z_{ij} - Z_{ij+1}$, $i,j \in N$, is an o-matrix transformation from c(X) to $c_o(x)$.

PROOF. Observe that we have one-one onto correspondence R between c(X) and cs(X), defined by $R(\bar{x}) = \bar{u}$, where $\bar{x} = \{x_j\} \in c(X)$ and $\bar{u} = \{u_j\}$ is given by $u_1 = x_1$ and $u_j = x_j - x_{j-1}$, $j \geq 2$.

Let us now consider a positive $\bar{x} = \{x_n\}$ in c(X) with $x_n \xrightarrow{(o)} x$ in X, for some $x \in X$. Then with $\bar{u}$ as defined above, $Z(\bar{u}) \in c(X)$ and $(Z(\bar{u}))_i \xrightarrow{(o)} x$ by the hypothesis. Also, from the proof of the part (ii) of the preceding theorem, $Z_{in}(x) \uparrow_n v_i$, for some $v_i \in X$ and for each $i \in N$. Further, $Z_{in}(x_n - x) \xrightarrow{(o)} \theta$ as $n \to \infty$, $\forall$ $i \geq 1$. Hence

$$(Z(\bar{u}))_i = \text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} (Z_{ij}-Z_{ij+1})(x_j) + v_i , \quad \forall \quad i \geq 1. \qquad (*)$$

Note that $v_i \downarrow x$ under the additional restriction, namely $Z_{ij} \geq Z_{i+1,j}$, $\forall$ $i,j \geq 1$. Hence if

$$s_i = \text{o-}\lim_{n\to\infty} \sum_{j=1}^{n} (Z_{ij}-Z_{ij+1})(x_j) = (Z(\bar{u}))_i - v_i \ , \quad i \geq 1,$$

then $s_i \xrightarrow{(o)} \theta$ in X. Thus $A(\bar{x}) = \{s_i\} \in c_o(X)$ for $\bar{x} \in c(X)$.

Lastly, we prove

THEOREM 4.12. Let X be an order complete Riesz space such that $<X,X^{so}>$ forms a dual pair and $\sigma(X^{so},X)$-bounded sets are order bounded in X^{so}. Also, assume that $\Lambda(X)$ is an ideal in $\Omega(X)$ and is a simple space for the topology $\sigma(\Lambda(X),\Lambda^{x}(X^{so}))$. For a matrix $Z \equiv [Z_{ij}]$ of positive linear operators from X into itself, consider the following statements:

(i) Z is an order bounded sequentially order continuous, o-matrix transformation from $\ell^1(X)$ to $\Lambda(X)$.

(ii) The adjoint Z^* of Z is an order bounded, sequentially order continuous o-matrix transformation from $\Lambda^{x}(X^{so})$ to $\ell^{\infty}(X^{so})$.

(iii) For $x \in X$, the sequence $\{a_j^x : j \geq 1\}$, where $a_j^x = \{Z_{ij}(x) : i \geq 1\}$, is an order bounded set in $\Lambda(X)$.

Then (i) => (ii) and (i) => (iii). Further, if (iii) holds, then the partial statement of (ii), namely, "Z^* is an o-matrix transformation from $\Lambda^{x}(X^{so})$ to $\ell^{\infty}(X^{so})$" holds.

PROOF. (i) => (ii). Since $(\ell^1(X))^x = \ell^{\infty}(X^{so})$, cf. [11]. Proposition 3.5, the implication follows from Proposition 4.3.

(i) => (iii) For a given j in N and $x \in K$, we first prove that $a_x^j \in \Lambda(X)$. Indeed, it is immediate as $a_x^j = Z(\delta_j^x)$ and $\delta_j^x \in \ell^1(X)$.

In order to show that the set $\{a_x^j : j \geq 1\}$ is $\sigma(\Lambda(X), \Lambda^{x}(X^{so}))$ bounded, consider a positive element $\bar{f} = \{f_j\}$ in $\Lambda^{x}(X^{so})$. Then from the preceding implication, $Z^*(\bar{f}) \in \ell^{\infty}(X^{so})$ and so there exists $g \in X^{so}$ such that

$$|<(Z^*(\bar{f}))_i,x>| = \Big|\sum_{j=1}^{\infty} f_j(Z_{ji}(x))\Big| \leq g(x), \quad \forall \quad i \geq 1. \tag{+}$$

Hence $\{a_x^j : j \geq 1\}$ is $\sigma(\Lambda(X), \Lambda^{x}(X^{so}))$-bounded and so order bounded in $\Lambda(X)$ by the hypothesis.

For the last statement, we first show that $\{\sum_{j=1}^{n} Z_{ji}^*(f_j)\}$ order converges in X^{so}, for a positive element $\bar{f} = \{f_j\}$ in $\Lambda^{x}(X^{so})$. Indeed, for $x \geq \theta$ and $n \in N$,

$$\sum_{j=1}^{n} Z_{ji}^* f_j(x) = \sum_{j=1}^{n} f_j \, a_{j,x}^i \ ,$$

=> the set $\{\sum_{j=1}^{n} Z_{ji}^* f_j(x): n \geq 1\}$ is bounded in R and so $\sum_{j=1}^{n} Z_{ji}^*(f_j)$ order converges in X^{so} by Theorem 2.2 and Proposition 2.3.

For proving that $Z^*(\bar{f}) \in \ell^{\infty}(X^{so})$, for $x \geq \theta$, choose $\bar{y} \in \Lambda(X)$ such that $|a_x^j| \leq \bar{y}$, for every $j \geq 1$. Hence

$$|<(Z^*(\overline{f}))_i, x>| = |\sum_{j=1}^{\infty} f_j(Z_{ji}(x)| = |\sum_{j=1}^{\infty} f_j(a^i_{j,x})|$$

$$\leq \sum_{j=1}^{\infty} f_j(y_j), \quad \forall \quad i \geq 1.$$

Thus $\{(Z^*(\overline{f}))_i : i \geq 1\}$ is order bounded in X^{so} and the result follows.

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