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RELATIVE INJECTIVITY AND CS-MODULES

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ABSTRACT. In this paper we show that a direct decomposition of modules $M \oplus N$, with N homologically independent to the injective hull of M, is a CS-module if and only if N is injective relative to M and both of M and N are CS-modules. As an application, we prove that a direct sum of a non-singular semisimple module and a quasi-continuous module with zero socle is quasi-continuous. This result is known for quasi-injective modules. But when we confine ourselves to CS-modules we need no conditions on their socles. Then we investigate direct sums of CS-modules which are pairwise relatively injective. We show that every finite direct sum of such modules is a CS-module. This result is known for quasi-continuous modules. For the case of infinite direct sums, one has to add an extra condition. Finally, we briefly discuss modules in which every two direct summands are relatively injective.

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INTRODUCTION.

Let R be a ring and M be a right R-module. The module M is a CS-module (for complement submodules are direct summands) provided every submodule of M is essential in a direct summand of M, or equivalently, every closed submodule of M is a direct summand. This is the terminology of Chatters and Hajarnavis [2], one of the first papers to study this concept.

Later other terminology, such as extending module, has been used in place of CS. CS-modules are generalizations of (quasi) continuous modules, which, in turn, are generalizations of injective and quasi-injective modules.

All modules will be unital right modules over a ring R with unit.

A submodule N of a module M is closed in M, if it has no proper essential extensions in M. $X \leq^{e} M$ and $Y \leq^{\oplus} M$ signify that X is an essential submodule, and Y is a direct summand, of M. The injective hull of a module M will be denoted by E(M). A module M is quasi-continuous if it is a CS-module and has the following property (C_3): for all $X, Y \leq^{\oplus} M$, with $X \cap Y = 0$, one has $X \oplus Y \leq^{\oplus} M$. M is continuous if it is a CS-module and satisfy (C_2): if a submodule N of M is isomorphic to a direct summand of M, then N is a direct summand of M.

For modules M, N and for any $f \in \mathrm{Hom}(M,E(N))$, let $X_f =: \{ m \in M : f(m) \in N \}$, and define the submodule B_f of $M \oplus N$ by $B_f = \{ m + f(m) : m \in X_f \}$. It is clear that X_f is an essential submodule of M, and that $X_f \cap B_f = \ker f$. If $\pi : M \oplus N \to M$ is the projection, then $\pi|B_f$ is a monomorphism and $X_f = \pi(B_f)$.

LEMMA 1. Let M, N be R-modules. Then for every $f \in \mathrm{Hom}(M,E(N))$, B_f and N are complements, of each other, in $M \oplus N$. If $\mathrm{Hom}(N,E(M)) = 0$, then N is the unique complement of B_f in $M \oplus N$.

PROOF. It is clear that $B_f \cap N = 0$. Now let L be a submodule of $M \oplus N$ such that $L \cap N = 0$, and that $B_f \subseteq L$. Let π and π^* be the natural projections of $M \oplus N$ onto M and N respectively. Then $B_f = L$, once we show that $\pi^*(l) = f\pi(l)$, for all $l \in L$. To this end, let $0 \neq (\pi^* - f\pi)(l)$ for some $l \in L$. By the essentiality of E(N) over N, there exists $r \in R$, such that $0 \neq \pi^*(lr) - f\pi(lr) \in N$. But $\pi^*(lr) - f\pi(lr) = lr - [\pi(lr) + f\pi(lr)] \in N \cap L = 0$ which is a contradiction.

For the second part of the lemma, let Y be a submodule of $M \oplus N$ such that $Y \cap B_f = 0$. If $Y \cap X_f \neq 0$, then the restriction of f to $Y \cap X_f$ provides a non-zero element of $\mathrm{Hom}(N,E(M))$, which contradicts our assumption. Then $Y \cap X_f = 0$, and thus $Y \cap M = 0$ (due to $X_f \subseteq^e M$). It follows that $\pi^*|_Y$ is a monomorphism, and thus $\pi(Y) = 0$ (otherwise it leads to a contradiction). Therefore $Y \subseteq N$.

LEMMA 2. Let M and N be modules. Then N is M-injective if and only if $M \oplus N = B_f \oplus N$; for every $f \in \mathrm{Hom}(M,E(N))$.

PROOF. N is M-injective if and only if $X_f = M$, and $M \oplus N = B_f \oplus N$ if and only if $X_f = M$; for every $f \in \mathrm{Hom}(M,E(N))$.

REMARK 3. It is known that a module M is quasi-continuous if and only if $M = X \oplus Y$, for any submodules X, Y which are complements of each other. An immediate consequence of Lemma 1, and Lemma 2, is that if $M \oplus N$ is quasi-continuous, then M and N are relatively injective ([10], Proposition 2.1).

The uniqueness, in the second part of Lemma 1, is used in Proposition 9 to obtain a generalization of the result given in Kamal and Müller [7].

LEMMA 4. ([3], Proposition 1.5) Let A and B be submodules of a module M, with $A \subseteq B$. If A is closed in B and B is closed in M, then A is closed in M.

COROLLARY 5. Every direct summand of a CS-module is a CS-module.

PROOF. Is obvious.

LEMMA 6. Let M and N be modules, and let A be a submodule of $M \oplus N$, with $A \cap N = 0$. Then A is closed in $M \oplus N$ if and only if $A = \{ x + f(x) : x \in X \}$, where X is a closed submodule of X_f, for some $f \in \mathrm{Hom}(M,E(N))$. Moreover, if M is uniform, then A is non-zero closed in $M \oplus N$ if and only if $A = B_f$, for some $f \in \mathrm{Hom}(M,E(N))$.

PROOF. Let π be the projection of $M \oplus N$ onto M. Since $A \cap N = 0$, there exists $f \in \mathrm{Hom}(M,E(N))$ such that $f\pi(a) = (1-\pi)(a)$ (i.e. $f\pi(a) + \pi(a) = a$) for all $a \in A$. Hence $A = \{ x + f(x) : x \in \pi(A) \}$. It is easy to check that $\pi(A)$ is contained in X_f. Now if $\pi(A) \subseteq^e Y \subseteq X_f$, then $A \subseteq^e \{ y + f(y) : y \in Y \} \subseteq M \oplus N$. Since A is closed in $M \oplus N$, it follows that $Y = \pi(A)$; and thus $\pi(A)$ is closed in X_f. Now if M is uniform, and A is non-zero closed in $M \oplus N$, then $0 \neq \pi(A)$ is closed in the uniform module X_f, and thus $\pi(A) = X_f$. Therefore $A = B_f$.

Conversely, let $A = \{ x + f(x) : x \in X \}$ where X is closed in X_f, and $f \in Hom(M,E(N))$. It is clear that $A \leq B_f$, and that A has a proper essential extension in B_f if and only if X has a proper essential extension in X_f. Since X is closed in X_f, it follows that A is closed in B_f. But, by Lemma 1, B_f is closed in $M \oplus N$. Therefore A is closed in $M \oplus N$.

Observe that, the major step in studying the property CS for modules is the one that deals with the characterization of all closed submodules. So that Lemma 6 (including its special case, i.e. when M is uniform), can be used in characterizing CS-modules, which are direct sums of two uniform modules (see [8]).

COROLLARY 7. Let M and N be modules. Then N is M-injective if and only if any closed submodule A of $M \oplus N$, with $A \cap N = 0$, must have the following form $A = \{ x + f(x) : x \in X \}$, where X is closed in M and $f \in Hom(M,E(N))$.

PROOF. ($\Rightarrow$). By Lemma 6, and since N is M-injective if and only if $X_f = M$; for every $f \in Hom(M,E(N))$.

($\Leftarrow$): Let $f \in Hom(M,E(N))$ be an arbitrary element. By Lemma 1, B_f is a closed submodule of $M \oplus N$ with $B_f \cap N = 0$. Then B_f has the form above for some $g \in Hom(M,E(N))$, and for some closed submodule Y of M. It follows that, $X_f = \pi(B_f) = Y$ is closed in M; where $\pi : M \oplus N \to M$ is the projection onto M. Since X_f is essential in M, we deduce $X_f = M$.

COROLLARY 8. Let M be a CS-module, and let N be M-injective. Then every closed submodule A of $M \oplus N$, with $A \cap N = 0$ is a direct summand.

PROOF. Let A be a closed submodule of $M \oplus N$, with $A \cap N = 0$. Then, by Corollary 7, $A = \{x + f(x) : x \in X\}$, where X is closed in M and $f \in Hom(M,E(N))$. Since M is a CS-module, we have that $M = X \oplus M^*$. It is easy to check that $A \oplus N = X \oplus N$; and thus $M \oplus N = A \oplus M^* \oplus N$.

PROPOSITION 9. Let M and N be modules. Let $Hom(N,E(M)) = 0$. Then N is M-injective and M is a CS-module if and only if every closed submodule A, of $M \oplus N$, with $A \cap N = 0$, is a direct summand.

PROOF. The necessary condition follows immediately from Corollary 8.

The sufficient condition: By Lemma 4, and since $A \cap N = 0$, for every closed submodule A of M, M is a CS-module. To show that N is M-injective it is enough to show $M \oplus N = B_f \oplus N$, for every $f \in Hom(M,E(N))$. By Lemma 1, B_f is a closed submodule of $M \oplus N$, with $B_f \cap N = 0$; and hence B_f is a direct summand. Since, by Lemma 1 N is the unique complement of B_f in $M \oplus N$, we have that $M \oplus N = B_f \oplus N$.

Theorem 10. Let M and N be modules. Let $Hom(N,E(M)) = 0$. Then $M \oplus N$ is a CS-module if and only if M and N are CS-modules, and N is M-injective.

PROOF. ($\Rightarrow$) Corollary 5, and Proposition 9.

($\Leftarrow$) By Proposition 9, it is enough to show that any closed submodule A of $M \oplus N$, with $A \cap N \neq 0$, is a direct summand. To this end, let A_1 be a maximal essential extension of $A \cap N$ in A . By Lemma 4, A_1 is closed in $M \oplus N$, with $A_1 \cap M = 0$. By Lemma 6 and since $Hom(N,E(M)) = 0$, it follows that $A_1 \leq N$. Since N is a CS-module, we get that $N = A_1 \oplus N^*$. Thus $A = A_1 \oplus A_2$, where $A_2 =: A \cap (M \oplus N^*)$ is a closed submodule of $M \oplus N^*$, with $A_2 \cap N^* = 0$. Since N^* is M-injective, it follows, by Corollary 8, that $A_2 \leq^{\oplus} M \oplus N^*$. Therefore A is a direct summand of $M \oplus N$.

The following are immediate consequence of Theorem 10.

COROLLARY 11. ([7], Theorem 1) Let M and N be modules, where M is non-singular and N is singular. Then $M \oplus N$ is a CS-module if and only if N is M-injective, and M, N are CS-modules.

COROLLARY 12. Let M and N be modules, where N is semisimple and M with zero socle. Then $M \oplus N$ is a CS-module if and only if M is a CS-module and N is M-injective.

PROPOSITION 13. Let M be a non-singular semisimple R-module, and N be an R-module, with Soc(N) = 0. Then N is quasi-continuous if and only if $M \oplus N$ is quasi-continuous.

PROOF. Let N be quasi-continuous. We show that Hom(N,E(M)) = 0. Let f be an arbitrary element of Hom(N,E(M)), and let $\operatorname{Ker} f \leq^e N_1 \leq N$. Then, for every $n_1 \in N_1$, there exists an essential right ideal I of R such that $f(n_1)I = 0$. Since E(M) is non-singular, it follows that $f(n_1) = 0$; and thus $N_1 = \operatorname{Ker} f$. Hence Kerf has no proper essential extensions in N; i.e. Ker f is closed in N. Since N is quasi-continuous, hence a CS-module, we have $N = \operatorname{Ker} f \oplus N^*$. Since Soc(N) = 0, it follows that $N^* = 0$; and thus f = 0. Then M and N are relatively injective quasi-continuous modules; and therefore $M \oplus N$ is quasi-continuous (see [10], Corollary 2.14).

REMARK 14. In Proposition 13, if M is semisimple but not non-singular or $Soc(N) \neq 0$, then $M \oplus N$ need not be quasi-continuous. This is illustrated in the following examples.

EXAMPLE 1. Let $M = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number; and let $N = \mathbb{Z}$. Then, as $\mathbb{Z}$-modules, M is singular semisimple and Soc(N) = 0, while $M \oplus N$ is not even a CS-module (by Corollary 12).

EXAMPLE 2. Let F be a field, $R = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$. Let $M = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}$. Then M is a non-singular simple R-module, and N is uniform, hence a quasi-continuous R-module, with non-zero socle, where $R_R = M \oplus N$. One can easily show that $I \leq^{\oplus} R_R$, $I \cap M = 0$, while $I \oplus M \lneq^e R_R$; where $I = \{ \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} : a \in F \}$. This shows that R_R does not satisfy (C_3), i.e. $M \oplus N$ is not quasi-continuous.

PROPOSITION 15. Let M and N be R-modules, where M is non-singular and N is M-injective. Then $M \oplus N$ is a CS-module if and only if M and N are both CS-modules.

PROOF. (⇐) Let A be a closed submodule of $M \oplus N$. Let A_1 and A_2 be maximal essential extensions in A of $A \cap M$ and $A \cap N$, respectively. Then A_i (i =1,2) are closed in $M \oplus N$, by Lemma 4. For each $a_2 \in A_2$, $a_2 = m + n$; $m \in M$ and $n \in N$. Since $A \cap N$ is essential in A_2, there exists an essential right ideal I of R such that $a_2 I \leq A \cap N$. It follows that mI = 0. Since M is non-singular, we deduce m = 0; and thus $A_2 \leq N$. Since N is a CS-module, and A_2 is closed in N, we get $N = A_2 \oplus N^*$, for some submodule N^* of N. By the essentiality of A_1 over $A \cap M$, we have $A_1 \cap N = 0$. Since N is M-injective, it follows that $M \oplus N = A_1 \oplus A_2 \oplus M^* \oplus N^*$ for some submodule M^* of M, by Corollary 7. Hence $A = \oplus_{i=1}^{3} A_i$, where $A_3 =: A \cap [M^* \oplus N^*]$. It is clear that A_3 is closed in $M^* \oplus N^*$, with $A_3 \cap N^* = 0$. By Corollary 5, M^* and N^* are CS-modules, where N^* is M^*-injective.

Thus, by Corollary 8, $A_3 \le^{\oplus} M^* \oplus N^*$; and therefore $A \le^{\oplus} M \oplus N$.

($\Rightarrow$) Is obvious.

REMARK 16. If M is not non-singular and N is M-injective, where both of M and N are CS-modules, then $M \oplus N$ need not be a CS-module. This is illustrated in Remark 14 (Example 1) by taking $M = \mathbb{Z}/p\mathbb{Z}$, and $N = \mathbb{Z}$.

In Remark 14 (Example 2), we have shown that Soc(N) = 0 is not avoidable condition for Proposition 13. This is not the case for CS-modules, as it is shown in the following.

COROLLARY 17. Let M be a non-singular semisimple module. Then $M \oplus N$ is a CS-module for any CS-module N.

THEOREM 18. Let $M = \oplus_{i=1}^{n} M_i$, where the M_i are M_j-injective for all $i \neq j$. Then M is a CS-module if and only if M_i are CS-modules for all i.

PROOF. If M is a CS-module, then, by Corollary 5, M_i is a CS-module for all i. We show the converse by induction. It is sufficient to prove the result when n = 2. Let $M = M_1 \oplus M_2$, where the M_i are CS-modules and M_j-injective for $i \neq j$ (i,j = 1,2). Let A be a closed submodule of M. Let $A_2 =: A \cap M_2$, and B_2 be a maximal essential extension of A_2 in A. Hence B_2 is closed in M, with $B_2 \cap M_1 = 0$. Since M_1 is M_2-injective, it follows by Corollary 7, that $B_2 = \{ x + f(x) : x \in X_2\}$; for some closed submodule X_2 of M_2 , and for some $f \in \mathrm{Hom}(M_2, E(M_1))$. Since M_2 is a CS-module, $M_2 = X_2 \oplus M_2^*$. Since $B_2 \le X_2 \oplus M_1$, it follows that $X_2 \oplus M_1 = B_2 \oplus M_1$; and hence $M = M_1 \oplus B_2 \oplus M_2^*$. Thus $A = B_2 \oplus B_1$, where $B_1 =: A \cap [M_1 \oplus M_2^*]$. It is clear that $B_1 \cap M_2^* = 0$, and that M_2^* is M_1-injective. Since M_1 is a CS-module; we have $B_1 \le^{\oplus} M_1 \oplus M_2^*$ (Corollary 8). Then A is a direct summand of M.

A module M is a DRI-module provided that any two submodules of M are relatively injective, whenever they form a direct decomposition of M, i.e. M_i is M_j-injective ($i \neq j = 1,2$) whenever $M = M_1 \oplus M_2$.

From Remark 3, every quasi-continuous module is a DRI- module. There are DRI-modules which are not even CS-modules. In fact every indecomposable module is a DRI-module. For an example of a decomposable DRI-module which is not a CS-module, let K be a field, and let $R = K[x,y]/\langle x^2, xy, y^2\rangle$. Let S be any simple injective R-module, and consider $M = R \oplus S$. M is not a CS-module (due to R indecomposable and not uniform). Now R, S are relatively injective, and any two docompositions of M are isomorphic (due to R and end(S) local rings); i. e. M is a DRI-module.

PROPOSITION 19. Every direct summand of a DRI-module is a DRI-module.

PROOF. Is obvious.

PROPOSITION 20. A module M is a quasi-continuous module if and only if M is a DRI-CS-module.

PROOF. Let $X , Y \le^{\oplus} M$, with $X \cap Y = 0$. Write $M = X \oplus M^*$. Since M is a DRI-module, X is M^*-injective. By Corollary 7, $Y = \{ a + f(a) : a \in A \}$ where A is a closed submodule of M^*, and $f \in \mathrm{Hom}(M^*, X)$. By Corollary 5, $M^* = A \oplus B$, and therefore $M = X \oplus Y \oplus B$.

The converse is obvious.

PROPOSITION 21. Let $M = \oplus_{i\in I}M_i$, where M_i are indecomposables. If $\{M_i\}_{i\in I}$ is a homologically independent family (i.e. $Hom(M_i,M_j) = 0$ for all $i \neq j \in I$), then M is a DRI-module.

PROOF. Let $M = K \oplus K^*$ be a decomposition of M. Let $\pi:M \to K$, $\pi^*:M \to K^*$ and $\pi_i:M \to M_i (i\in I)$ be the canonical projections. Let $\Lambda =:\{\alpha\in I: \pi(M_\alpha) \neq 0\}$. We show that $K = \oplus_{\alpha\in\Lambda}M_\alpha$. Since $Hom(M_i,M_j) = 0$, we have that $\pi_i\pi|M_j = 0$ for all $i \neq j$; and hence $\pi(M_j) \leq M_j$ for all $j \in I$. Now we have $K \leq \oplus_{j\in I}\pi(M_j) \leq \oplus_{j\in I}(M_j \cap K) \leq K$; and hence $K = \oplus_{j\in I}\pi(M_j)$. Since $\pi(M_j) \leq^{\oplus} K \leq^{\oplus} M$, it follows that $\pi(M_j) \leq^{\oplus} M_j$ for all $j \in I$. Since the M_j are indecomposables, we have $\pi(M_\alpha) = M_\alpha$ for all $\alpha \in \Lambda$. Therefore $K = \oplus_{\alpha\in\Lambda}M_\alpha$. By the same argument we can show that $K^* = \oplus_{s\in S} M_s$, where $S =:\{ s \in I: \pi^*(M_s) \neq 0 \}$. This shows that K and K^* are relatively injective.

THEOREM 22. ([10], Theorem 2.13) Let $\{M_i: i \in I\}$ be a family of quasi-continuous modules. Then the following are equivalent:

1. $M = \oplus_{i\in I} M_i$ is quasi-continuous;
2. $\oplus_{j\neq i\in I} M_i$ is M_j-injective for every $j \in I$.

COROLLARY 23. Let $M = \oplus_{i\in I} M_i$, where the M_i are quasi-continuous for all $i \in I$. Then M is a DRI-module if and only if M is quasi continuous.

PROOF. Is obvious.

REFERENCES

1. Anderson, F. W. and Fuller, K. R. Rings and Categories of Modules, Springer Verlage, New York (1973).

2. Chatters, A.W. and Hajarnavis, C.R. Rings in which every Complement Right Ideal is a Direct Summand, Quart. J. Math. Oxford 28 (1977), 61-80.

3. Goodearl, K. R. Ring Theory, Marcel Dekker, Inc. New York (1976).

4. Harada, M. On Modules with Extending Property, Osaka J. Math. 19(1982), 203-215.

5. Harada, M. and Oshiro, K. On Extending Property of Direct Sums of Uniform Modules, Osaka J. Math. 18 (1981), 767-785).

6. Kamal, M. A. Modules in which Complements are Summands, Ph. D. Thesis McMaster University, (1986).

7. Kamal, M. A. and Müller, B. J. Extending Modules over Commutative Domains, Osaka J.Math. 25 (1988), 531-538).

8. Kamal, M. A. and Müller, B. J. The Structure of Extending Modules over Noetherian Rings, Osaka J. Math. 25(1988),539-551.

9. Kamal, M. A. and Müller, B. J. Torsion Free Extending Modules, Osaka J. Math. 25 (1988).

10. Mohamed, S. H. and Müller, B. J. Continuous and Discrete Modules, London Math. Soc. Lec. N., Cambridge Univ. Press, 147 (1990).

11. Müller, B.J. and Rizvi, S.T. On Injective and Quasi-continuous Modules, J.Pure Applied Algebra 28(1983),197-210.

12. Müller, B.J. and Rizvi, S.T. Direct Sum of Indecomposable Modules, Osaka J. Math. 21 (1984), 365-374.

13. Smith, P.F. CS-modules and Weak CS-modules [to appear]