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# RESEARCH NOTES

# A NOTE ON $p$-SOLVABLE AND SOLVABLE FINITE GROUPS

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**ABSTRACT.** The notion of normal index is utilized in proving necessary and sufficient conditions for a group $G$ to be respectively, $p$-solvable and solvable where $p$ is the largest prime divisor of $|G|$. These are used further in identifying the largest normal $p$-solvable and normal solvable subgroups, respectively, of $G$.

**KEY WORDS AND PHRASES.** Solvable, $p$-solvable.
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## 1. INTRODUCTION AND NOTATION.

Structures of solvable and $p$-solvable finite groups are closely related to the indices and normal indices of various kinds of maximal subgroups. The largest and also the smallest prime divisors of the order of a group seem to play, in this connection, important roles in the investigation of these structures. This is precisely the focus of the present note.

The following standard notations and terminologies have been used throughout.

(a) $M$ is a maximal subgroup of $G$: $M <\cdot G$

(b) Normal index of a maximal subgroup $M$: $\eta(G{:}M)$

(c) The $p$-part of normal index: $\eta(G{:}M)_p$

(d) The $p$-part of index of maximal subgroup $M$ of $G$: $[G{:}M]_p$

(e) $\phi_p(G) = \cap\{M <\cdot G \mid [G{:}M]_p = 1\}$

(f) A maximal subgroup whose index is a composite number: $c$-maximal subgroup

## 2. PRELIMINARIES.

If $\mathcal{H}$ is a minimal normal supplement to $M <\cdot G$ then for any chief factor $\mathcal{H}/K$, $K \subseteq M$ and $G = M\mathcal{H}$. Evidently, $[G{:}M]$ divides $|\mathcal{H}/K| = \eta\ (\mathrm{G} : \mathrm{M})$ and if $G$ is simple then obviously $\eta(G{:}M) = |G|, \forall M <\cdot G$. the integer $\eta(G{:}M)$ is unique $\forall M <\cdot G$. [2].

**LEMMA 2.1** [2, Lemma 2]

If $N \triangleleft G$, $M$ is a maximal subgroup of $G$ such that $N \subseteq M$, then $\eta(G/N{:}M/N) = \eta(G{:}M)$.

**LEMMA 2.2** [5, Theorem 3]

In any group $G$ the following are equivalent.

(1) $\eta(G{:}M)_2 = [G{:}M]_2$, for all maximal subgroup $M$ of $G$.

(2) $G$ is solvable.

(3) $\eta(G{:}M)$ is power of a prime for all maximal subgroups $M$ of $G$.

(4) $\eta(G{:}M) = [G{:}M]$ for all maximal subgroups $M$ of $G$.

**LEMMA 2.3.** [1, Lemma 3]

If $G$ is a group with a maximal core free subgroup then the following are equivalent:

(i) There exists a unique minimal normal subgroup of $G$ and there exists a common prime divisor of the indices of all maximal core free subgroup of $G$.

(ii) There exists a nontrivial solvable normal subgroup of $G$.

(ii) The indices of all maximal core free subgroup of $G$ are powers of a unique prime.

**THEOREM 2.4** [6, Theorem 8]

If $p$ is the largest prime dividing the order of a group $G$ then $S_p(G) = \cap\{M <\!\cdot\, G \mid M$ is $c$-maximal and $[G{:}M]_p = 1\}$ is solvable.

## 3. $p$-SOLVABILITY CONDITIONS.

While the equality of the indices and normal indices of each maximal subgroup $M$ of $G$ is both necessary and sufficient for $G$ to be solvable, $\eta(G{:}M)_p = [G{:}M]_p$ $\forall M <\!\cdot\, G$ does not necessarily imply $G$ is $p$-solvable. This condition holds in $G = PSL(2,7)$ for $p = 2$. However $G$ is not 2-solvable.

**THEOREM 3.1.** Let $p$ be the largest prime divisor of the order of a group $G$. Then $G$ is $p$-solvable if and only if $\eta(G{:}M)_p = [G{:}M]_p$ $\forall$ $c$-maximal subgroup $M$ of $G$.

**PROOF.** Let $M$ be a $c$-maximal subgroup of the $p$-solvable group $G$ and consider $G/N$ where $N$ is a minimal normal subgroup of $G$. Case I $|N| = p'$. If $N \subset M$ then by induction it follows that $\eta(G/N{:}M/N)_p = [G/N{:}M/N]_p$ i.e., $\eta(G{:}M)_p = [G{:}M]_p$. On the other hand if $N \not\subset M$ then $\eta(G{:}M)_p = [G{:}M]_p = 1$ since $G = MN$. Case II. $|N| = p^\alpha$. Observe that $N$ is elementary abelian and if $N \subset M$ then $\eta(G{:}M)_p = |N| = [G{:}M]_p$. Now, suppose $N \subset M$ and consider $G/N$. If $p$ divides $|G/N|$ then the equality $\eta(G{:}M)_p = [G{:}M]_p$ follows by induction and if $p \nmid |G/N|$ then $\frac{G}{N}$ is a $p'$-group and trivially, $\eta(G/N{:}M/N)_p = [G/N{:}M/N]_p$, i.e., $\eta(G{:}M)_p = [G{:}M]_p$.

Conversely, let $\eta(G{:}M)_p = [G{:}M]_p \forall$ $c$-maximal subgroup of $G$.

Step I. $G$ is not simple. Let $K <\!\cdot\, G$ and $[G{:}K]_p = 1$. $K$ cannot be $c$-maximal as otherwise $G$ is simple implies $|G|_p = 1$ and trivially $G$ is $p$-solvable. Suppose $[G{:}K] = q$, a prime. By representing $G$ on the cosets of $K$ it follows that core $K \neq 1$ and so $G$ cannot be simple. Let $N$ be a minimal normal subgroup of $G$ and consider $G/N$.

Step II. $\frac{G}{N}$ is $p$-solvable. If $p$ divides $|G/N|$ and $\frac{M}{N}$ is a $c$-maximal subgroup of $G/N$ then $\eta(G/N{:}M/N)_p = [G/N{:}M/N]_p$ and by induction $G/N$ is $p$-solvable. Observe that, if $G/N$ has no $c$-maximal subgroup then $G/N$ is supersolvable and so $G/N$ is $p$-solvable. However, if $p \nmid |G/N|$ then $G/N$ is a $p'$-group and trivially, therefore $G/N$ is $p$-solvable.

Step III. $G$ is $p$-solvable. If $N \subset S_p(G)$ then by theorem 2.4 [6] it follows that $N$ is solvable and therefore is elementary abelian. Consequently, $G$ is $p$-solvable. If $N \not\subset S_p(G)$ then $G = MN$, $[G{:}M]_p = 1$ and $M$ is a $c$-maximal subgroup of $G$. This implies $\eta(G{:}M)_p = |N|_p = [G{:}M]_p = 1$, i.e., $N$ is a $p'$-group and $G$ is consequently, $p$-solvable.

Theorem 3.1 can be used to identify the largest normal $p$-solvable subgroup of $G$ when $p$ is the largest prime divisor of $|G|$.

**THEOREM 3.2.** Let $C$ denote the class $\{M$ is $c$-maximal in $G \mid \eta(G.M)_p \neq [G{:}M]_p$, $p$ is the largest prime divisor of $|G|\}$ of maximal subgroups of a group $G$. Then $T = \cap\{M <.G \mid M \in C\}$ is the largest normal $p$-solvable subgroup of $G$.

**PROOF.** If $G$ has no $c$-maximal subgroup then $G$ is supersolvable and on the other hand if for each $c$-maximal subgroup $M$ of $G$. $\eta(G{:}M)_p = [G{:}M]_p$ then by theorem 3.1 $G$ is $p$-solvable. Therefore, for $C = \phi$, the assertion in the theorem follows since in that event $T = G$. Now, suppose $C \neq \phi$ and $N$ is a minimal normal subgroup of $G$ included in $T$. If $p$ does not divide $|G/N|$ then $G/N$ is trivially $p$-solvable and so $T/N$ is $p$-solvable. Otherwise, by induction, it follows that $T/N$ is $p$-solvable and $N$ may be treated as the only minimal normal subgroup of $G$ in $T$.

If $V \neq N$ is another minimal normal subgroup of $G$ then consider $\overline{G} = \frac{G}{V}$. Suppose, $p$ divides $|\overline{G}|$, and set $C^* = \{\frac{X}{V} = \overline{X}$ is $c$-max in $\overline{G} \mid \eta(\overline{G}:\overline{X})_p \neq [\overline{G}:\overline{X}]_p\}$. If $\overline{G}$ has no $c$-maximal subgroup then $\overline{G}$ is supersolvable. This implies $\frac{TV}{V} \cong T$ is supersolvable and consequently, $T$ is $p$-solvable. Again, if $\forall$ $c$-max subgroup $\overline{X}$ in $\overline{G}$, $\eta(\overline{G}:\overline{X})_p = [\overline{G}:\overline{X}]_p$ then by theorem 3.2, $\overline{G} = \frac{G}{V}$ is $p$-solvable and this will imply, as before, $T$ is $p$-solvable. $C^*$ may therefore be assumed nonempty. Set $\frac{R}{V} = \cap\{X$ is $c$-max in $\overline{G} \mid \overline{X} \in C^*\}$. Note that if $X \in C^*$ then $X \in C$ and this implies $R \supset T$. By induction $R/V$ is $p$-solvable and therefore $\frac{TV}{V} \cong \frac{T}{T \cap V} \cong T$ is $p$-solvable.

If $p$ does not divide $|\overline{G}| = |G/V|$ then $G/V$ is trivially $p$-solvable and consequently, $T$ is $p$-solvable, $N$ may therefore be viewed as the unique minimal normal subgroup of $G$. It may be assumed that $p$ divides $|N|$ as otherwise $N$ is a $p'$-group and $p$-solvability of $T$ follows.

Since $\phi_p$ is solvable ([4], 1.1) one may assume $N \subset \phi_p$ and $G = YN, Y <.G, [G{:}Y]_p = 1$. If $[G{:}Y]$ is composite then $\eta(G{:}Y)_p = |N|_p = [G{:}Y]_p = 1$ and $N$ is a $p'$-group. This implies, however, that $T$ is $p$-solvable. Assume, therefore, $[G{:}Y] = q$, a prime. By representing $G$ on the cosets of $Y$ it follows that core $Y \neq 1$ and this contradicts the fact that the unique minimal normal subgroup $N \subset Y$. Consequently, it now follows that $T$ is $p$-solvable.

We shall now show that $T$ is indeed the largest normal $p$-solvable subgroup of $G$. Suppose $K$ is the largest normal $p$-solvable subgroup of $G$ and let $D$ be a minimal normal subgroup of $G$ in $K$. Then $D$ is either a $p'$-group or is an elementary abelian $p$-group. If $D$ is a $p'$-group and $M \in C$ then $D \subset M$ implies $G = MD$. But then $\eta(G{:}M)_p = [G{:}M]_p = 1$, a contradiction and so $D$ is included in each $M \in C$. Similarly, if $D$ is an elementary abelian $p$-group then also $\forall M \in C, D \subset M$. By induction, $\frac{T}{D} = \frac{K}{D}$, i.e., $T = K$ and the assertion in the theorem is proved completely. (If $p \nmid |\frac{G}{D}|$ then trivially $G$ is $p$-solvable and $T = G$).

Applying similar techniques it is not difficult to prove the following results.

**THEOREM 3.3.** Let $p$ be the largest prime divisor of the order of a group $G$ and $C$ be the class $\{M <.G \mid [G{:}M]_p = 1\}$ of maximal subgroups of $G$. Then $G$ is $p$-solvable if and only if $\eta(G{:}M)_p = 1 \forall M \in C$.

**COROLLARY.** $\cap\{M \in C \mid \eta(G{:}M)_p \neq 1\} = S$ is $p$-solvable. By induction it can be shown as in above that $S$ is indeed the largest normal $p$-solvable subgroup of $G$.

## 4. SOLVABILITY CONDITIONS.

If the indices of all maximal subgroups of a group $G$ is prime then it is well known that $G$ is supersolvable. $G$ turns out to be solvable if only a subclass of maximal subgroups has prime indices.

**THEOREM 4.1.** Let $C$ be the class $\{M <.G \mid [G{:}M]_p = 1$, $p$ is the largest prime divisor of

$|G|\}$ of maximal subgroups of a group $G$. Then $G$ is solvable if each maximal subgroup in $C$ has prime index.

**PROOF.** Note, $C \neq \phi$ and if $M \in C$ then by representing $G$ on the cosets of $M$ it follows that $G$ is not simple. Consider $G/N$ where $N$ is a minimal normal subgroup of $G$. If $pX \mid G/N \mid$ then $\forall \frac{M}{N} < \cdot G/N, [G/N: M/N]_p = 1$, i.e., $[G:M]_p = 1$ which however implied $[G/N: M/N]$ is of prime index. Consequently, $G/N$ is supersolvable. On the other hand, if $p$ divides $|G/N|$ then by induction $G/N$ is solvable.

$N$ may therefore be treated as the unique minimal normal subgroup of $G$. If $N \not\subseteq \phi_p$ then $G = MN$ and by representing $G$ and the cosets of $M$ it follows that core $M \neq 1$ and we have a contradiction. Hence $N \subset \phi_p$ and therefore $G$ is solvable.

**REMARK.** The theorem does not hold if $p$ is note the largest prime divisor. In $PSL(2,7) = G, \forall M < .G$ and $[G:M]_2 = 1$, $[G:M] =$ a prime. But $G$ is not solvable.

The equality of the index and the normal index $\forall M < .G$ is both necessary and sufficient for the solvability of a group $G$. The result remains valid if this holds for the subclass of maximal subgroups with odd indices.

**THEOREM 4.2.** A group $G$ is solvable iff $\eta(G:M) = [G:M] \forall M < .G$ such that $[G:M] = \text{odd}$.

**COROLLARY.** $W = \cap \{M < .G \mid \eta(G:M) \neq [G:M], [G:M] = \text{odd}\}$ is solvable and is the largest normal subgroup of $G$.

The above theorem can be easily proved using induction and the fact that every odd ordered group is solvable.

In proving the corollary, one uses the same techniques as in the proof of theorem 3.2. That $W$ is the largest normal solvable subgroup of $G$ follows from lemma 3 in [10]. For the sake of completeness Lemma 3 mentioned above is stated below.

**LEMMA.** In any group $G, W_3$ is the largest normal solvable subgroup.

**REMARKS.**

1. $W_3 = \cap \{M < .G \mid \eta(G:M) \neq [G:M]\}$
2. $\cap \{M < .G \mid \eta(G:M) = \text{even}\}$ is solvable.

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