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RESEARCH NOTES

EXTREMAL PROBLEMS FOR COMPLETELY POSITIVE MAPS

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ABSTRACT. In this note, we study the faces of some convex subsets of $CP_c(A, B(\mathcal{H}))$(the continuous completely positive linear maps from pro-C^*-algebra A to $B(\mathcal{H})$).
KEY WORDS AND PHRASES: Pro-C^*-algebras, completely positive operators, faces.
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A pro-C^*-algebra is a complete Hausdorff topological *-algebra over $\mathcal{C}$ containing identity 1 whose topology is determined by its continuous C^*-seminorms in the sense that a net a_λ converges to 0 if and only if $p(a_\lambda) \to 0$ for every continuous C^*-seminorm p on A. From [4], we see this is a generalization of C^*-algebras.

First we recall the following analogue of Stinespring's representation theorem from [3].

Theorem 1. Let A be a pro-C^*-algebra, and $B(\mathcal{H})$ denote the set of all bounded linear operators on Hilbert space $\mathcal{H}$. If $\phi : A \to B(\mathcal{H})$ is a continuous completely positive liear map, then there exists a Hilbert space $\mathcal{K}$, a continuous *-representation $\pi : A \to B(\mathcal{K})$, and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ such that $\phi(a) = V^*\pi(a)V$ for all $a \in A$.

Remark 2. Let $\phi(a) = V^*\pi(a)V$ be as in the theorem. Letting $\mathcal{K}_0 = [\pi(A)V\mathcal{H}]$, the restriction π_0 of π to $\mathcal{K}_0$ also satisfies $\phi(a) = V^*\pi_0(a)V$, and so there is no essential loss if we require that $[\pi(A)V\mathcal{H}] = \mathcal{K}$. Such a pair (π, V) will be called minimal.

Recall from elementary convexity theory that a closed, non-empty subset F of a convex subset C is called a face if F is convex, and if $ax + (1-a)y$ in F for $0 < a < 1$ implies that $x \in F$ and $y \in F$, for all elements x, y in C. A minimal (i.e. one-point) face of C is called an extreme point.

Lemma 3. Let $T \in B(\mathcal{H})$, $T \geq 0$. The map $S \to T^{\frac{1}{2}}ST^{\frac{1}{2}}$ is an affine isomorphism of $[0, R_T]$ onto $[0, T]$, where R_T denotes the range projection of T.

Proof. For $S \in [0, R_T]$ and $\xi \in \mathcal{H}$, $< T^{\frac{1}{2}}ST^{\frac{1}{2}}\xi, \xi > = < ST^{\frac{1}{2}}\xi, T^{\frac{1}{2}}\xi > \leq < R_TT^{\frac{1}{2}}\xi, T^{\frac{1}{2}}\xi > = < T^{\frac{1}{2}}\xi, T^{\frac{1}{2}}\xi > = < T\xi, \xi >$, thus $T^{\frac{1}{2}}ST^{\frac{1}{2}} \leq T$, also one sees that $T^{\frac{1}{2}}ST^{\frac{1}{2}} \geq 0$, so $T^{\frac{1}{2}}ST^{\frac{1}{2}} \in [0, T]$, The map is clearly affine and, for $S_1, S_2 \in [0, R_T]$, if $T^{\frac{1}{2}}S_1T^{\frac{1}{2}} = T^{\frac{1}{2}}S_2T^{\frac{1}{2}}$, then, for all $\xi, \eta \in \mathcal{H}$, $< S_1T^{\frac{1}{2}}\xi, T^{\frac{1}{2}}\eta > = < S_2T^{\frac{1}{2}}\xi, T^{\frac{1}{2}}\eta >$. This implies S_1 and S_2 agree on $[T^{\frac{1}{2}}\mathcal{H}] = [T\mathcal{H}]$. Since they are both 0 on $[T\mathcal{H}]^\perp$, $S_1 = S_2$. Therefore the map is one to one. It remines to show that it is onto. For $\eta \in T(\mathcal{H})$, say $\eta = T\xi$, $\xi \in \mathcal{H}$, let $T^{-\frac{1}{2}}\eta = T^{\frac{1}{2}}\xi$, since $T\xi_1 = T\xi_2$ implies $T^{\frac{1}{2}}\xi_1 = T^{\frac{1}{2}}\xi_2$,$T^{-\frac{1}{2}}\eta$ is well defined for all $\xi \in T(\mathcal{H})$, now let $A \in [0, T]$. Define a sesqui-linear form B on $T(\mathcal{H}) \times T(\mathcal{H})$ by $B(\xi, \eta) = < AT^{-\frac{1}{2}}\xi, T^{-\frac{1}{2}}\eta >$. Using the polarization identity and the fact $A \leq T$, one sees that B is bounded on $T(\mathcal{H}) \times T(\mathcal{H})$ and thus defines a bounded linear operator S_0 on $[T\mathcal{H}]$ such that $< S_0\xi, \eta > = B(\xi, \eta)$ for all $\xi, \eta \in T(\mathcal{H})$. Define $S\xi = S_0(R_T\xi)$, for all $\xi \in \mathcal{H}$. Thus $S \in B(\mathcal{H})$. For all $\xi \in T(\mathcal{H})$, $< S\xi, \xi > = < S_0\xi, \xi > = < AT^{-\frac{1}{2}}\xi, T^{-\frac{1}{2}}\xi > \leq < TT^{-\frac{1}{2}}\xi, T^{-\frac{1}{2}}\xi > = < R_T\xi, \xi >$. Thus $< S\xi, \xi > \leq < R_T\xi, \xi >$, for all $\xi \in [T\mathcal{H}]$. For $\xi \in \mathcal{H}$, $< S\xi, \xi > = < S(R_T\xi + (I - R_T)\xi), R_T\xi + (I - R_T)\xi > = < SR_T\xi, R_T\xi > \leq < R_T\xi, \xi >$. Therefore, $S \leq R_T$ and a similar argument shows $S \geq 0$. Finally, $T^{\frac{1}{2}}ST^{\frac{1}{2}} = A$ by construction.

Theorem 4. If B_+ is the positive part of the unit ball in a von Neumann algebra A, then each weakly closed face F of B_+ has form $F = \{L \in B_+ | \, p \leq L \leq q\}$ for a unique pair of projections such that $p \leq q$ in A.

Corollary 5. Each weakly closed face of $[0, T]$ has form $\{L : T^{\frac{1}{2}}pT^{\frac{1}{2}} < L < T^{\frac{1}{2}}qT^{\frac{1}{2}}\}$, where p and q are projections, and $p \leq R_T$ and $q \leq R_T$.

We recall certain topological properties of the space of all operator-valued linear maps.

Let A be a pro-C^*-algebra, and let $\mathcal{H}$ be a Hilbert space, $B(A, B(\mathcal{H}))$ will denote the vector space of all continuous linear maps of A into $B(\mathcal{H})$. We shall endow $B(A, B(\mathcal{H}))$ with a certain weak topology, namely BW-topology. For $r \geq 0$, let $B_r(A, B(\mathcal{H}))$ denote the closed ball of radius r: $B_r(A, B(\mathcal{H})) = \{\phi \in B(A, B(\mathcal{H})); ||\phi(a)|| \leq rp(a), a \in A\}$, where because ϕ is continuous, there exists $p \in S(A)$ such that $||\phi(a)|| \leq Mp(a)$. First we topologize B_r as follows, by definition, a net $\phi_v \in B_r(A, B(\mathcal{H}))$ converges to $\phi \in B_r(A, B(\mathcal{H}))$ if $\phi_v(a) \rightarrow \phi(a)$ in the weak operator topology, for every $a \in A$. A convex subset $\mathcal{U}$ of $B(A, B(\mathcal{H}))$ is open if $\mathcal{U} \bigcap B_r(A, B(\mathcal{H}))$ is an open subset of $B_r(A, B(\mathcal{H}))$ for every $r \geq 0$. The convex open sets form a base for a locally convex Hausdorff topology on $B(A, B(\mathcal{H}))$, which we shall call the BW-topology.

Now we come to discuss the facial structure of completely positive operators. First we give a lemma. Let $\phi(a) = V^*\pi(a)V$ be a continuous completely positive linear map as in Theorem 1.

Lemma 6. The mapping from $\{T \in \pi(A)' : 0 \leq T \leq I\}$ to $[0, \phi]$ defined by $\phi_T(a) = V^*T\pi(a)V$ is a homeomorphism related to the restriction of weak operator topology of von Neumann algebra $\pi(A)'$ and BW-topology of $B(A, B(\mathcal{H}))$.

Proof. $[0, \phi]$ is a BW-closed subset of $B(A, B(\mathcal{H}))$. If $\{\phi_v\}$ is a net in $[0, \phi]$, and ϕ_v converges to $\phi_0 \in [0, \phi]$ in BW-topology. We have for every $a \in A$, $\phi_v(a) \rightarrow \phi_0(a)$ in weak operator topology. That is, for every $\xi, \eta \in \mathcal{H}$, $< \phi_v(a)(\xi), \eta > \rightarrow < \phi_0(a)(\xi), \eta >$. But we have $\phi_v(a) = V^*T_v\pi(a)V$, and $\phi_0 = V^*T_0\pi(a)V$, where $T_0, T_v \in \pi(A)'$, and $0 \leq T_0, T_v \leq I$. So we have $< V^*T_v\pi(a)V(\xi), \eta > \rightarrow < V^*T_0\pi(a)V(\xi), \eta >$, for every $a, b \in A$, and $\xi, \eta \in \mathcal{H}$, we have $< T_v\pi(b)V(\xi), \pi(a)V(\eta) > \rightarrow < T_0\pi(b)V(\xi), \pi(a)V(\eta) >$, but $\mathcal{R} = [\pi(A)V\mathcal{H}]$, so $T_v \rightarrow T_0$ in the weak operator topology. The other direction is similar.

Theorem 7. Given two completely positive operators ψ and ϕ with $\psi \leq \phi$. Let $\phi = V^*\pi V$ be the minimal representation of ϕ, then the BW-closed faces in $[0, \phi]$ are of the form $\{\phi_L; L \in \pi(A)', (I-T)^{\frac{1}{2}}p(I-T)^{\frac{1}{2}} + T \leq L \leq (I-T)^{\frac{1}{2}}q(I-T)^{\frac{1}{2}} + T\}$, where p and q are projections in $\pi(A)'$ and $p \leq R_{I-T}$, $q \leq R_{I-T}$, and $\psi = V^*T\pi V$.

Proof. It is an easy consequence of lemma 6 and the above corollary 5.

Corollary 8. Let $\phi = V^*\pi V$ be the minimal representation of ϕ, then the BW-closed faces in $[0, \phi]$ are of the form $\{\phi_T; T \in \pi(A)', \ p \leq T \leq q\}$, where p and q are projections in $\pi(A)'$.

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