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A GENERALIZED FORMULA OF HARDY

GEOFFREY B. CAMPBELL

Institute of Advanced Studies
School of Mathematical Sciences
The Australian National University
GPO Box 4, Canberra, Australia 2601

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ABSTRACT. We give new formulae applicable to the theory of partitions. Recent work suggests they also relate to quasi-crystal structure and self-similarity. Other recent work has given continued fractions for the type of functions herein. Hardy originally gave such formulae as ours in early work on gap power series which led to his and Littlewood's "High Indices" Theorem. Over a decade ago, Mahler and then others proved results on irrationality of decimal fractions applicable to types of functions we consider.

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1. INTRODUCTION.

Hardy in his classical book on Ramanujan ([12], p 39), gives the formula, essentially (for $Re\ x > 0$),

$$\sum_{k=-\infty}^{\infty} 2^k e^{-2^k x} = \frac{1}{x \log 2}\left\{1 - \sum{}' \,\Gamma\left(1 + \frac{2ki\pi}{\log 2}\right) x^{\frac{-2ki\pi}{\log 2}}\right\}, \tag{1.1}$$

with $\sum'$ over all non-zero integers k. The oscillatory part on the right side eluded Ramanujan, perhaps because the Euler Maclaurin formula ([1], p 806, 23.1.30) yields $1/(x \log 2)$ exactly. This assumption led Ramanujan astray in his theory of distribution of prime numbers.

The series $\sum exp(2^k x)$ in various forms has been considered by numerous authors with varying perspectives during the past century. Hardy considered it whilst examining gap power series in the context of the so called *converse of Abel's Theorem.* He gave formulae such as in [9]: *If a is an integer greater than* 1, *and* $Re\ x > 0$,

$$\sum_{k=0}^{\infty} e^{-a^k x} + \sum_{k=1}^{\infty}\left(e^{-a^{-k}x} - 1\right) = \frac{-\log x}{\log a} - \frac{\gamma}{\log a} + \frac{1}{2} - \frac{1}{\log a}\sum{}' \,\Gamma\left(\frac{2ki\pi}{\log a}\right) x^{\frac{-2ki\pi}{\log 2}} \tag{1.2}$$

The $\sum'$ summed part here oscillates between finite limits, or "wobbles." These wobbles are self-similar and clearly the function in question here satisfies the functional equation

$$F(x) = F(ax). \tag{1.3}$$

As mentioned recently in Ninham [25], and also in his work with Frankel, Glasser and Highes [8], these wobbling functions scale at every level, but are *not* fractal in the sense of Mandelbrot, being everywhere differentiable. Both types of functions are continuous, although it appears that the functions we consider here may have physical applications to at least quasi-crystal spectra (see [25]), and furthermore, since they arise from Fourier transforms of periodic delta functions they may be of great fundamental significance in other ways.

Mahler [20] considered cases of the function

$$f(z) = f(-t + \phi i) = \sum_{k=1}^{\infty} e^{-2^k z} \quad , \quad (0 < t \leq 1) \tag{1.4}$$

where it happens exceptionally that $\lim_{t \to +0} f(z)$ exists, and tends to a definite finite value.

Central to the considerations of Hardy and Littlewood [8, 11, 13, 16] was the idea that as $z \to 0$ in (1.4) the series does not tend to a limit consistent with a *simple* converse of Abel's Theorem (see also Pólya [34]). In [20], Mahler showed that if

$$N = \left[\frac{\log(1/t)}{\log 2} \right] \quad , \tag{1.5}$$

with $[x]$ as usual the greatest integer in x, and both

$$\lim_{t \to +0} \frac{\log 2}{\log(1/t)} f(e^{-t+i\phi}) := g_1(\phi) \quad , \tag{1.6}$$

and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2^n \phi i} := g_2(\phi) \quad ; \tag{1.7}$$

then, for $0 < p \leq q-1$, $(p, q) = 1$, with q odd,

$$\begin{aligned} g_1(2\pi\, p/q) &= g_2(2\pi\, p/q) \\ &= \frac{1}{r} \sum_{n=1}^{r} \exp(2^n \pi\, i\, p/q) \quad . \end{aligned} \tag{1.8}$$

Here $r = \varphi(q)$, the Euler totient function of q. The sum of (1.8) is a Gaussian period from the theory of cyclotomy (see Kummer [15]).

Mahler [20] showed that if $g \geq 2$ is a fixed integer, the decimal fraction given by

$$0.(1)(g)(g^2)(g^3)\ldots,$$

where (g^n) is to mean the number g^n written in decimal form, is irrational. His methods are often simply applied to the gap series of our present note. Several authors have since worked on similar problems (see [4, 5, 6, 24, 26 to 29, 34, 35, 38]).

In recent years interest has arisen in the continued fraction forms of series such as $f(z)$ of (1.4). For an account of such work see [4, 23, 30, 32, 33, 36, 37].

2. SOME NEW RESULTS.

As an undergraduate student in 1979 the author gave the following formula (see [7]), which has Hardy's formula (1.2) as the case $m = 0$.

THEOREM 2.1. *For non-negative integers m, $\operatorname{Re} x > 0$, $\operatorname{Re} a > 1$,*

$$\sum_{k=0} k^m e^{-a^k x} + \sum_{k=1} (-k)^m (e^{-a^{-k} x} - 1) \tag{2.1.1}$$

$$= \frac{1}{(m+1)(\log a)^{m+1}} \sum_{j=0}^{m+1} \binom{m+1}{j} \Gamma^{(m+1-j)}(1) \left(\log \frac{1}{x}\right)^j \tag{2.1.2}$$

$$- \frac{1}{m+1} B_{m+1} \tag{2.1.3}$$

$$- \frac{1}{(\log a)^{m+1}} \sum{}' \int_0^\infty \left(\log \frac{t}{x}\right)^m \left(\frac{t}{x}\right)^{\frac{2ki\pi}{\log a}} \frac{dt}{t\, e^t} \quad , \tag{2.1.4}$$

where $\Gamma^{(j)}$ is the j-th derivative of the gamma function, B_m the m-th Bernoulli number, $\sum'$ is the sum over nonzero integers k, and $k^m := 1$ at $k = m = 0$.

Although [7] was accepted for publication it never actually appeared in print. There are several features of this theorem worth noting at this point:-

(i) The derivatives at unity of the gamma function as in (2.1.2) are easily found from the expansion valid for $|z| < 1$, (see [1, pp259, 6.3.14; and [3])

$$\frac{\Gamma'(1+z)}{\Gamma(1+z)} = -\gamma + \zeta(2)z - \zeta(3)z^2 + \zeta(4)z^3 - \dots \quad , \tag{2.2}$$

so that $\Gamma'(1) = -\gamma$, $\Gamma^{(2)}(1) = \gamma^2 + \frac{1}{6}\pi^2$, $\Gamma^{(3)}(1) = -\gamma^3 - \frac{1}{2}\gamma\pi^2 - 2\zeta(3)$, and so on.

(ii) The relationship of the integral in (2.1.4) to the gamma function is obviated by realising that

$$\int_0^\infty \left(\log \frac{t}{x}\right)^m \left(\frac{t}{x}\right)^{\frac{2ki\pi}{\log a}} \frac{dt}{t\, e^t} = x^{\frac{-2ki\pi}{\log a}} \sum_{j=0}^{m} \binom{m}{j} \left(\log \frac{1}{x}\right)^j \Gamma^{(m-j)}\left(\frac{2ki\pi}{\log a}\right) \quad . \tag{2.3}$$

Indeed, the version of Theorem 2.1 given in [7] has the sum of (2.3) in place of the integral. Taking (2.3) into account, the summation (2.1.4) is seen to converge bilaterally, owing to the fact *[1, pp 257, 6.1.45]* that for fixed m and $1 + \delta \le \operatorname{Re} t \le \operatorname{Re} a\, (\delta > 0)$,

$$\left|\Gamma^{(m)}\left(1 + \frac{2ki\pi}{\log t}\right)\right| \sim \exp\left(-\frac{1}{2}\pi \left|\frac{2k\pi}{\log t}\right|\right) \sum_{j=0}^{m} A(m, j) \left|\frac{2k\pi}{\log t}\right|^{\frac{1}{2}-j} \quad , \tag{2.4}$$

where

$$A(m, j) = (2\pi)^{\frac{1}{2}} \binom{m}{j} \frac{\Gamma(\frac{1}{2}+j)}{\Gamma(\frac{1}{2})} \left(-\frac{1}{2}\pi\right)^{m-j}$$

(iii) Hardy's method of proving (1.2) relied on a simple Mellin inversion applied to the left side series. Such a straightforward application designed to arrive at Theorem 2.1 would involve the rather non-trivial evaluation of residues leading to

$$\frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} \Gamma(-u) x^u \left\{ \sum_{j=0}^{m} S_m^{(j)} \frac{a^{uj}\, j!}{(1-a^u)^{j+1}} \right\} du = \sum_{k=1}^{\infty} (-k)^m \left(1 - e^{-a^{-k} x}\right)$$

+ the addition of (2.1.2), (2.1.3) and (2.1.4), in which $S_m^{(j)}$ are Stirling numbers of the second kind.

(iv) It can be seen from the above, that (2.1.4) is a sum of $m+1$ oscillating functions, each multiplied by a power of $\log x$. Each oscillating function is of order $o(1)$ and oscillates between finite limits. In Hardy's original paper [9] estimates were given for these limits in terms of a. In Mahler [20] an elementary method for obtaining such estimates was also given. However, if $a = 2$ in (1.2) it is easy to use a small calculator to establish that the oscillatory term only enters in after about the fourth decimal place, so for $a = 2$, $o(0.001)$ seems a conservative estimate for the oscillatory function under $\sum'$.

3. PROOFS OF THEOREM 2.1.

The first proof of Theorem 2.1 appeared in Campbell [7]. It depended on Lemma 3.1 below, and comparing partial derivatives with respect to the different variables, then comparison to obtain a constant of integration (2.1.3). This was somewhat of a departure from the approach taken by Hardy, since the method in [7] required a continuous differentiable function of a, as distinct to a as an integer greater than unity. (1.1) as it appears in Hardy's book on Ramanujan [12] actually has a misprint indicating the author was not thinking of a continuous variable a. The landmark paper of Hardy and Littlewood [13] contains theorems of sufficient generality to justify all of the differentiations of the series of kind (2.1.1) as they occur in [7] and in the present note. (1.1) is a case of the following.

Lemma 3.1. *If* $\operatorname{Re} x > 0$, $\operatorname{Re} a > 1$, $\operatorname{Re} n > 0$,

$$\sum_{k=-\infty}^{\infty} a^{kn}\, e^{-a^k x} = \frac{1}{x^n \log a}\left\{\Gamma(n) - \sum{}' \,\Gamma\left(n + \frac{2ki\pi}{\log a}\right) x^{\frac{-2ki\pi}{\log a}}\right\} \tag{3.1}$$

PROOF. This comes easily from both straightforward application of the residue theorem, and by summing on the Mellin inversion formula, so the integral

$$\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} \Gamma(-u)\, x^u\, (1-a^{nu})^{-1}\, du \quad , \tag{3.2}$$

is given the two expressions

$$\sum_{k=0}^{\infty} a^{kn}\, e^{-a^k x} \quad , \tag{3.3}$$

and

$$-\sum_{k=1}^{\infty} a^{-kn}\, e^{-a^{-k} x} + R.H.S.\ of\ (3.1)$$

PROOF OF THEOREM 2.1. We consider the behaviour of the terms in the bilateral series on the left side of (3.1) as $n \to 0$. In the direction with positive k terms the series clearly converges, however for negative k terms the series

$$\sum_{k=1}^{\infty} e^{-a^{-k} x} \tag{3.4}$$

diverges, each term approaching unity successively. We may compensate for this by subtracting the function

$$\sum_{k=1}^{\infty} a^{-kn} = \frac{a^{-n}}{1-a^{-n}}$$

from both sides of (3.1) so that (3.4) becomes the convergent

$$\sum_{k=1}^{\infty} (e^{-a^{-k} x} - 1)$$

when $n \to 0$, and we are left with Hardy's formula (1.2), after calculating the corresponding limit for the right side of the compensated version of (3.1). Next, we see that if (3.1) be written in the form

$$\sum_{k=0}^{\infty} a^{kn}\, e^{-a^k x} + \sum_{k=1}^{\infty} a^{-kn}(e^{-a^{-k}x} - 1)$$

$$= \frac{\Gamma(n)}{x^n \log a} - \frac{a^{-n}}{1-a^{-n}} - \frac{1}{x^n \log a} \sum{}' \, \Gamma\left(n + \frac{2ki\pi}{\log a}\right) x^{\frac{-2ki\pi}{\log a}} \quad , \tag{3.5}$$

we can formally expand each side into power series in n and equate coefficients to arrive at the theorem. This is justified since (3.5) holds for any $n > 0$ in the neighbourhood of $n = 0$. ■

SKETCH OF THE EARLIER UNPUBLISHED PROOF. Whilst the proof in Campbell [7] is longer and more laboured, it has the merit of showing an interplay between the independent variables x, a, and n. It starts with our Lemma 3.1 and uses the fact that

$$(\log a)^{-m} \frac{\partial^m}{\partial n^m} \sum_{k=-\infty}^{\infty} a^{kn}\, e^{-a^k x} \Big|_{n=1}$$

$$= \sum_{k=-\infty}^{\infty} k^m\, a^k\, e^{-a^k x}$$

$$= \frac{-\partial}{\partial x} \left\{ \sum_{k=0}^{\infty} k^m\, e^{-a^k x} + \sum_{k=1}^{\infty} (-k)^m \,(e^{-a^{-k}x} - 1) \right\} \quad .$$

Integration with respect to x gives the major terms of Theorem 2.1, with the constant term (2.1.3) obtained from the functional equation

$$f_m\,(a,\, x/a) = \sum_{j=0}^{m} \binom{m}{j} f_j\,(a,\, x)$$

clearly satisfied by the series (2.1.1). Of course the crucial part of all this is Lemma 3.1, and neither of the two proofs could stand without it.

4. CONNECTIONS WITH PARTITION THEORY.

Theorem 2.1 and indeed Lemma 3.1 itself may be used to derive identities for generating functions of partitions of various sorts into a-th powers. To begin just with (1.2), we can easily show that

THEOREM 4.1. *If* $\operatorname{Re} a > 1$, *and* $\operatorname{Re} t > 0$ *with* $x = e^{-t}$,

$$\prod_{k=0}^{\infty} \left(1 + x^{a^k}\right) \prod_{k=1}^{\infty} \left(\frac{1 + x^{a^{-k}}}{2}\right) \tag{4.1}$$

$$= \left(\frac{a}{2}\right)^{\frac{1}{2}\frac{\log 2}{\log a}} \left(\frac{1}{\log \frac{1}{x}}\right)^{\frac{\log 2}{\log a}}$$

$$\times \exp\left\{ \frac{-1}{\log a} \sum{}' \, \Gamma\left(\frac{2ki\pi}{\log a}\right) \zeta\left(1 + \frac{2ki\pi}{\log a}\right) \left(1 - 2^{\frac{-2ki\pi}{\log a}}\right) \left(\log \frac{1}{x}\right)^{\frac{-2ki\pi}{\log a}} \right\} \quad .$$

We note that as a corollary, if $a = 2$ we have the elementary result

$$\prod_{k=0}^{\infty}\left(1+x^{2^k}\right)\prod_{k=1}^{\infty}\left(\frac{1+x^{2^{-k}}}{2}\right)=\frac{1}{\log\frac{1}{x}} \quad , \tag{4.2}$$

or simply

$$\prod_{k=1}^{\infty}\left(\frac{1+x^{2^{-k}}}{2}\right)=\frac{1-x}{\log\frac{1}{x}} \quad .$$

PROOF OF THEOREM 4.1. (4.1) may be known, since it is so easily accessible by (1.2). The proof involves application of the operation

$$\sum_{j=1}^{\infty}(-1)^{j+1}f(a,\,xj)\,j^{-1}$$

to (1.2), letting either side of (1.2) be $f(a,\,x)$. This, for the left side gives the logarithm of the left side of (4.1). For the right side the same procedure leads to the required result if we know that

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\log k}{k}=\frac{1}{2}(\log 2)^2-\gamma\log 2 \quad , \tag{4.3}$$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\,k^{-(1+in)}=\zeta(1+in)(1-2^{-in}) \quad ,$$

for real $n\neq 0$. (4.3) is given in Hardy's note on Vacca's series [10]. A little simplification thereafter gives the theorem. ■

Just as Theorem 2.1 generalizes Hardy's formula (1.2), the following formula, which appears to be new, generalizes Theorem 4.1.

THEOREM 4.2. *If m, a, x, are as in Theorem 2.1,*

$$\prod_{k=0}^{\infty}\left(1+e^{-a^k x}\right)^{k^m}\prod_{k=1}^{\infty}\left(\frac{1+e^{-a^{-k}x}}{2}\right)^{(-k)^m}=\exp\Big(G_1(x)+{\sum}'\,G_2(x)\Big) \quad , \tag{4.4}$$

where ${\sum}'$ is over non-zero integers k, and $k^m:=1$ at $k=m=0$,

$$G_1(x)=\frac{-1}{m+1}\,B_{m+1}\log 2 + \left(\frac{1}{\log a}\right)^{m+1}\sum(m;\,m_1,\,m_2\,m_3)\,\Gamma^{(m_1)}(1)\,\eta^{(m_2)}(1)\left(\log\frac{1}{x}\right)^{m_3} \quad , \tag{4.5}$$

$$G_2(x)=\left(\frac{1}{\log a}\right)^{m+1}\int_0^{\infty}\left\{\sum_{r=1}^{\infty}\frac{(-1)^r(\log(t/rx))^m}{r^{(1+2ki\pi/\log a)}}\right\}\left(\frac{t}{x}\right)^{\frac{2ki\pi}{\log a}}\frac{dt}{t\,e^t} \quad , \tag{4.6}$$

where the sum in (4.5) is over $m=m_1+m_2+m_3$ in non-negative integers and

$$(m;\,m_1,\,m_2,\,m_3)=\frac{m!}{m_1!\,m_2!\,m_3!} \quad ,$$

and also

$$\eta^{(m)}(1)=\sum_{r=1}^{\infty}(-1)^{r+1}\frac{1}{r}\left(\log\frac{1}{r}\right)^m \quad . \tag{4.7}$$

As with Theorem 2.1, Theorem 4.2 has several notable features:-

(i) The function

$$\prod_{k=0}^{\infty}\left(1+e^{-a^k x}\right)^{k^m}=\sum_{k=0}^{\infty}P_a(k)\,e^{-kx}\quad, \tag{4.8}$$

clearly generates the number of partitions into no more than k^m terms of kind a^k, according to the methods in Andrews *[2]*.

(ii) Estimates as $x \to 0$ for the function

$$\prod_{k=1}^{\infty}\left(\frac{1+e^{-a^{-k}x}}{2}\right)^{k^m} \tag{4.9}$$

are easily obtained, together with appropriate inequalities, enabling us, if we choose, to focus on (4.8).

(iii) Some modification of Mahler's method [20] may be applicable to Theorem 4.2, thus giving elementary estimates for the functions concerned.

(iv) If

$$\eta(n)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^n}$$

then for the terminology of $\sum$ in (4.5),

$$\begin{aligned} &G_2(x)\\ &=\left(\frac{1}{\log a}\right)^{m+1}\left\{\sum(m;m_1,m_2,m_3)\Gamma^{(m_1)}\left(\frac{2ki\pi}{loga}\right)\eta^{(m_2)}\left(1+\frac{2ki\pi}{\log a}\right)\left(\log\frac{1}{x}\right)^{m_3}\right\}x^{\frac{-2ki\pi}{\log a}}\\ &=\text{R.H.S. of (4.6)}\end{aligned}$$

The clue to this is the identity

$$(x_1+x_2+x_3)^m=\sum(m;\,m_1,\,m_2,\,m_3)\,x_1^{m_1}\,x_2^{m_2}\,x_3^{m_3}\quad. \tag{4.10}$$

(v) It is clear from Theorem 4.2 that $G_1(x)$ is a polynomial in $\log\frac{1}{x}$ of degree m, and that $\Gamma^{(m)}(1)$ and $\eta^{(m)}(1)$ can be evaluated as finite combinations of Riemann zeta functions, and the constants $(\gamma=\gamma_0)$ (see [3])

$$\gamma_n=\lim_{r\to\infty}\left\{\sum_{k=1}^{r}\frac{(\log k)^n}{k}-\frac{(\log r)^{n+1}}{n+1}\right\} \tag{4.11}$$

which occur naturally in the expansion *[1, pp 807, 23.2.5]*,

$$\zeta(s)=\frac{1}{s-1}+\gamma+\sum_{k=1}^{\infty}(-1)^k\frac{(s-1)^k}{k!}\gamma_k\quad. \tag{4.12}$$

Hardy [10] applied (4.11); and recently in [8] it is explained that (4.12) is necessary for obtaining precise results for thermodynamic functions near the so-called "Hagedorn temperature" which occurs in particle physics (hadron physics), related to prime and integer gases.

(vi) As a curiosity, if x is, say, equal to $\log 10$, the decimal fraction obtained from (4.8) is trivially calculated, and the corresponding (4.9) type function can be found easily using a small calculator. The oscillating functions are not so easily calculated, but the major terms of $G_1(x)$ are, as are the *order* of each of the $m+1$ oscillating functions in $G_2(x)$.

PROOF OF THEOREM 4.2. This proof follows an analogous line to our proof of Theorem 2.1. Lemma 3.1 easily transforms (under the same conditions) to

$$\prod_{k=-\infty}^{\infty}\left(\frac{1}{1-e^{-a^k x}}\right)^{a^{kn}} \tag{4.13}$$
$$= \exp\left\{\frac{\Gamma(n)\zeta(n+1)}{x^n \log a} - \frac{1}{\log a}\sum{}' \Gamma\left(n+\frac{2ki\pi}{\log a}\right)\zeta\left(n+1+\frac{2ki\pi}{\log a}\right) x^{-\left(n+\frac{2ki\pi}{\log a}\right)}\right\} \quad ,$$

which easily becomes (after dividing B.S. by the case with $2x$ for x)

$$\prod_{k=0}^{\infty}(1+e^{-a^{-k}x})^{a^{kn}} \prod_{k=1}^{\infty}\left(\frac{1+e^{-a^{-k}x}}{2}\right)^{a^{-kn}} \tag{4.14}$$
$$= \exp\left\{\frac{\Gamma(n)\eta(n+1)}{x^n \log a} - \frac{\log 2}{a^n-1} - \frac{1}{\log a}\sum{}' \Gamma\left(n+\frac{2ki\pi}{\log a}\right)\eta\left(n+1+\frac{2ki\pi}{\log a}\right)\left(\frac{1}{x}\right)^{n+\frac{2ki\pi}{\log a}}\right\}$$

when the compensating factor

$$\prod_{k=1}^{\infty}\left(\frac{1}{2}\right)^{a^{-kn}} = \left(\frac{1}{2}\right)^{\frac{1}{a^n-1}} = \exp\left\{\frac{-\log 2}{a^n-1}\right\}$$

is multiplied to each side. Theorem 4.2 comes easily from equating coefficients of like powers of n in the logarithm of each side of 4.14. ■

5. CLOSING REMARKS.

The results of this paper evidently have consequences in number theory [12], and also in physics as shown from [8] and [25]. Related topics have held the interest of mathematicians such as Mahler for over 50 years. (See [17 to 22], especially [22], where Mahler reminisces about the subject.) The approaches taken by Hardy, Littlewood and later Pólya suggest that series of the kinds represented in this paper have been considered important for various reasons. In [25] the self-similarity property of the oscillating "wobble" function was applied in an ingenious fashion to Penrose tilings and quasi-crystalline structures having fivefold symmetries connected with equi-angular spirals and Fibonacci sequences. Using methods of our current work, results can be obtained for series involving Fibonacci numbers such as

$$\sum_{k=0}^{\infty} F_k\, e^{-a^k x} + \sum_{k=1}^{\infty} F_{-k}\left(e^{-a^{-k}x}-1\right) \quad ,$$

which satisfies the functional equation

$$1 + f(a\,x) + f(a^2\,x) = f(x)$$

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REFERENCES

1. ABRAMOWITZ, M., and STEGUN, I.A., (1972), Handbook of Mathematical Functions, Dover Publications Inc., New York.

2. ANDREWS, G.E., (1976), The Theory of Partitions, Encyclopedia of Mathematics, Vol. 2, Addison Wesley.

3. APOSTOL, Tom M., (1985), Formulas for higher derivatives of the Riemann zeta function, Math. of Comp., 44, 223-232.

4. BUNDSCHUH, P., (1984), Trancendental Continued Fractions, J. Number Theory, 18, 91-98.

5. BUNDSCHUH, P., (1984), Generalization of a Recent Irrationality Result of Mahler, J. Number Theory, 19, 248-253.

6. BUNDSCHUH, P., (1990), Indépendance algébrique par des d'approximation, méthodes in Lecture Notes in Math., Vol. 1415, Springer Verlag, 116-122.

7. CAMPBELL, G.B., (1979), Generalization of a Formula of Hardy, Pure Math. Research Paper, 795, LaTrobe Univ., Melbourne, Australia.

8. FRANKEL, N.E., GLASSER, M.L., HUGHES, B.D., and NINHAM, B.W., (1992), Möbius, Mellin, and Mathematical Physics, Physica A. 186, (1992), 441-481.

9. HARDY, G.H., (1907), On certain Oscillating Series, Qtly. J. Math. 38, 269-288. (Also in Collected Papers, Vol. VI, Clarendon Press, Oxford, 146-167, (1974)).

10. HARDY, G.H., (1912), Note on Dr. Vacca's Series for γ, Qtly. J. Math., 43, 215-216. (Also in Collected Papers, Vol. IV, ibid., 475-476.)

11. HARDY, G.H., (1913), An Extension of a Theorem on Oscillating Series, Proc. Lond. Math. Soc., (2), 12, 174-180. (Also in Collected Papers, Vol.VI, ibid., 500-506.)

12. HARDY, G.H., (1940), Ramanujan, Cambridge Univ. Press, London and New York (reprinted by Chelsea, New York).

13. HARDY, G.H., and LITTLEWOOD, J.E., (1926), A Further Note on the Converse of Abel's Theorem, Proc. Lond. Math. Soc., (2), 25, 219-236. (Also in Collected Papers of Hardy, Vol. VI, ibid., 699-716.)

14. HARDY, G.H., and WRIGHT, E.M., (1971), An Introduction to the Theory of Numbers, Oxford Univ. Press (Clarendon), London.

15. KUMMER, E., (1975), Theorie der idealen Primfaktoren der complexen Zahlen, welche aus den Wurzeln der Gleichung $\omega^n = 1$ gebildet sind, wenn n eine zusammengesetzte Zahl ist. Collected Papers, Vol. 1, 583-629, Springer Verlag, Berlin/New York.

16. LITTLEWOOD, J.E., (1910), The Converse of Abel's Theorem on Power Series, Proc. Lond. Math. Soc., (2), 9, 434-448.

17. MAHLER, K., (1929), Arithmetische Eigenschaften der Lösungen einer Klasse von Funktional-gleichungen. Math. Ann., 101, 342-366.

18. MAHLER, K., (1930), Über das Verschwinden von Potenzreihen mehrer Veränderlichen in Gewissen Punktfolgen. Math. Ann., 103, 573-587.

19. MAHLER, K., (1930), Arithmetische Eigenschaften einer Klasse transzendental-transzendenter Funktionen. Math. Z., 32, 545-585.

20. MAHLER, K., (1980), On a Special Function. J. Number Theory, 12, 20-26.

21. MAHLER, K., (1981), On Some Irrational Decimal Fractions. J. Number Theory, 13, 268-269.

22. MAHLER, K., (1982), Fifty Years as a Mathematician. J. Number Theory, 14, 121-155.

23. MENDES FRANCE, M., and van der POORTEN, A.J., (1991), Some Explicit Continued Fraction Expansions. Mathematika, 38, 1-9.

24. NIEDERREITER, H., (1986), On an Irrationality Theorem of Mahler and Bundschuh, J. Number Theory, 24, 197-199.

25. NINHAM, B.W., and LIDIN, S., (1992), Some Remarks on Quasi-Crystal Structure, Acta Cryst., A48, (to appear).

26. NISHIOKA, K., (1987), Conditions for Algebraic Independence of Certain Power Series of Algebraic Numbers, Compositio Math., 62, 53-61.

27. NISHIOKA, K., (1990), A New Approach in Mahler's Method, J. Reine Angew. Math., 407, 202-219.

28. NISHIOKA, K., (1990), Algebraic Independence of Certain Power Series, Séminaire de Théorie des Nombres, Paris 1987-88, Prog. Math., 81, Birkhäuser Boston, Boston, MA., 201-212.

29. ORYAN, M.H., (1989), On Power Series and Mahler's U-Numbers, Math. Scand., 65, No. 1, 143-151.

30. PETHŐ, A., (1982), Simple Continued Fractions for the Fredholm Numbers, J. Number Theory, 14, 232-236.

31. PÓLYA, G., (1942), On Converse Gap Theorems, Trans. Amer. Math. Soc., 52, 65-71.

32. SHALLIT, J., (1979), Simple Continued Fractions for Some Irrational Numbers, J. Number Theory, 11, 209-217.

33. SHALLIT, J.O., (1982), Simple Continued Fractions for Some Irrational Numbers II, J. Number Theory, 14, 228-231.

34. SHAN, Z., (1987), A Note on Irrationality of Some Numbers, J. Number Theory, 25, 211-212.

35. SHAN, Z., and WANG, E.T.H., (1989), Generalization of a Theorem of Mahler, J. Number Theory, 32, 111-113.

36. van der POORTEN, A.J., (1986), An Introduction to Continued Fractions, Diophantine Analysis, London Math. Soc., Lecture Note Series 109, Cambridge Univ. Press.

37. WU, T., (1986), On the Proof of Continued Fraction Expansions for Irrationals, J. Number Theory, 23, 55-59.

38. ZHU, Y.C., (1985), Algebraic Independence Property of Values of Certain Gap Series, Kexue Tongbao, 30, 293-297.