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ASYMPTOTIC EQUIVALENCE AND SUMMABILITY

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ABSTRACT. This paper is a study of the relationships between the asymptotic equivalence of two sequences ($\lim_n x_n/y_n = 1$) and three variations of this equivalence. For a sequence-to-sequence transformation A, the three variations are given by the ratios $R_m Ax/R_m Ay$, $S_m Ax/S_m Ay$, and $\mu_m Ax/\mu_m Ay$, where $R_m Az := \sum_{n\geq m} |(Az)_n|$, $S_m Az := \sum_{n\leq m} |(Az)_n|$, and $\mu_m Az := \sup_{n\geq m} |(Az)_n|$.

KEY WORDS AND PHRASES. Asymptotically regular, Asymptotic equivalence, Nörland-type matrix $\bar{N}_p$.
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1. INTRODUCTION.

In [2] Pobyvanets introduced the concept of *asymptotically regular* matrices, which preserve the asymptotic equivalence of two nonnegative number sequences; that is, if the quotient x_n/y_n has limit 1 then the quotient $(Ax)_n/(Ay)_n$ of the transformed sequences also has a limit 1. Stated symbolically this is $x \sim y$ implies $Ax \sim Ay$. The frequent occurrence of terms having zero value makes a term-by-term ratio x_n/y_n inapplicable in many cases. Therefore in [1] Fridy introduced new ways of comparing rates of convergence. For x in ℓ^1 he used the remainder sum, whose m-th term is $R_m x := \sum_{n=m}^{\infty} |x_n|$, and examined the ratio $R_m x/R_m y$ as $m \to \infty$. For the case where x is not in ℓ^1 he used the sequence of partial sums of moduli which is given by $S_m x := \sum_{n\leq m} |x_n|$. If x is a bounded sequence he used the supremum of the remaining terms which is given by $\mu_m t := \sup_{n\geq m} |t_n|$. In each case the ratio x_n/y_n can be replaced by the appropriate new ratio: $R_m x/R_m y$, $S_m x/S_m y$, or $\mu_m x/\mu_m y$. The goal of the work in [1] was to study the relationship of these ratios when they tend to zero. In the present study we investigate the relationships of these ratios when they have limit one.

2. BASIC THEOREMS.

The following results are stated here for the convenience of the reader and to illustrate our notation.

THEOREM 1. (Pobyvanets [2]). The nonnegative matrix A is asymptotically regular if and only if for each fixed integer $m, \lim_{n\to\infty} a_{nm}/\sum_{k=1}^{\infty} a_{nk} = 0$.

THEOREM 2. The Nörland type matrix $\bar{N}_p$ is regular if and only if $\lim_n P_n = \infty$.

THEOREM 3. The matrix A is a $c_0 - c_0$ matrix (i.e.. A preserves zero limits) if and only if

(a) $\lim_n a_{nk} = 0$ for $k = 1, 2, \ldots$, and

(b) There exists a number $M > 0$ such that for each $n, \sum_{k=1}^{\infty} |a_{nk}| < M$.

3. ASYMPTOTIC EQUIVALENCE.

In order to give the main results we first introduce some notation.

NOTATIONS:

P_δ = {the set of all real number sequences such that $x_k \geq \delta > 0$ for all k}

P_0 = {the set of all nonnegative sequences which have at most a finite number of zero terms}

P = {the set of all sequences x such that $x_k > 0$ for all k}

$$\bar{N}_p \text{ is defined by } \bar{N}_p[n,k] := \begin{cases} \frac{p_k}{P_n} & , \text{ if } k \leq n, \\ 0 & , \text{ if } k > n, \end{cases}$$

where $\{p_n\}_{n=0}^{\infty}$ is the sequence of nonnegative real numbers with $p_0 > 0$ and $P_n := \sum_{k=0}^{n} p_k$.

THEOREM 1. Let A be a nonnegative $c_0 - c_0$ matrix, and let x and y be bounded sequences such that $x \sim y$ and $x, y \in P_\delta$ for some $\delta > 0$, then $\mu Ax \sim \mu Ay$.

PROOF. Since $x \sim y$ we can write $x_n = y_n(1 + z_n)$ for some null sequence z. For each n we have

$$\begin{aligned} \frac{(\mu Ax)_n}{(\mu Ay)_n} &= \frac{\sup\limits_{k\geq n} (Ax)_k}{\sup\limits_{k\geq n} (Ay)_k} = \frac{\sup\limits_{k\geq n} \sum\limits_{i=0}^{\infty} a_{ki}x_i}{\sup\limits_{k\geq n} \sum\limits_{i=0}^{\infty} a_{ki}y_i} \\ &= \frac{\sup\limits_{k\geq n} \sum\limits_{i=0}^{\infty} a_{ki}(y_i + y_i z_i)}{\sup\limits_{k\geq n} \sum\limits_{i=0}^{\infty} a_{ki}y_i} \leq 1 + \frac{\sup\limits_{k\geq n} \sum\limits_{i=0}^{\infty} a_{ki}y_i|z_i|}{\sup\limits_{k\geq n} \sum\limits_{i=0}^{\infty} a_{ki}y_i} \\ &\leq 1 + \frac{\sup\limits_{k\geq n} \sum\limits_{i=0}^{\infty} a_{ki}y_i|z_i|}{\delta \sup\limits_{k\geq n} \sum\limits_{i=0}^{\infty} a_{ki}}. \end{aligned}$$

Since y is a positive bounded sequence, z is a null sequence, and A is a $c_0 - c_0$ matrix, it follows that $\lim_n \sum_{i=0}^{\infty} a_{ki}y_i|z_i| = 0$, and therefore $\lim_n ((\mu Ax)_n/(\mu Ay)_n) \leq 1$. In a similar manner we can get $\lim_n \{(\mu Ax)_n/(\mu Ay)_n\} \geq 1$, hence $\lim_n \{(\mu Ax)_n/(\mu Ay)_n\} = 1$, i.e, $\mu Ax \sim \mu Ay$.

THEOREM 2. If A is a nonnegative $\ell^\infty - \ell^1$ summability matrix then the following are equivalent:

(i) if x and y are bounded sequences such that $x \sim y$ and $y \in P_\delta$ for some $\delta > 0$, then $RAx \sim RAy$;

(ii) for each m,

$$\lim_{n\to\infty} \left(\frac{\sum\limits_{k=n}^{\infty} a_{km}}{\sum\limits_{k=n}^{\infty} \sum\limits_{j=0}^{\infty} a_{kj}} \right) = 0.$$

PROOF. [(ii) $\Rightarrow$ (i)] The hypothesis $x \sim y$ implies that for $\varepsilon > 0$ there exists a number J such that if $k \geq J$ then $|(x_k/y_k) - 1| < \varepsilon$. Hence, for all $k \geq J$ we have

$$(1-\varepsilon)y_k \leq x_k \leq (1+\varepsilon)y_k. \tag{3.1}$$

First,

$$R_n Ax = \sum_{k=n}^{\infty} (Ax)_k \leq \sum_{k=n}^{\infty} \sum_{j=0}^{J-1} a_{kj}x_j + \sum_{k=n}^{\infty} \sum_{j=J}^{\infty} a_{kj}x_j.$$

By using (3.1) we have

$$R_n Ax \leq \left(\sum_{j=0}^{J-1} x_j \right) \sum_{k=n}^{\infty} \left(\max_{0\leq j\leq J-1} a_{kj} \right) + (1+\varepsilon) \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}y_j$$

Hence,

$$\frac{R_nAx}{R_nAy} \le \frac{\left(\sum_{j=0}^{J-1} x_j\right)\sum_{k=n}^{\infty}\left(\max_{0\le j\le J-1} a_{kj}\right)}{\delta\sum_{k=n}^{\infty}\sum_{j=0}^{\infty} a_{kj}} + 1 + \varepsilon$$

Now we claim that

$$\lim_{n\to\infty}\left[\frac{\left(\sum_{k=n}^{\infty}\left(\max_{0\le j\le J-1} a_{kj}\right)\right)}{\sum_{k=n}^{\infty}\sum_{j=0}^{\infty} a_{kj}}\right] = 0.$$

Using (ii), we get

$$\sum_{j=0}^{J-1}\left(\frac{\left(\sum_{k=n}^{\infty} a_{kj}\right)}{\sum_{k=n}^{\infty}\sum_{j=0}^{\infty} a_{kj}}\right) = 0,$$

where J is finite; hence,

$$0 \ge \lim_{n\to\infty}\left[\frac{\left(\sum_{k=n}^{\infty}\left(\max_{0\le j\le J-1} a_{kj}\right)\right)}{\sum_{k=n}^{\infty}\sum_{j=0}^{\infty} a_{kj}}\right].$$

Since A is a nonnegative matrix we have proved the claim, and it follows that

$$\limsup_n \left(\frac{R_nAx}{R_nAy}\right) \le 1 + \varepsilon. \tag{3.2}$$

Second, we seek a lower bound for R_nAx/R_nAy. Consider

$$R_nAx = \sum_{k=n}^{\infty}\sum_{j=0}^{\infty} a_{kj}x_j \ge \sum_{k=n}^{\infty}\left(\min_{0\le j\le J-1} a_{kj}\right)\sum_{j=0}^{J-1} x_j + \sum_{k=n}^{\infty}\sum_{j=J}^{\infty} a_{kj}x_j.$$

Using Inequality (3.1) again we get

$$R_nAx \ge \left(\sum_{j=0}^{J-1} x_j\right)\sum_{k=n}^{\infty}\left(\min_{0\le j\le J-1} a_{kj}\right) + (1-\varepsilon)\sum_{k=n}^{\infty}\sum_{j=0}^{\infty} a_{kj}y_j - (1-\varepsilon)\sum_{k=n}^{\infty}\sum_{j=n}^{J-1} a_{kj}y_j.$$

Therefore

$$\frac{R_nAx}{R_nAy} \ge \left(\frac{\sum_{j=0}^{J-1} x_j}{\sup_j y_j}\right)\left(\frac{\sum_{k=n}^{\infty}\min_{0\le j\le J-1} a_{kj}}{\sum_{k=n}^{\infty}\sum_{j=0}^{\infty} a_{kj}} - \right) + (1-\varepsilon)$$

$$- \left(\frac{(1-\varepsilon)\sum_{j=0}^{J-1} y_j}{\delta}\right)\left(\frac{\sum_{k=n}^{\infty}\max_{0\le j\le J-1} a_{kj}}{\sum_{k=n}^{\infty}\sum_{j=0}^{\infty} a_{kj}}\right).$$

Using the previous claim we have

$$\liminf_n \left(\frac{R_nAx}{R_nAy}\right) \ge 1 - \varepsilon \tag{3.3}$$

From (3.2) and (3.3) we conclude that $\lim_n (R_nAx/R_nAy) = 1$, whence $RAx \sim RAy$.

To show that (i) implies (ii) we choose a fixed positive integer m and define x and y as follows: $y_k := 1$ for all k and

$$x_k := \begin{cases} 0, & \text{if } k \le m, \\ 1, & \text{if } k > m. \end{cases}$$

It is obvious that the conditions in (i) are satisfied by the sequences x and y. Consider

$$R_n Ax = \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} x_j = \sum_{k=n}^{\infty} \sum_{j=m+1}^{\infty} a_{kj} = \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} - \sum_{k=n}^{\infty} \sum_{j=0}^{m} a_{kj}.$$

Thus

$$\left(\frac{R_n Ax}{R_n Ay}\right) = 1 - \left[\frac{\left(\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}\right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj} y_i}\right].$$

The hypothesis that A is a nonnegative matrix implies

$$\left(\frac{R_n Ax}{R_n Ay}\right) \le 1 - \left[\frac{\left(\sum_{k=n}^{\infty} a_{km}\right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}}\right].$$

Hence,

$$\liminf_n \left(\frac{R_n Ax}{R_n Ay}\right) \le 1 - \limsup_{n\to\infty} \left[\frac{\left(\sum_{k=n}^{\infty} a_{km}\right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}}\right].$$

From (i) we get $\limsup_n \left[\left(\sum_{k=n}^{\infty} a_{km}\right) / \sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}\right] = 0$, and since A is nonnegative matrix we have

$$\lim_{n\to\infty} \left[\frac{\left(\sum_{k>n} a_{km}\right)}{\sum_{k=n}^{\infty} \sum_{j=0}^{\infty} a_{kj}}\right] = 0.$$

Hence, the proof of the theorem is complete.

In the following theorem we use another notation that was introduced by Fridy in [1] to study sequences that converge at different rates. Now, we use the same notation to study sequences that converge at the same rate. For sequences x and y that are not in ℓ^1 Fridy used the partial sum $S_N x := \sum_{k \le N} |x_k|$ to say that y diverges faster than x provided that $S_N x = o(S_N y)$. Here we use $S_N Ax \sim S_N Ay$ to indicate that $\sum_{n \le N} (Ax)_n$ and $\sum_{n \le N} (Ay)_n$ diverge at the same rate.

THEOREM 3. If A is a nonnegative matrix, then the following statements are equivalent:

(a) if x and y are sequences such that $x \sim y$, $x \in P_0$, and $y \in P_\delta$ for some $\delta > 0$, then Ax and Ay are not in ℓ^1 and $SAx \sim SAy$;

(b) A satisfies the following conditions:

(i) $\left\{\sum_{j=0}^{\infty} a_{kj}\right\}_{k=0}^{\infty} \notin \ell^1$;

(ii) for each m,

$$\lim_{N\to\infty} \left(\frac{\sum_{k=1}^{N} a_{km}}{\sum_{k=1}^{N} \sum_{j=0}^{\infty} a_{kj}}\right) = 0.$$

PROOF. Suppose that condition (b) is satisfied and assume that x and y are sequences such that $x \sim y$, $x \in P_0$, and $y \in P_\delta$ for some $\delta > 0$. Using condition (i), it is clear that Ay is not in ℓ^1. Also, the fact that $x \sim y$ implies that for each given $\varepsilon > 0$ there exists an integer J such that $|x_k/y_k - 1| < \varepsilon$ whenever $k \geq J$, or equivalently, if $k \geq J$ then

$$(1-\varepsilon)y_k \leq x_k \leq (1+\varepsilon)y_k \tag{3.4}$$

From this it is easy to prove that Ax is not in ℓ^1.

Now, we need to show that $SAx \sim SAy$. Consider

$$S_N Ax = \sum_{k=1}^{N}\sum_{j=1}^{\infty} a_{kj}x_j = \sum_{k=1}^{N}\sum_{j=1}^{J-1} a_{kj}x_j + \sum_{k=1}^{N}\sum_{j=J}^{\infty} a_{kj}x_j.$$

By using (3.4) we get

$$S_N Ax \leq \sum_{k=1}^{N}\sum_{j=1}^{J-1} a_{kj}x_j + (1+\varepsilon)\sum_{k=1}^{N}\sum_{j=J}^{\infty} a_{kj}y_j$$

$$\leq \left(\sum_{j=1}^{J-1} x_j\right)\sum_{k=1}^{N}\left(\max_{0\leq j\leq J-1} a_{kj}\right) + (1+\varepsilon)\sum_{k=1}^{N}\sum_{j=0}^{\infty} a_{kj}y_j$$

$$\leq \left(\sum_{j=1}^{J-1} x_j\right)\sum_{k=1}^{N}\left(\max_{0\leq j\leq J-1} a_{kj}\right) + (1+\varepsilon)S_N Ay.$$

Therefore

$$\frac{S_N Ax}{S_N Ay} \leq \left(\frac{\sum_{j=1}^{J-1} x_j}{\delta}\right)\left(\frac{\sum_{k=1}^{N}\left(\max_{0\leq j\leq J-1} a_{kj}\right)}{\sum_{k=1}^{N}\sum_{j=0}^{\infty} a_{kj}}\right) + (1+\varepsilon).$$

Hence, from condition (ii) we get

$$\lim_{N\to\infty} \frac{S_N Ax}{S_N Ay} \leq 1+\varepsilon, \tag{3.5}$$

similarly, using the left-hand inequality in (3.4) we have

$$S_N Ax \geq \sum_{k=1}^{N}\sum_{j=0}^{J-1} a_{kj}x_j + (1-\varepsilon)\sum_{k=1}^{N}\sum_{j=J}^{\infty} a_{kj}y_j$$

$$\geq \left(\sum_{k=1}\left(\min_{0\leq j\leq J-1} a_{kj}\right)\right)\sum_{j=0}^{J-1} x_j + (1-\varepsilon)\sum_{k=1}^{N}\sum_{j=0}^{\infty} a_{kj}y_j - (1-\varepsilon)\sum_{k=1}^{N}\sum_{j=0}^{J-1} a_{kj}y_j.$$

Thus

$$\frac{S_N Ax}{S_N Ay} \geq \left(\frac{\sum_{j=0}^{J-1} x_j}{\sup_j y_j}\right)\frac{\sum_{k=1}^{N}\left(\max_{0\leq j\leq J-1} a_{kj}\right)}{\sum_{k=1}^{N}\sum_{j=0}^{\infty} a_{kj}} + 1 - \varepsilon$$

$$- \left(\frac{(1-\varepsilon)\sum_{j=0}^{J-1} y_j}{\delta}\right)\frac{\sum_{k=0}^{N}\left(\max_{0\leq j\leq J-1} a_{kj}\right)}{\sum_{k=1}^{N}\sum_{j=0}^{\infty} a_{kj}}.$$

As before, from condition (ii) we see that the first and the third terms in the previous inequality tend to zero as $N \to \infty$. Hence,

$$\lim_{N\to\infty} \frac{S_N Ax}{S_N Ay} \geq 1 - \varepsilon, \tag{3.6}$$

From (3.5) and (3.6) we conclude that $\lim_{N\to\infty} S_N Ax/S_N Ay = 1$, i.e., $SAx \sim SAy$.

To prove that (a) implies (b), we first need to show the necessity of condition (ii). Thus, we form two sequences x and y such that $x \sim y$ and both satisfy the hypothesis in (a), but SAx and SAy are not asymptotically equivalent if condition (ii) does not hold. Let m be an arbitrary fixed positive integer and define $y_k := 1$ for all k, and

$$x_k := \begin{cases} 0, & \text{if } k \leq m, \\ 1, & \text{if } k > m. \end{cases}$$

It is clear that $x \sim y, x \in P_0$, and $y \in P_1$. Assume condition (ii) does not hold. For any such m we have

$$\begin{aligned} S_N Ax &= \sum_{k=1}^{N} \sum_{m+1}^{\infty} a_{kj} = \sum_{k=1}^{N} \sum_{j=0}^{n} a_{kj} - \sum_{k=1}^{N} \sum_{j=0}^{m} a_{kj} \\ &\leq \sum_{k=1}^{N} \sum_{j=0}^{\infty} a_{kj} - \sum_{k=1}^{N} a_{km}. \end{aligned}$$

Therefore,

$$\liminf_{N} \frac{S_N Ax}{S_N Ay} \leq 1 - \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} a_{km}}{\sum_{k=1}^{N} \sum_{j=0}^{\infty} a_{kj}}.$$

Since A is a nonnegative matrix and condition (ii) does not hold, we conclude that

$$\lim_{N\to\infty} \frac{S_N Ax}{S_N Ay} < 1,$$

where SAx and SAy are not asymptotically equivalent.

It remains to show that the sequence $\{\sum_{j=0}^{\infty} a_{kj}\}_{k=1}^{\infty}$ is not in ℓ^1. Since y is identically 1, we get $(Ay)_k = \sum_{j=0}^{\infty} a_{kj}$, and since the sequence Ay is not in ℓ^1, we see that $\{\sum_{j=0}^{\infty} a_{kj}\}_{k=0}^{\infty}$ is not in ℓ^1. Therefore the proof is complete.

In the following theorem we study the relationship between the original asymptotic equivalence that uses term by term comparison and the asymptotic equivalence that uses the remainder sum comparison.

THEOREM 4. If x and y are nonnegative sequences in ℓ^1 such that $x \sim y$, then $Rx \sim Ry$.

PROOF. The assumption that $x \sim y$ implies that for a given $\varepsilon > 0$ there exists an integer N such that $|x_n/y_n - 1| \leq \varepsilon$ for all $n \geq N$.

Hence,

$$R_n x \leq (1+\varepsilon) \sum_{k \geq n} y_k = (1+\varepsilon) R_n y$$

and,

$$R_n x \geq (1-\varepsilon) \sum_{k \geq n} y_k = (1-\varepsilon) R_n y.$$

Therefore, for all $n \geq N$,

$$(1-\varepsilon) \leq \frac{R_n x}{R_n y} \leq (1+\varepsilon),$$

and this implies that $\lim_{n\to\infty} R_n x/R_n y = 1$, i.e., $Rx \sim Ry$.

The next theorem shows the relationship between the original symptotic equivalence and the asymptotic equivalence that uses the partial sum comparison.

THEOREM 5. If x and y are nonnegative sequences not in ℓ^1 such that $x \sim y$, then $Sx \sim Sy$.

PROOF The assumption that $x \sim y$ implies that $x_n/y_n - 1 = \varepsilon_n$, where $\lim_{n\to\infty} \varepsilon_n = 0$. Since x is nonnegative we have

$$S_m x = \sum_{n\le m} |x_n| = \sum_{n\le m} (y_n + \varepsilon_n y_n) = \sum_{n\le m} y_n + \sum_{n\le m} \varepsilon_n y_n.$$

Therefore,

$$\frac{S_m x}{S_m y} = 1 + \frac{\sum_{n\le m} \varepsilon_n y_n}{\sum_{n\le m} y_n} = 1 + \sum_{n\le m} \left(\frac{y_n}{\sum_{j\le m} y_j} \right) \varepsilon_n$$

$$= 1 + \left(\bar{N}y\varepsilon\right)_m,$$

where $\bar{N}y$ is the Nörland-type matrix [3, p. 45]. Since y is not in $\ell^1, \bar{N}y$ is regular; therefore $\lim_{m\to\infty} \left(\bar{N}y\varepsilon\right)_m = 0$. Thus we have $\lim_{m\to\infty} S_m x/S_m y = 1$, i.e., $Sx \sim Sy$.

In the last result we study the relationship between the original asymptotic equivalence and the asymptotic equivalence that compares the "supremum of the remaining terms."

THEOREM 6. If x and y are nonvanishing null sequences such that $x \sim y$, then $\mu x \sim \mu y$.

PROOF. The hypothesis $x \sim y$ implies that for $\varepsilon > 0$ there exists an integer N such that if $k \ge N$ then

$$(1-\varepsilon)y_k \le x_k \le (1+\varepsilon)y_k.$$

For $n \ge N$ we have

$$\mu_n x = \sup_{k\ge n} |x_k| \le (1+\varepsilon) \sup_{k\ge n} |y_k|.$$

Hence,

$$\limsup_n \frac{\mu_n x}{\mu_n y} \le 1 + \varepsilon.$$

Also,

$$\mu_n x = \sup_{k\ge n} |x_k| \ge (1-\varepsilon) \sup_{k\ge n} |y_k|.$$

Hence,

$$\liminf_n \frac{\mu_n x}{\mu_n y} \ge 1 - \varepsilon.$$

Therefore we conclude that $\lim_{n\to\infty} \mu_n x/\mu_n y = 1$, i.e., $\mu x \sim \mu y$.

REMARK. The converse of Theorems 5 and 6 is not true i.e., there exist sequences x and y such that $Sx \sim Sy$ and $\mu x \sim \mu y$ but x is not asymptotically equivalent to y. It is enough here to give the two sequences x and y and leave the proof to the reader.

$$x_n := \begin{cases} \dfrac{2}{n+1}, & \text{if } n \text{ is odd,} \\ \dfrac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

$$y_n := \begin{cases} \dfrac{1}{n+1}, & \text{if } n \text{ is odd,} \\ \dfrac{2}{n}, & \text{if } n \text{ is even.} \end{cases}$$

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