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RESEARCH NOTES

ON ANALYTIC CONTINUATION AND FUNCTIONAL EQUATION OF CERTAIN DIRICHLET SERIES

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ABSTRACT. Analytic continuation and functional equation of Riemann's type are proved for a class of Dirichlet series associated to rational functions.

KEY WORDS AND PHRASES. Dirichlet series, analytic continuation, functional equation, Hurwitz zeta function.

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1. INTRODUCTION.

In this paper we are concerned with analytic continuation and functional equation (of Riemann's type) of Dirichlet series. Our objective is to show how a very classical method, which is one of Riemann's methods (see [1]), works for a quite general class of Dirichlet series. In [2] this method is applied to Dirichlet L-series $L(s,\chi)$ where χ is the non-principal Dirichlet character *mod* 3.

In the course of the proof it turns out that the Dirichlet series we consider in our paper are expressed in terms of Hurwitz zeta functions. This result should be compared with Theorem 1 of Arakawa [3] where a representation of $\sum_{n=1}^{\infty} cotg\alpha n\pi\, n^{-s}$ in terms of Barnes zeta functions is given.

Let $L(s)=\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with $a_n \in \mathbb{C}$, having finite abscissa of absolute convergence σ_o. We assume that the power series $\sum_{n=1}^{\infty} a_n z^n$, associated to $L(s)$, defines a (non constant) rational function $G(z)=\frac{P(z)}{Q(z)}$ such that:

(1) $\partial P \leq \partial Q = M$

(2) $Q(z)$ has zeros $\tau_1,\cdots,\tau_m$, where $\tau_j = exp(2\pi i\vartheta_j), \vartheta_j \in [-\frac{1}{2}, \frac{1}{2})$ for $1 \leq j \leq m$ with multiplicity n_j

(3) $P(\tau_j) \neq 0$ for $1 \leq j \leq m$.

From the above assumptions it follows that the radius of convergence ρ of $\sum_{n=1}^{\infty} a_n z^n$ is 1. We prove the following:

THEOREM. Let $L(s)$ and $G(z)$ be functions with the above hypotheses. Then we have

(1) if $z=1$ is a regular point for $G(z)$, then $L(s)$ can be continued to an entire function;

(2) if $z=1$ is a pole for $G(z)$ of order $n_o \geq 1$, then $L(s)$ can be continued to a meromorphic function over $\mathbb{C}$ with a simple pole at $s=n_o$ and possibly simple poles at $s=1,2,\cdots,n_o-1$.

Furthermore, if we put

$$\Phi(s) = \sum_{\vartheta_j > 0} \frac{1}{(n_j-1)!} \sum_{h=0}^{n_j-1} (2\pi)^{-h} \binom{-s}{h} h!\, T_j^h \,[(-i)^{-s-h}\zeta(s+h, 1-\vartheta_j) +$$

$$+ i^{-s-h}\zeta(s+h,\vartheta_j)] + \sum_{\vartheta_j < 0} \frac{1}{(n_j-1)!} \sum_{h=0}^{n_j-1} (2\pi)^{-h} \binom{-s}{h} h!\, T_j^h \,[(-i)^{-s-h}\zeta(s+h, -\vartheta_j) +$$

$$+ i^{-s-h}\zeta(s+h, 1+\vartheta_j)] + \frac{1}{(n_o-1)!} \sum_{h=0}^{n_o-1} (2\pi)^{-h} \binom{-s}{h} h!\, T_o^h \,\zeta(s+h)[(-i)^{-s-h} + i^{-s-h}],$$

where the last sum appears if $Q(1) = 0$ with multiplicity n_o, $\zeta(s,a)$ is the Hurwitz zeta function with $0 < a \le 1$ and T_j^h are suitable constants computed in §3, then $\Phi(s)$ is an entire function and the function

$$\xi(s) = (2\pi)^{-s}\,\Gamma(s)L(s)\Phi(s)$$

satisfies the functional equation $\xi(s) = \xi(1-s)$.

We note that the class of Dirichlet series which satisfy our hypotheses contains strictly that of linear combination of shifted Dirichlet L-series $L(s-k,\chi)$, k non-negative integer.

2. SOME LEMMAS.

LEMMA 2.1. Let $G(z) = \sum_{n=1}^{\infty} a_n z^n$ be a complex power series with radius of convergence $\rho = 1$. If $z = 1$ is a pole of order $n_o \ge 1$ for $G(z)$, then the Dirichlet series $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has a meromorphic continuation over $\mathbb{C}$ with simple poles at $s = n_o$ and possibly at $s = 1, \cdots, n_o - 1$. If $z = 1$ is a regular point, then $L(s)$ is continuable as an entire function.

PROOF. From the classical integral representation of $\Gamma(s)$ one gets

$$L(s)\Gamma(s) = \sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-nt}\, t^{s-1}\, dt \tag{2.1}$$

for $\sigma = Re\ s$ sufficiently large.

The function $G(e^{-t})$ is $0(e^{-Kt})$ for a suitable $K > 0$, as $t \to +\infty$, and is infinite of order n_o, as $t \to 0^+$, provided $z = 1$ is a pole. So we get from (2.1)

$$L(s)\Gamma(s) = \int_0^1 G(e^{-t})\, t^{s-1}\, dt + \int_1^{\infty} G(e^{-t})\, t^{s-1}\, dt \qquad (\sigma\ large). \tag{2.2}$$

The last integral is an entire function which will be denoted by $M(s)$.

Let $G(e^{-t}) = \sum_{n=-n_o}^{\infty} \alpha_n t^n$ be the Laurent expansion at $t = 0$; we can suppose that its outer radius λ is greater than 1 (for otherwise we write $\frac{\lambda}{2}$ instead 1 in (2.2)). Then

$$\int_0^1 G(e^{-t})\, t^{s-1}\, dt = \int_0^1 \sum_{n=-n_o}^{\infty} \alpha_n\, t^{n+s-1}\, dt = \sum_{n=-n_o}^{\infty} \frac{\alpha_n}{n+s}. \tag{2.3}$$

The last series in (2.3) defines a meromorphic function with simple poles at $s = n_o$ and possibly at $s = n < n_o$, since it converges uniformly on any compact subset of $\{s \in \mathbb{C}: |s+n| \ge C, n \ge -n_o\}$, C being fixed positive constant.

From (2.2) and (2.3) one has

$$L(s)\Gamma(s) = M(s) + \sum_{n=-n_o}^{\infty} \frac{\alpha_n}{n+s},$$

so that our first claim follows, because of the poles of $\Gamma(s)$ at non-positive integers.

The second claim follows from the above argument with $n_o = 0$.

LEMMA 2.2. Let $L(s)$ and $G(z)$ satisfy the hypotheses of the Theorem. Let us consider the entire function

$$I(s) = -\frac{1}{\pi} L(s)\, \Gamma(s)\, \sin \pi s$$

then for $\sigma > 1$

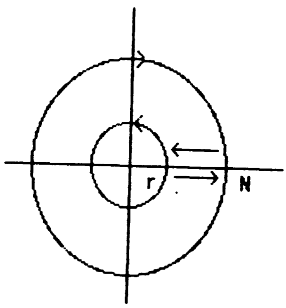
$$I(1-s) = \sum_{j=1}^{M} \sum_{k=-\infty}^{+\infty} Res(H(z)(-z)^{-s}, z_{j,k}) ,$$

where $H(z) = G(e^{-z})$ and $z_{j,k} = 2\pi i(-\vartheta_j + k)$ are the non zero complex numbers such that $exp(-z_{j,k}) = r_j$, $1 \le j \le m$.

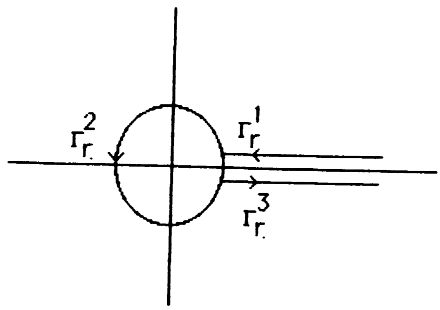
PROOF. Let $C_{N,r}$ and Γ_r be the contours drawn below. We put

$$I_{N,r}(s) = \frac{1}{2\pi i} \int_{C_{N,r}} H(z)(-z)^{s-1}\, dz , \quad I_r(s) = \frac{1}{2\pi i} \int_{\Gamma_r} H(z)(-z)^{s-1}\, dz.$$

Here $(-z)^{s-1}$ is defined to be $exp((s-1)log(-z))$, where the logarithm is real on the positive real axis. If r is sufficiently small and $N \neq |z_{j,k}|$ for each j,k, then the above functions are analytic functions of s, because they are expressed as sums of absolutely convergent integrals with parameter s of univalent holomorphic functions.



$C_{N,r}$



Γ_r

We know that, by a standard argument based on Cauchy's theorem, $I_r(s)$ is independent on r. Moreover, since $H(z)(-z)^{n_o}$ is holomorphic, one gets

$$\left| \int_{|z|=r} H(z)(-z)^{s-1}\, dz \right| \le 2\pi L\, e^{|t|} r^{\sigma - n_o}$$

where $L = sup\, \{ | H(z)(-z)^{n_o} | : |z| = r \}$ and $s = \sigma + it$. It follows that, for $\sigma > n_o$,

$$\lim_{r \to 0} \int_{|z|=r} H(z)(-z)^{s-1}\, dz = 0.$$

Hence, for $\sigma > n_o$

$$I(s) = -\frac{1}{\pi} L(s)\Gamma(s) sin\, \pi s = -\frac{1}{\pi} sin\, \pi s \int_0^\infty H(t) t^{s-1}\, dt$$

$$= \frac{1}{2\pi i} \int_\infty^0 H(t) exp((s-1)(log t - \pi i)) dt - \frac{1}{2\pi i} \int_0^\infty H(t) exp((s-1)(log t + \pi i))\, dt$$

$$= \lim_{r \to 0} \frac{1}{2\pi i} \int_{\Gamma_r} H(z)(-z)^{s-1}\, dz = \lim_{r \to 0} I_r(s).$$

The above equality holds on the whole complex plane for analytic continuation.

Now, we can fix some $\delta > 0$ and choose N such that $|N - |z_{j,k}|| \geq \delta$ in order to obtain $|H(z)| \leq K$, with $K = K(\delta)$, for $|z| = N$. We obtain, from

$$\left| \int_{|z| = N} H(z)(-z)^{s-1}\, dz \right| \leq 2\pi K e^{|t|} N^{\sigma}$$

that, for $\sigma < 0$

$$\lim_{N \to \infty} \int_{|z| = N} H(z)(-z)^{s-1}\, dz = 0.$$

Finally for $\sigma > 1$

$$I(1-s) = \lim I_r(1-s) = \lim I_{N,r}(1-s) = \sum_{j=1}^{M} \sum_{k=-\infty}^{+\infty} Res(H(z)(-z)^{-s}, z_{j,k}).$$

3. PROOF OF THE THEOREM.

It is clear that Lemma 2.1 gives immediately the analytic (meromorphic) continuation.

It remains to prove functional equation. An easy computation shows

$$Res(H(z)(-z)^{-s}, z_{j,k}) = \frac{1}{(n_j - 1)!} \sum_{h=0}^{n_j - 1} (2\pi)^{-h-s} \binom{-s}{h} h!\, T_j^h (i(-\vartheta_j + k))^{-s-h},$$

where $T_j^h = (-1)^h \binom{n_j - 1}{h} S^{(n_j - 1 - h)}(z_{j,k})$ with $S(z) = H(z)(z - z_{j,k})^{n_j}$. We note that T_j^h does not depend on k.

From the definition of $I(s)$ and well-known properties of $\Gamma(s)$ it follows that

$$L(1-s) = -I(1-s)\Gamma(s). \tag{3.1}$$

For $\sigma > 1$ we can apply Lemma 2.2 to (3.1) so that we get

$$L(1-s) = \Gamma(s) \Bigg\{ \sum_{\vartheta_j > 0} \frac{1}{(n_j - 1)!} \sum_{h=0}^{n_j - 1} (2\pi)^{-h-s} \binom{-s}{h} h!\, T_j^h \Bigg[\sum_{k \leq 0} i^{-s-h} (\vartheta_j - k)^{-s-h} +$$

$$+ \sum_{k>0} (-i)^{-s-h} (k - \vartheta_j)^{-s-h} \Bigg] + \sum_{\vartheta_j < 0} \frac{1}{(n_j - 1)!} \sum_{h=0}^{n_j - 1} (2\pi)^{-h-s} \binom{-s}{h} h!\, T_j^h$$

$$\Bigg[\sum_{k \geq 0} (-i)^{-s-h} (k - \vartheta_j)^{-s-h} + \sum_{k<0} i^{-s-h} (\vartheta_j - k)^{-s-h} \Bigg] +$$

$$+ \frac{1}{(n_o - 1)!} \sum_{h=0}^{n_o - 1} (2\pi)^{-h-s} \binom{-s}{h} h!\, T_o^h\, \zeta(s+h)[(-i)^{-s-h} + i^{-s-h}] \Bigg\}$$

$$= \Gamma(s)(2\pi)^{-s} \Bigg\{ \sum_{\vartheta_j > 0} \frac{1}{(n_j - 1)!} \sum_{h=0}^{n_j - 1} (2\pi)^{-h} \binom{-s}{h} h!\, T_j^h\, [i^{-s-h} \zeta(s+h, \vartheta_j) + (-i)^{-s-h}$$

$$\zeta(s+h, 1 - \vartheta_j)] + \sum_{\vartheta_j < 0} \frac{1}{(n_j - 1)!} \sum_{h=0}^{n_j - 1} (2\pi)^{-h} \binom{-s}{h} h!\, T_j^h\, [(-i)^{-s-h} \zeta(s+h, -\vartheta_j) + i^{-s-h}$$

$$\zeta(s+h,1+\vartheta_j)]+\frac{1}{(n_o-1)!}\sum_{h=0}^{n_o-1}(2\pi)^{-h}\binom{-s}{h}h!\,T_o^h\,\zeta(s+h)[(-i)^{-s-h}+i^{-s-h}]\Bigg\}$$

$$=(2\pi)^{-s}\Gamma(s)\Phi(s).$$

The function $\Phi(s)$ is entire because of well-known properties of Hurwitz and Riemann zeta functions. So the equality

$$L(1-s)=(2\pi)^{-s}\Gamma(s)\Phi(s) \tag{3.2}$$

holds, by analytic continuation, on the whole complex plane. In particular

$$L(s)=(2\pi)^{s-1}\Gamma(1-s)\Phi(1-s)\ ,$$

hence

$$\xi(s)=(2\pi)^{-s}\Gamma(s)\Phi(s)L(s)\ .$$

(which is meromorphic on $\mathbb{C}$) is invariant by $s\to 1-s$. We note that the set of (simple) poles of $\xi(s)$ is contained in $\{-n_o+1,\cdots,n_o\}$ if $n_o\geq 1$ and it is empty if $n_o=0$.

4. EXAMPLES.

The hypotheses of the Theorem are fulfilled by those Dirichlet series whose coefficients are polynomials having rational generating function. For instance, let $T_n(x)$ (resp. $U_n(x),C_n^p(x)$) be Tchebychev polynomials of the first kind (resp. of the second kind, Gegenbauer polynomials), i.e., for $|x|<1$ (see [4])

$$\sum_{n\geq 1}T_n(x)z^n=\frac{-2z^2+2xz}{z^2-2xz+1}\,,\qquad \sum_{n\geq 1}U_n(x)z^n=\frac{(1-x^2)^{\frac{1}{2}}(-z^2+2xz)}{z^2-2xz+1}.$$

$$\sum_{n\geq 0}C_n^p(x)z^n=\frac{1}{(z^2-2xz+1)^p}\,,\ p \text{ positive integer.}$$

The corresponding Dirichlet series

$$\sum_{n\geq 1}T_n(x)\,n^{-s},\qquad \sum_{n\geq 1}U_n(x)\,n^{-s},\qquad \sum_{n\geq 1}C_n^p(x)\,n^{-s} \tag{4.1}$$

have analytic continuation and functional equation.

Let χ be a Dirichlet character *mod* q (not necessarily primitive). Then $L(s,\chi)=\sum_{n\geq 1}\chi(n)n^{-s}$ is associated to the rational function $G(z)=\frac{P(z)}{Q(z)}$ where $Q(z)=z^q-1$ and $P(z)=-\sum_{n=1}^{q}\chi(n)z^n$. If χ is not principal then $P(1)=0$ and so $G(z)$ must be expressed as quotient of two coprime polynomials by dividing $P(z)$ and $Q(z)$ by $z-1$. In this case the function $\Phi(s)$ takes a very simple form

$$\Phi(s)=\sum_{j=1}^{[q/2]}[\lambda_j(s)\ \zeta(s,\tfrac{j}{q})+\mu_j(s)\zeta(s,1-\tfrac{j}{q})]$$

with

$$\lambda_j(s)=Re\left(\frac{R_j}{\tau_j}\cos\pi s/2\right)-Im\left(\frac{R_j}{\tau_j}\sin\pi s/2\right),\ \mu_j(s)=Re\left(\frac{R_j}{\tau_j}\cos\pi s/2\right)+Im\left(\frac{R_j}{\tau_j}\sin\pi s/2\right)$$

where $R_j=Res\left(\frac{P(z)}{Q(z)},\tau_j\right)$.

If $L(s)=L(s,\chi)$, then (3.2) can be deduced from well-known properties as a long but straightforward computation. In particular, our functional equation reduces to the classical one if

χ is primitive and real, whereas it takes an apparently different form if χ is primitive and complex. Indeed it connects $L(1-s,\chi)$ and $L(s,\chi)$ instead of $L(s,\bar{\chi})$.

We point out that if $arg(x+i(1-x^2)^{\frac{1}{2}})$ is not a rational multiple of π, each of series in (4.1) is not a linear combination of shifted Dirichlet L-series.

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