A METRIC SPACE ASSOCIATED WITH A PROBABILITY SPACE

KEITH F. TAYLOR and XIKUI WANG

Department of Mathematics
University of Saskatchewan
Saskatoon, Sask. Canada S7N 0W0

ABSTRACT. For a complete probability space (Ω, Σ, P), the set of all complete sub-σ-algebras of Σ, $S(\Sigma)$, is given a natural metric and studied. The questions of when $S(\Sigma)$ is compact or connected are awswered and the important subset consisting of all continuous sub-σ-algebras is shown to be closed. Connections with Christensen's metric on the von Neumann subalgebras of a Type II$_1$-factor are briefly discussed.

1. INTRODUCTION.

Let (Ω, Σ, P) be a complete probability space and let $S(\Sigma)$ denote the set of all complete sub-σ-algebras of Σ. We will define a metric d on $S(\Sigma)$ and investigate some topological properties of the resulting metric space $(S(\Sigma), d)$. In [1], Boylan introduced a metric d' on $S(\Sigma)$ for the purpose of studying convergence properties of the conditional expectation operators defined by varying sub-σ-algebras of Σ. Our metric turns out to be equivalent to Boylan's and seems to be more convenient for study using functional analytic methods.

We prove, in section 3, that $(S(\Sigma), d)$ is compact if and only if Σ is purely atomic. We also show that $(S(\Sigma), d)$ is connected if and only if Σ has at most one atom.

In section 4, we consider the continuous sub-σ-algebras of Σ and show that they form a closed nowhere dense subset of $(S(\Sigma), d)$.

There is a close analogy between probability spaces and von Neumann algebras with a faithful finite normal trace. In fact, our definition of d is modeled on a metric defined by Christensen [2] on the set of all von Neumann subalgebras of a Type II$_1$-factor. Christensen's metric has been useful in the study of the index in II$_1$-factors (see [3] and [5]). We give a short discussion of a common generalization of our metric and Christensen's in section 5.

2. THE METRICS.

In this section, we define the metric d and Boylan's metric d' and show that they are equivalent.

If $\Sigma_0 \in S(\Sigma)$, then let $L^\infty(\Sigma_0) = L^\infty(\Omega, \Sigma_0, P)$ be considered as a closed subspace of $L^\infty(\Omega, \Sigma, P)$ in the natural way. Since bounded functions are square intergrable on a probability space, we may consider $L^\infty(\Sigma_0)$ as a (non-closed) subspace of the Hilbert space $L^2(\Omega, \Sigma, P)$. It is easy to check that the unit ball in $L^\infty(\Sigma_0)$ is closed in $L^2(\Omega, \Sigma, P)$. The metric d is essentially the Hausdorff metric on the unit balls in the L^2-norm. Let

$$L^\infty(\Sigma_0)_1 = \{f \in L^\infty(\Sigma_0) : \|f\|_\infty \leq 1\}.$$

For $\Sigma_1, \Sigma_2 \in S(\Sigma)$, let

$$d(\Sigma_1, \Sigma_2) = max\{\sup_{f \in L^\infty(\Sigma_1)_1} \inf_{g \in L^\infty(\Sigma_2)_1} \|f - g\|_2, \sup_{g \in L^\infty(\Sigma_2)_1} \inf_{f \in L^\infty(\Sigma_1)_1} \|f - g\|_2\}.$$

It is easy to check that d is a metric on $S(\Sigma)$.

For the metric d', we use a definition due to Rogge [6] which is a slight variation on Boylan [1]. For $\Sigma_1, \Sigma_2 \in S(\Sigma)$, let

$$d'(\Sigma_1, \Sigma_2) = \sup_{A \in \Sigma_1} \inf_{B \in \Sigma_2} P(A \triangle B) \vee \sup_{B \in \Sigma_2} \inf_{A \in \Sigma_1} P(A \triangle B).$$

(Boylan's metric d^* has $\vee$ replaced by $+$ and clearly $\frac{1}{2}d^* \leq d' \leq d^*$, so d' and d^* are essentially the same). The arguments in [1] show that $(S(\Sigma), d')$ is a complete metric space.

For any $\Sigma_0 \in S(\Sigma)$, let ϵ^{Σ_0} denote the orthogonal projection of $L^2(\Omega, \Sigma, P)$ onto $L^2(\Omega, \Sigma_0, P)$, considered as a subspace of $L^2(\Sigma)$. Let E^{Σ_0} denote the restriction of ϵ^{Σ_0} to $L^\infty(\Sigma)$. As is well known, ϵ^{Σ_0} and E^{Σ_0} are restrictions of the conditional expectation mapping of $L^1(\Omega, \Sigma, P)$ onto $L^1(\Omega, \Sigma_0, P)$. We will use any of the well known special properties of conditional expectation without giving references.

We now consider the relationship between the metric d and d'.

THEOREM 1. For any $\Sigma_1, \Sigma_2 \in S(\Sigma)$,

$$d'(\Sigma_1, \Sigma_2) \leq d(\Sigma_1, \Sigma_2) \leq 2\sqrt{2d'(\Sigma_1, \Sigma_2)(1 - d'(\Sigma_1, \Sigma_2))}$$

Thus d' and d are equivalent metrics.

PROOF. To prove the left hand inequality, fix $A \in \Sigma_1$. It is shown in 2.1 of [4] that $\inf_{B \in \Sigma_2} P(A \triangle B)$ is achieved at the set $C = \{E^{\Sigma_2}(\chi_A) > \frac{1}{2}\}$, where χ_A is the indicator function of the set A. Some elementary manipulations show that

$$P(A \triangle C) = \frac{1}{2} - \int_\Omega |\frac{1}{2} - E^{\Sigma_2}(\chi_A)| dP$$

Now $\|\frac{1}{2} - \chi_A\|_1 = \frac{1}{2}$, so

$$P(A \triangle C) = \|\frac{1}{2} - \chi_A\|_1 - \|\frac{1}{2} - E^{\Sigma_2}(\chi_A)\|_1 \leq \|\chi_A - E^{\Sigma_2}(\chi_A)\|_1 \leq \|\chi_A - E^{\Sigma_2}(\chi_A)\|_2$$

Thus $\inf_{B \in \Sigma_2} P(A \triangle B) \leq d(\Sigma_1, \Sigma_2)$, for all $A \in \Sigma_1$. Symmetric arguments apply for Σ_2, which gives $d'(\Sigma_1, \Sigma_2) \leq d(\Sigma_1, \Sigma_2)$.

For the right inequality, note that, for any $f \in L^\infty(\Sigma_1)_1, E^{\Sigma_1}(f) = f$ and $E^{\Sigma_2}(f)$ is the (L^2-norm) closest element of $L^\infty(\Sigma_2)_1$ to f. Thus,

$$\sup_{f \in L^\infty(\Sigma_1)_1} \inf_{g \in L^\infty(\Sigma_2)_1} \|f - g\|_2 = \sup_{f \in L^\infty(\Sigma_1)_1} \|f - E^{\Sigma_2}(f)\|_2 \leq \sup_{f \in L^\infty(\Sigma_1)_1} \|E^{\Sigma_1}(f) - E^{\Sigma_2}(f)\|_2$$

Similarly for the other term in $d(\Sigma_1, \Sigma_2)$, so we have the inequality

$$d(\Sigma_1, \Sigma_2) \leq \sup_{f \in L^\infty(\Sigma)_1} \|E^{\Sigma_1}(f) - E^{\Sigma_2}(f)\|_2$$

However, for any $f \in L^\infty(\Sigma)_1$, the function $f' = \frac{1}{2}(1 + f)$ takes values in $[0, 1]$ almost everywhere

and theorem 3 in [6] shows that

$$\|E^{\Sigma_1}(f') - E^{\Sigma_2}(f')\|_2 \leq \sqrt{2d'(\Sigma_1, \Sigma_2)(1 - d'(\Sigma_1, \Sigma_2))}$$

Thus,

$$\|E^{\Sigma_1}(f) - E^{\Sigma_2}(f)\|_2 \leq 2\sqrt{2d'(\Sigma_1, \Sigma_2)(1 - d'(\Sigma_1, \Sigma_2))}$$

This completes the proof of the theorem.

COROLLARY. The identity map is a homeomorphism of $(S(\Sigma), d')$ with $(S(\Sigma), d)$.

REMARK 1. Not only are d and d' equivalent in the sense that they define the same open sets in $S(\Sigma)$, but the same sequences are Cauchy in each of $(S(\Sigma), d)$ and $(S(\Sigma), d')$. Since $(S(\Sigma), d')$ is complete, we have that $(S(\Sigma), d)$ is a complete metric space.

REMARK 2. The right hand inequality in theorem 1 cannot be improved to one of the form $d \leq kd'$ for some positive constant k, as is shown by the following example.

EXAMPLE. Let $\Omega = [0, 1]$, with P denoting Lebesgue measure on Σ, the Lebesgue measurable subsets of $[0, 1]$. For each $a > 0$, let Σ_a denote the σ-algebra generated by an atom $[0, a]$ and $\Sigma \bigcap (a, 1] = \{B \in \Sigma : B \subseteq (a, 1]\}$. For any $A \in \Sigma$, let $B_1 = A \bigcap (a, 1]$ and $B_2 = [0, a] \bigcup (A \bigcap (a, 1])$. Then $B_1, B_2 \in \Sigma_a$ and either $P(A \Delta B_1)$ or $P(A \Delta B_2)$ is less than or equal to $\frac{a}{2}$. If $A = [0, \frac{a}{2}]$, then $P(A \Delta B) \geq \frac{a}{2}$, for all $B \in \Sigma_a$. Thus $d'(\Sigma_a, \Sigma) = \frac{a}{2}$.

To compute $d(\Sigma_a, \Sigma)$, note that, for any $f \in L^\infty(\Sigma)_1$, the function $f\chi_{(a,1]}$ is in $L^\infty(\Sigma_a)_1$ and $\|f - f\chi_{(a,1]}\|_2 \leq \sqrt{a}$. On the other hand, if $f = \chi_{[0,\frac{a}{2}]} - \chi_{(\frac{a}{2},a]}$, then $E^{\Sigma_a}(f) = 0$ almost everywhere. This implies that $\|f - g\|_2 \geq \|f\|_2 = \sqrt{a}$, for any $g \in L^\infty(\Sigma_a)_1$. Thus $d(\Sigma_a, \Sigma) = \sqrt{a}$.

With $\Sigma_1 = \Sigma_a$ and $\Sigma_2 = \Sigma$, the inequalities in theorem 1 become $\frac{a}{2} \leq \sqrt{a} \leq 2\sqrt{\frac{a}{2} - (\frac{a}{2})^2}$ ($\leq \sqrt{2a}$). Thus no k exists with $d(\Sigma_a, \Sigma) \leq kd'(\Sigma_a, \Sigma)$ for all $a \in (0, 1)$.

3. THE MAIN RESULTS.

We consider the metric space $(S(\Sigma), d)$ and establish the following two theorems. Since $(S(\Sigma), d')$ is homeomorphic to $(S(\Sigma), d)$, these characterizations hold for either metric.

THEOREM 2. $(S(\Sigma), d)$ is compact if and only if Σ is purely atomic.

THEOREM 3. $(S(\Sigma), d)$ is connected if and only if Σ has at most one atom.

Before proving these theorems, we need the following lemma which is a folklore result of measure theory and can be proven with an easy application of Zorn's lemma.

LEMMA If $(X, \mathcal{F}, \mu)$ is a finite measure space with no atoms, then there exists a map $F : [0, 1] \to \mathcal{F}$ such that $F(s) \subseteq F(t)$ and $\mu(F(t)) = t\mu(X)$ for all $s, t \in [0, 1]$ with $s \leq t$.

We will use the notation, for $A \in \Sigma$ and $\Sigma_0 \in S(\Sigma)$,

$$A \bigcap \Sigma_0 = \{A \bigcap B : B \in \Sigma_0\}$$

PROOF OF THEOREM 2. Assume that (Ω, Σ, P) is not completely atomic. Let Ω_c and Ω_a be the continuous and atomic parts of Ω respectively. We are assuming $P(\Omega_c) > 0$. Let $F : [0, 1] \to \Sigma$ be such that $F(t) \subseteq \Omega_c, P(F(t)) = tP(\Omega_c)$, and $F(s) \subseteq F(t)$ for $s, t \in [0, 1], s \leq t$. Define $F[s, t] = F(t) \backslash F(s)$ and let

$$A_n = \bigcup \{F[(k-1)/2^{n-1}, (2k-1)/2^n] : k = 1, 2, \ldots, 2^{n-1}\}$$

for each $n = 1, 2, \ldots$ Then $P(A_n) = P(\Omega_c)/2$, for each n and $P(A_n \Delta A_m) = P(\Omega_c)/2$, for $n \neq m$. Let Σ_n debote the σ-algebra generated by $A_n \cap \Sigma$ and the atom $\Omega \backslash A_n$, for each n. Let f denote the indicator function of $\Omega \backslash A_n$. Then $f \in L^\infty(\Sigma_n)_1$ and for any $g \in L^\infty(\Sigma_m)_1$, if $m \neq n$, then g differs from f by at least $\frac{1}{2}$ on either $(\Omega \backslash A_m) \cap A_n$ or $(\Omega \backslash A_m) \cap (\Omega \backslash A_n)$. Each of these sets has probability $P(\Omega_c)/4$. Thus $\|f - g\|_2^2 \geq P(\Omega_c)/16$, which implies $d(\Sigma_n, \Sigma_m) \geq \sqrt{P(\Omega_c)}/4$, if $n \neq m$. Hence, the sequence $(\Sigma_n)_{n=1}^\infty$ has no cluster points and $S(\Sigma)$ is not compact.

Conversely, suppose (Ω, Σ, P) is completely atomic. Since $(S(\Sigma), d)$ is complete, compactness will follow if we show that $S(\Sigma)$ is totally bounded. If Σ has only a finite number of atoms, then $S(\Sigma)$ is a finite set and obviously compact. So suppose $\{A_1, A_2, \ldots\}$ are disjoint atoms generating Σ. For $\epsilon > 0$, choose N such that $\sum_{n=N+1}^\infty P(A_n) < \epsilon^2$. There are only a finite number of algebras of sets contained in the algebra generated by $\{A_1, \ldots, A_N\}$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ denote these algebras. For $1 \leq j \leq k$, let Σ_j denote the σ-algebra generated by $\mathcal{A}_j$ and $\{A_{N+1}, A_{N+2}, \ldots\}$. Let $A = \bigcup_{n=1}^N A_n$. For any $\Sigma' \in S(\Sigma), A \cap \Sigma' = \mathcal{A}_j$, for some $j, 1 \leq j \leq k$. Then $\Sigma' \subseteq \Sigma_j$, and it is easy to check that $d(\Sigma', \Sigma_j) < \epsilon$. Therefore, $S(\Sigma)$ is covered by the finite set of ϵ-balls centered at $\Sigma_j, 1 \leq j \leq k$. Since $\epsilon > 0$ was arbitrary, $S(\Sigma)$ is totally bounded.

PROOF OF THEOREM 3. Suppose A and B are two distinct atoms in Σ. Let S_1 denote the set of all Σ_0 in $S(\Sigma)$ which have an atom containing both A and B, let S_2 denote the set of all Σ_0 in $S(\Sigma)$ which have two disjoint atoms containing A and B, respectively. It is easy to see that $S(\Sigma) = S_1 \cup S_2$. Let $\Sigma_1 \in S_1$ and $\Sigma_2 \in S_2$. Let A' and B' be disjoint atoms of Σ_2 containing A and B, respectively. Let C be an atom of Σ_1 which contains $A \cup B$. Let $f = \chi_{A'} - \chi_{B'} \in L^\infty(\Sigma_2)_1$. For any $g \in L^\infty(\Sigma_1)_1, g = a$, almost everywhere on C, for some constant $a, |a| \leq 1$. Thus

$$
\begin{aligned}
\|f - g\|_2 &\geq \sqrt{\int_A |f - g|^2 dP + \int_B |f - g|^2 dP} \\
&= \sqrt{(1-a)^2 P(A) + (1+a)^2 P(B)} \\
&\geq \sqrt{2\min\{P(A), P(B)\}}
\end{aligned}
$$

Therefore, $d(\Sigma_1, \Sigma_2) \geq \sqrt{2\min\{P(A), P(B)\}}$, for all $\Sigma_1 \in S_1$ and $\Sigma_2 \in S_2$. Notice also that the trivial σ-algebra $\Sigma_\tau = \{\Omega, \emptyset\} \in S_1$ and $\Sigma \in S_2$. So S_1 and S_2 are nonempty disjoint open subsets of $S(\Sigma)$ with union $S(\Sigma)$. Hence, $S(\Sigma)$ is not connected if there are two or more atoms in Σ.

Conversely, suppose (Ω, Σ, P) has at most one atom. If there is an atom, call it A and if there is no atom, let $A = \emptyset$. If $P(A) = 1$, then $S(\Sigma)$ is a one point space, which is connected. So assume $P(\Omega \backslash A) > 0$. We will show that any $\Sigma_0 \in S(\Sigma)$ can be connected to Σ by an arc in $S(\Sigma)$.

First assume $\Sigma_0 \in S(\Sigma)$ and Σ_0 is completely atomic, generated by the disjoint atoms $A_0, A_1, \ldots$ (a finite or infinite collection). We may assume $A \subseteq A_0$. For each $n \geq 1$, by the lemma, there exists $F_n : [0, 1] \to A_n \cap \Sigma$ such that $P(F_n(t)) = tP(A_n)$ and $F_n(s) \subseteq F_n(t)$ if $s, t \in [0, 1]$ with $s \leq t$. Similarly, let $F_0 : [0, 1] \to A_0 \cap \Sigma$ be such that F_0 is increasing and $P(F_0(t)) = P(A) + tP(A_0 \backslash A)$, for all $t \in [0, 1]$. For each $t \in [0, 1]$, define $\Sigma(t) \in S(\Sigma)$ as the σ-algebra generated by $\{F_n(t) \cap \Sigma, A_n \backslash F_n(t) : n = 0, 1, 2, \ldots\}$. It is clear that $\Sigma(0) = \Sigma_0, \Sigma(1) = \Sigma$ and $\Sigma(s) \subseteq \Sigma(t)$ if $s \leq t$. Fix $s, t \in [0, 1]$ with $s \leq t$.

For any $f \in L^\infty(\Sigma(t))_1$ and for each $n = 0, 1, 2, \ldots, f$ is constant, say a_n, almost everywhere on the atom $A_n \backslash F_n(t)$ in $\Sigma(t)$. Define $g \in L^\infty(\Sigma(s))_1$, by making $g = a_n$ on $A_n \backslash F_n(s)$ and $g = f$ on $F_n(s)$, for $n = 0, 1, 2, \ldots$ Then f and g differ only on $\bigcup(F_n(t) \backslash F_n(s))$ and there by at most 2. Now

$$P(\bigcup(F_n(t) \backslash F_n(s))) = (t - s)[P(A_0 \backslash A) + P(A_1) + P(A_2) + \ldots]$$

Thus $\|f - g\|_2 \le 2\sqrt{t - s}$.

This implies that $d(\Sigma(s), \Sigma(t)) \le 2\sqrt{t - s}$. Therefore, any atomic element Σ_0 of $S(\Sigma)$ is connected by an arc to Σ.

If Σ_0 is not completely atomic, let $\Omega = \Omega_c \bigcup \Omega_a$, where $\Omega_c \bigcap \Sigma_0$ is continuous and $\Omega_a \bigcap \Sigma_0$ is atomic. Let Σ_1 be the atomic σ-algebra generated by an atom equal to Ω_c and $\Omega_a \bigcap \Sigma_0$. By a method similar to the above, but just working with Ω_c, an arc from Σ_1 to Σ_0 can be found. Also, Σ_1 being atomic is connected by an arc to Σ. Thus any element Σ_0 of $S(\Sigma)$ can be connected by an arc to Σ. Therefore $S(\Sigma)$ is connected, in fact, arcwise connected if (Ω, Σ, P) has at most one atom.

The two theorems of this section show that there is some meaningful connection between the topology of $S(\Sigma)$ and the structure of the original probability space. Since this topology is really an embodiment of the equiconvergence property for conditional expectations, we feel that $S(\Sigma)$ with this metric d (or if one prefers d') will provide a good locale for studying conditional expectations.

4. CONTINUOUS SUB-σ-ALGEBRAS.

Let $S_c(\Sigma) = \{\Sigma_0 \in S(\Sigma) : (\Omega, \Sigma_0, P) \text{ has no atoms}\}$, the space of continuous sub-σ-algebras of Σ. Of course, if Σ is not a continuous σ-algebra, then $S_c(\Sigma) = \emptyset$. On the other hand, if Σ is continuous, then $S_c(\Sigma)$ is a very rich set to look at. For example, if G is a group of Σ-measurable transformations of Ω, let Σ_G denote the σ-algebra of G-invariant sets in Σ. If G is a finite group, then $\Sigma_G \in S_c(\Sigma)$. This provides a way in which to construct many interesting examples of elements of $S_c(\Sigma)$. In spite of this, $S_c(\Sigma)$ is a "small" subset of $S(\Sigma)$.

THEOREM 4. $S_c(\Sigma)$ is a closed nowhere dense subset of $S(\Sigma)$.

PROOF. We first observe that if $\Sigma_0 \in S_c(\Sigma)$ and $A \in \Sigma$, then

$$\{P(A \bigcap B) : B \in \Sigma_0\} = [0, P(A)].$$

Now fix $\Sigma_1 \in S(\Sigma) \backslash S_c(\Sigma)$ and let A be an atom of Σ_1. By the above observation, for any $\Sigma_0 \in S_c(\Sigma)$, there exists $B \in \Sigma_0$ such that $P(A \bigcap B) = \frac{1}{2}P(A)$.

Let $f = \chi_B - \chi_{\Omega \backslash B} \in L_1^\infty(\Sigma_0)$. For any $g \in L_1^\infty(\Sigma_1), g$ is constant on A and one easily checks that

$$\|f - g\|_2 \ge (P(A))^{1/2}.$$

Thus $d(\Sigma_0, \Sigma_1) \ge (P(A))^{1/2}$ for any $\Sigma_0 \in S_c(\Sigma)$. Hence $S(\Sigma) \backslash S_c(\Sigma)$ is open and $S_c(\Sigma)$ is closed.

To see that $S_c(\Sigma)$ is nowhere dense, let $\Sigma_0 \in S_c(\Sigma)$ and $\epsilon > 0$. Choose $A \in \Sigma_0$ such that $0 < P(A) < \epsilon^2$. Define Σ_1 to be the σ-algebra generated by A and $\{B \in \Sigma_0 : B \subset \Omega \backslash A\}$. Then Σ_1 has an atom A. For any $f \in L_1^\infty(\Sigma_0)$, let $g = \chi_{\Omega \backslash A} f \in L_1^\infty(\Sigma_1)$. Since $\|f - g\|_2^2 = \int_A |f|^2 \le P(A)$ and $\Sigma_1 \subset \Sigma_0$, we have that $d(\Sigma_0, \Sigma_1) \le (P(A))^{1/2} < \epsilon$. Combined with the general estimate above,

for this particular Σ_0 and $\Sigma_1, d(\Sigma_0, \Sigma_1) = (P(A))^{1/2}$. Thus the open ball of radius ϵ centered at Σ_0 is not contained in $S_c(\Sigma)$ for any $\epsilon > 0$. Since $S_c(\Sigma)$ is closed, it is nowhere dense in $S(\Sigma)$.

Thus $(S_c(\Sigma), d)$ is itself a complete metric space whose topological properties should be interesting to study. We do not consider such a study in this paper.

5. CONNECTIONS WITH VON NEUMANN ALGEBRAS.

There is a one-to-one correspondence between the sub-σ-algebras of Σ and the von Neumann subalgebras of $L^\infty(\Omega, \Sigma, P)$. In this section, we assume the reader is familar with the basic theory of von Neumann algebras as found in a reference such as Sakai [7].

Let $\mathcal{M}$ be a fixed finite von Neumann algebra with a distinguished faithful, finite, normal trace τ. For example, $\mathcal{M}$ could be $L^\infty(\Omega, \Sigma, P)$ or a Type II$_1$-factor. Let $S(\mathcal{M})$ denote the set of all von Neumann subalgebras of $\mathcal{M}$ which have the same identity as $\mathcal{M}$.

In the case that $\mathcal{M}$ is a Type II$_1$-factor, Christensen defined a metric on $S(\mathcal{M})$ in [2]. He showed that $S(\mathcal{M})$ then became a complete metric space. He used his metric to study perturbation properties of subfactors and his metric has also been used in the study of the index of a subfactor in a Type II$_1$-factor (see [3] and [5]). Christensen's definition works well in the situation we are considering. For $x \in \mathcal{M}$, let $\|x\|_2 = \sqrt{\tau(xx^*)}$. For $\mathcal{N}, \mathcal{L} \in \mathcal{S}(\mathcal{M})$, let $\mathcal{N}_1$ and $\mathcal{L}_1$ denote the unit balls in $\mathcal{N}$ and $\mathcal{L}$, respectively.

Then define

$$d(\mathcal{N}, \mathcal{L}) = max\{\sup_{x \in \mathcal{N}_1} \inf_{y \in \mathcal{L}_1} \|x - y\|_2, \sup_{y \in \mathcal{L}_1} \inf_{x \in \mathcal{N}_1} \|x - y\|_2\}$$

It is routine to check that d is a metric on $S(\mathcal{M})$ and Christensen's proof that $S(\mathcal{M})$ is complete carries over to this more general situation.

Some of the proofs of section 3 can be adapted to the von Neumann algebra situation but not all. We list below what we can prove and leave the details of the adaption of the proofs to the interested reader.

THEOREM 5. Let $\mathcal{M}$ be a finite von Neumann algebra with a fixed faithful, finite trace τ. With the Christensen's metric, $S(\mathcal{M})$ is compact if and only if $\mathcal{M}$ is generated by its minimal projections.

PROPOSITION 6. If $S(\mathcal{M})$ is connected, then $\mathcal{M}$ has at most one minimal projection.

OPEN PROBLEM Is the converse of proposition 6 true?

REFERENCES

1. Boylan, E. S. Equiconvergence of Martingales, <u>Ann. Math. Stat.</u> <u>42</u> (1971), 552-559.

2. Christensen, E. Subalgebras of a Finite Algebra, <u>Math. Ann.</u> <u>243</u> (1979), 17-29.

3. Jones, V. F. R. Index for Subfactors, <u>Invent. Math.</u> <u>72</u> (1983), 1-25.

4. Kudo, H. A Note on the Strong Convergence of σ-algebras, <u>Ann. Prob.</u> <u>2</u> (1974), 76-83.

5. Mashhood, B. and Taylor, K. F. On Continuity of the Index of Subfactors of a Finite Factor, <u>J. Func. Anal.</u> <u>12</u> (1988), 56-66.

6. Rogge, L. Uniform Inequalities for Conditional Expectations, <u>Ann. Prob.</u> <u>2</u> (1974), 486-489.

7. Sakai, S. <u>C*-algebras</u> <u>and</u> <u>W*-algebras</u>, Ergebnisse der Math. 60, Berlin, Heidelley, New York: Springer-Verlag 1971.