Internat. J. Math. & Math. Sci.
VOL. 16 NO. 2 (1993) 401-402

A GENERALIZATION OF GELFAND-MAZUR THEOREM

SUNG GUEN KIM

Department of Mathematics
Pohang Institute of Science and Technology
P.O. Box 215
Pohang, 790-600 Korea

(Received August 12, 1991 and in revised form November 19, 1991)

ABSTRACT. In this paper, we show that if A is a unital semisimple complex Banach algebra with only the trivial idempotents and if $\sigma_A(x)$ is countable for each $x \in Fr(G(A))$, then $A \cong \mathbf{C}$, this generalizes the Gelfand-Mazur theorem.

1. INTRODUCTION.

In this paper, we show that if A is a unital semisimple complex Banach algebra with only the trivial idempotents and if $\sigma_A(x)$ is countable for each $x \in Fr(G(A))$, then $A \cong \mathbf{C}$; this generalizes the Gelfand-Mazur theorem. Throughout this paper we assume that all algebras are over the complex field. We denote by $G(A)$ the set of all invertible elements in A and by $\sigma_A(x)$ the spectrum of x in A. Let $\rho_A(x)$ denote the spectral radius of x in A. From the theory of Banach algebras, we know that $\sigma_A(x)$ is a nonempty, compact subset of the complex numbers, that $\rho_A(x) = \lim_{n \to \infty} \| x^n \|^{1/n}$, and that $G(A)$ is an open set in A. We denote by Rad(A) the radical of A and by $Fr(G(A))$ the boundary of $G(A)$ in A. From the theory of Banach algebras, we know that Rad(A) is a closed two sided ideal.

2. MAIN RESULTS.

LEMMA 1. Let A be a unital Banach algebra. Then Rad$(A) = \{x \in A \mid \rho_A(xy) = 0$ for all $y \in A\}$.

PROOF. See R.S. Doran and V.A. Belfi [1, p. 325].

LEMMA 2. Let A be an algebra with the identity. Let $x, y \in A$. If at least two points of the set $\{x, y, xy, yx\}$ belong to $G(A)$, then $\{x, y, xy, yx\} \subseteq G(A)$.

PROOF. See W. Rudin [2, p. 259].

LEMMA 3. Let A be a Banach algebra. If A has only the trivial idempotents, then $\sigma_A(x)$ is a connected set in $\mathbf{C}$ for each $x \in A$.

PROOF. See W. Rudin [2, p. 247].

LEMMA 4. Let A be a unital Banach algebra. Then Rad$(A) = G(A)^c$ if and only if for each $x \in Fr(G(A))$, $\sigma_A(x) = \{0\}$.

PROOF. ($\Rightarrow$) Suppose that Rad$(A) = G(A)^c$. Let $x \in Fr(G(A))$. Since $G(A)$ is open in $A, x \in G(A)^c$. Hence $x \in$ Rad(A) and so $\rho_A(x) = 0$ by Lemma 1. Therefore $\sigma_A(x) = \{0\}$ because $\sigma_A(x)$ is a nonempty set in $\mathbf{C}$.

($\Rightarrow$) Suppose that $\sigma_A = \{0\}$ for each $x \in Fr(G(A))$. Let $x \in$ Rad(A). Then $\rho_A(x) = 0$ by Lemma 1. We claim that $x \in G(A)^c$. Assume that $x \in G(A)$. Then $xy = e = yx$ for some $y \in A$, and hence $1 = \rho_A(e) = \rho_A(xy) \leq \rho_A(x) \cdot \rho_A(y) = 0$, which is impossible. Therefore $\mathrm{Rad}(A) \subseteq G(A)^c$. It suffices to show that $G(A)^c \subseteq \mathrm{Rad}(A)$. Let $x_0 \in G(A)^c$. Since $\sigma_A(x_0)$ is nonempty and compact, we can choose $\lambda \in \sigma_A(x_0)$ such that $\lambda_0 \in \mathrm{Fr}(\sigma_A(x_0))$. Then $x_0 - \lambda_0 e \in \mathrm{Fr}(G(A))$ by continuity. First we will show that $\sigma_A(x_0) = \{0\}$. Let $\lambda \in \sigma_A(x_0)$, then $x_0 - \lambda e = (x_0 - \lambda_0 e) - (\lambda - \lambda_0)e$. Hence $\lambda - \lambda_0 \in \sigma_A(x_0 - \lambda_0 e) = \{0\}$. Hence $\lambda = \lambda_0$. Since $x_0 \in G(A)^c$, $\lambda_0 = 0$. Hence $\sigma_a(x_0) = \{0\}$ for all $x_0 \in G(A)$. Let $y \in A$. We claim that $x_0 y \in G(A)^c$ or $yx_0 \in G(A)^c$. Assume otherwise. Then $\{x_0 y, yx_0\} \subseteq G(A)$, $\{x_0, y, x_0 y, yx_0\} \subseteq G(A)$ by Lemma 2, which is impossible, because $x_0 \in G(A)^c$. Hence $x_0 y \in G(A)^c$ or $yx_0 \in G(A)^c$. By the above argument, $\rho_A(x_0 y) = 0$ or $\rho_A(yx_0) = 0$. Since $\rho_A(u \cdot v) = \rho_A(v \cdot u)$ for all $u, v \in A$, $\rho_A(x_0 y) = 0 = \rho_A(yx_0)$. Since y is an arbitrary element in A, by Lemma 1, $x_0 \in \mathrm{Rad}(A)$. Therefore $G(A)^c \subseteq \mathrm{Rad}(A)$. Hence $\mathrm{Rad}(A) = G(A)^c$.

PROPOSITION 5. Let A be a unital semisimple Banach algebra. If A has only the trivial idempotents and $\sigma_A(x)$ is countable for each $x \in Fr(G(A))$, $A \cong \mathbb{C}$.

PROOF. Let $x \in Fr(G(A))$, then $\sigma_A(x)$ is a connected subset of $\mathbb{C}$ by Lemma 3. Since $\sigma_A(x)$ is a connected, separable metric space, $\sigma_A(x)$ has exactly one point or an uncountable number of points of $\mathbb{C}$. By our hypothesis, $\sigma_A(x)$ has exactly one point. Hence $\sigma_A(x) = \{0\}$, because $x \in G(A)^c$. Therefore we know that $\sigma_A(x) = \{0\}$ for all $x \in Fr(G(A))$. It follows that $\mathrm{Rad}(A) = G(A)^c$ by Lemma 4. Since A is an semisimple, $G(A)^c = \{0\}$. By the Gelfand-Mazur theorem, $A \cong \mathbb{C}$.

COROLLARY 6. If A is a unital Banach algebra with $G(A)^c = \{0\}$, then $A \cong \mathbb{C}$.

PROOF. By Proposition 5, it suffices to show that A is simisimple and that A has only the trivial idempotents and that $\sigma_A(x)$ is countable for each $x \in Fr(G(A))$. Since $\{0\} \subseteq \mathrm{Rad}(A) \subseteq G(A)^c = \{0\}$, $\mathrm{Rad}(A) = \{0\}$. Let x be an idempotent in A. Then $x(x - e) = 0$. Hence $x \in G(A)^c$ or $x - e \in G(A)^c$, hence $x = 0$ or $x = e$. Let $x \in Fr(G(A))$. Then $x \in G(A)^c$, hence $x = 0$ and so $\sigma_A(x) = \{0\}$; therefore $\sigma_A(x)$ is countable for each $x \in Fr(G(A))$.

COROLLARY 7. If A is a unital semisimple Banach algebra such that for each $x \in Fr(G(A))$, $\sigma_A(x) = \{0\}$, then $A \cong \mathbb{C}$.

PROOF. By Lemma 4, it follows.

COROLLARY 8. If A is a C^*-algebra such that for each $x \in Fr(G(A))$, $\sigma_A(x) = \{0\}$, then $A \cong \mathbb{C}$.

PROOF. Since C^*-algebra is a simisimple Banach algebra, it follows from Corollary 7.

REMARK 9. We give another proof of Corollary 8.

Let $x \in Fr(G(A))$. Since A is C^*-algebra, $\rho_A(xx^*) = \| xx^* \| = \| x \|^2$. Since $x \in G(A)^c; xx^* \in G(A)$ or $x^*x \in G(A)^c$ by Lemma 2. Hence by the continuity of involution $*$, $xx^* \in Fr(G(A))$ or $x^*x \in Fr(G(A))$. By our hypothesis, $\rho_A(xx^*) = 0 = \rho_A(x^*x)$. Therefore, $\| x \|^2 = 0$, i.e., $x = 0$. Since $Fr(G(A)) = \{0\}$, we can deduce that $G(A)^c = \{0\}$. By the Gelfand-Mazur theorem, $A \cong \mathbb{C}$.

REFERENCES

1. DORAN, R.S. and BELFI, V.A., Characterizations of C^*-algebras: The Gelfand-Naimark Theorems, Marcel Dekker, Inc., New York, (1986).

2. RUDIN, W., Functional Analysis, McGraw-Hill, New York, (1973).