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# APPROXIMATING FIXED POINTS OF NONEXPANSIVE AND GENERALIZED NONEXPANSIVE MAPPINGS

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**ABSTRACT.**

In this paper we consider a mapping $S$ of the form

$$S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \dots + \alpha_k T^k,$$

where $\alpha_i \geq 0$. $\alpha_1 > 0$ with $\sum_{i=0}^{k} \alpha_i = 1$, and show that in a uniformly convex Banach space the Picard iterates of $S$ converge to a fixed point of $T$ when $T$ is nonexpansive or generalized nonexpansive or even quasi-nonexpansive.

**KEY WORDS AND PHRASES:** Fixed points, nonexpansive mappings, uniformly convex Banach spaces.
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## 1. INTRODUCTION.

Let $B$ be a Banach space and $C$ a convex subset of $B$. A mapping $T$: $C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x$, $y \in C$. A mapping $T$: $C \rightarrow C$ is said to be quasi-nonexpansive if $T$ has a fixed point $p$ such that $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$. The concept of quasi-nonexpansiveness is more general than that of nonexpansiveness. Indeed, a nonexpansive mapping with at least one fixed point is quasi-nonexpansive, but there exists quasi-nonexpansive mappings which are not nonexpansive. See, for example, Petryshyn and Williamson [7].

If $T$ is nonexpansive, then the Picard iterates of $T$ may not converge and, even if they do converge, they may not converge to a fixed point of $T$. However, to circumvent the difficulty, one may consider the mapping

$$T_\lambda = (1 - \lambda)I + \lambda T, \tag{1.1}$$

where $I$ is the identity mapping and $0 < \lambda < 1$, and show that the Picard iterates of $T_\lambda$ converge to a fixed point of $T$ under certain restrictions, see [3,5,10]. Generalizing the idea Kirk [6] has introduced a mapping $S$ given by

$$S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \dots + \alpha_k T^k \tag{1.2}$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$ with $\sum_{i=0}^{k} \alpha_i = 1$, and has shown that the Picard iterates of $S$ converge to a fixed point of $T$ under conditions similar to those imposed in connection with the convergence of Picard iterates of $T_\lambda$.

Our purpose here is two-fold. First we show that the Picard iterates of $S$ converge to a fixed point of $T$ under conditions weaker than those imposed by Kirk [6]. Secondly, we establish that the Picard iterates of $S$ converge to a fixed point of $T$ even when $T$ is generalized nonexpansive, i.e., when $T$ satisfies

$$\|Tx - Ty\| \le a\|x - y\| + b\{\|x - Tx\| + \|y - Ty\|\} + c\{\|y - Tx\| + \|x - Ty\|\} \tag{1.3}$$

for all $x,\ y \in C$, where $a,b,c \ge 0$ with $a + 2b + 2c \le 1$. Then the analysis has been extended to a more general mapping resulting in generalization of some results obtained by Ray and Rhoades [9].

## 2. CONVERGENCE TO FIXED POINTS

It has been established by Kirk [6] that $S$ and $T$ have common fixed points if $T$ is nonexpansive. Let the common fixed point set be denoted by $F$. Further, the set $F$ is closed when $T$ is nonexpansive or even when $T$ is quasi-nonexpansive (see Dotson [2]). We now state the following conditions:

**CONDITION-A.** A mapping $T$: $C \to C$ with a nonempty fixed point set $F$ is said to satisfy Condition-A if there is a nondecreasing function $f$: $[0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Sx\| \ge f(d(x,F))$ for all $x \in C$, where $d(x,F) = \inf_{p \in F} \|x - p\|$.

**CONDITION-B.** A mapping $T$: $C \to C$ with a nonempty fixed point set $F$ is said to satisfy Condition-B if there exists a number $\alpha > 0$ such that $\|x - Sx\| \ge \alpha d(x,F)$ for all $x \in C$.

It may be remarked that the mappings which satisfy Condition-B also satisfy Condition-A. However, Condition-B may be verified easily by giving examples. It may be further remarked that Conditions I and II of Senter and Dotson [11] are identical with Conditions A and B when $\alpha_2 = \alpha_3 = \ldots = \alpha_k = 0$.

We now recall the following lemma due to Dotson [1]. This will be used later to establish our results.

**LEMMA.** If the sequences $\{s_n\}$ and $\{t_n\}$ are in the closed unit ball of a uniformly convex Banach space and $\{z_n\} = \{(1 - \alpha_n)s_n + \alpha_n t_n\}$ satisfies $\lim_{n \to \infty} \|z_n\| = 1$, where $0 < a \le \alpha_n \le b < 1$, then $\lim_{n \to \infty} \|s_n - t_n\| = 0$.

**THEOREM 1.** Let $C$ be a nonempty, closed, convex and bounded subset of a uniformly convex Banach space $B$ and $T$: $C \to C$ be a nonexpansive mapping. If $T$ satisfies Condition-A, where $F$ is the fixed point set of $T$ in $C$, then for an arbitrary $x_0 \in C$, the Picard iterates $(S^n x_0)$ converge to a member of $F$.

**PROOF.** If $x_0 \in F$, then the result is trivial. We assume that $x_0 \in C - F$. Then, setting $x_n = S^n x_0$, we have for an arbitrary $p \in F$

$$\begin{aligned}
\|x_{n+1} - p\| = \|S^{n+1}x_0 - p\| &= \|Sx_n - p\| \\
&= \|\alpha_0 x_n + \alpha_1 Tx_n + \alpha_2 T^2 x_n + \ldots + \alpha_k T^k x_n - p\| \\
&= \|\alpha_0(x_n - p) + \alpha_1(Tx_n - p) + \alpha_2(T^2 x_n - p) + \ldots + \alpha_k(T^k x_n - p\| \\
&\le \alpha_0\|x_n - p\| + \alpha_1\|Tx_n - p\| + \alpha_2\|T^2 x_n - p\| + \ldots + \alpha_k\|(T^k x_n - p)\| \\
&\le \|x_n - p\|.
\end{aligned}$$

This implies that $d(x_{n+1}, F) \le d(x_n, F)$ and hence that the sequence $\{d(x_n,F)\}$ is nonincreasing. Then $\lim_{n \to \infty} d(x_n,F)$ exists. In the sequel we shall show that this limit is zero.

Suppose that $\lim_{n \to \infty} d(x_n,F) = b > 0$. Then, for a $p \in F$, $\lim_{n \to \infty} \|x_n - p\| = b' \ge b > 0$. Choose a positive integer $N$ such that $\|x_n - p\| \le 2b'$ for $n \ge N$. Set $y_n^i = (T^i x_n - p)/\|x_n - p\|$ for all $n$ and all $i = 0,1,2,\ldots,k$ with $T^0 x_n = x_n$. Then $\|y_n^i\| \le 1$. Further, set $z_n = \alpha_0 y_n^0 + (1 - \alpha_0)t_n$, where $t_n = \sum_{i=1}^{k} (\alpha_i y_n^i)/(1 - \alpha_0)$ with $\|t_n\| \le 1$. Then $\{y_n^0\}$ and $\{T_n\}$ are in the closed unit ball. Now

$$\|z_n\| = \left\| \alpha_0 \frac{x_n - p}{\|x_n - p\|} + \alpha_1 \frac{Tx_n - p}{\|x_n - p\|} + \alpha_2 \frac{T^2x_n - p}{\|x_n - p\|} + \ldots + \alpha_k \frac{T^kx_n - p}{\|x_n - p\|} \right\|$$

$$= \frac{\|\alpha_0 x_n + \alpha_1 Tx_n + \alpha_2 T^2 x_n + \ldots + \alpha_k T^k x_n - p\|}{\|x_n - p\|} = \frac{\|x_{n+1} - p\|}{\|x_n - p\|}, \tag{2.1}$$

implying $\|z_n\| \to 1$ as $n \to \infty$. But, for $n \geq N$, we have

$$\|y_n^0 - t_n\| = \left\| \frac{x_n - p}{\|x_n - p\|} - \frac{1}{(1-\alpha_0)} \sum_{i=1}^{k} \frac{\alpha_i(T^ix_n - p)}{\|x_n - p\|} \right\|$$

$$= \left\| \frac{x_n - p}{\|x_n - p\|} - \frac{Sx_n - \alpha_0 x_n - (1-\alpha_0)p}{(1-\alpha_0)\|x_n - p\|} \right\|$$

$$= \frac{\|x_n - Sx_n\|}{(1-\alpha_0)\|x_n - p\|}$$

$$\geq \frac{f(d(x_n, F))}{(1-\alpha_0)\|x_n - p\|} \geq \frac{f(b)}{2b'(1-\alpha_0)} > 0, \tag{2.2}$$

implying $\lim_{n\to\infty} \|y_n^0 - t_n\| \neq 0$, which contradicts the lemma. Hence $\lim_{n\to\infty} d(x_n, F) = 0$. This implies that $\{x_n\}$ converges to a member of $F$, since $F$ is closed.

**REMARK 1.** It is obvious that the above theorem holds if Condition-B is satisfied instead of Condition-A.

**REMARK 2.** Condition-A is more general than the condition imposed by Kirk [6] in establishing the convergence of Picard iterates $\{S^n x_0\}$, see Senter and Dotson [11].

**REMARK 3.** In the above theorem the existence of a nonempty fixed point set $F$ is not assumed and is ensured by the conditions assumed therein. However, if we assume that $T$ has a nonempty fixed point set $F$, then $T$ need not be assumed to be nonexpansive and it is enough for $T$ to be quasi-nonexpansive. Further, $C$ need not be bounded. Indeed, the following result holds.

**THEOREM 2.** Let $C$ be a nonempty, closed and convex subset of a uniformly convex Banach space $B$ and $T\colon C \to C$ be a quasi-nonexpansive mapping. If $T$ satisfies Condition-A, where $F$ is the fixed point set of $T$ in $C$, then, for an arbitrary $x_0 \in C$, the Picard iterates $\{S^n x_0\}$ converge to a member of $F$.

The proof may be established exactly in the same way as in Theorem 1. It only remains to be shown here that $S$ and $T$ have common fixed points. A fixed point of $T$ is obviously a fixed point of $S$. We now show that the converse is also so. Let $p$ be a fixed point of $S$. Then from Condition-A it is obvious that $d(p, F) = 0$, implying $p \in \overline{F}$. Since $T$ is quasi-nonexpansive, $F$ is closed and hence $p \in F$, i.e., $p$ is a fixed point of $T$. Hence the result.

Next, we show that the Picard iterates of $S$ converge to a fixed point of $T$ even when $T$ is generalized nonexpansive. However, one need not assume Condition-A or Condition-B in this case. These conditions are automatically satisfied.

**THEOREM 3.** Let $C$ be a nonempty, bounded, closed and convex subset of a uniformly convex Banach space $B$ and $T\colon C \to C$ be a continuous mapping such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\{\|x - Tx\| + \|y - Ty\|\} + c\{\|x - Ty\| + \|y - Tx\|\} \tag{2.3}$$

for all $x,\ y \in C$, where $a,\ c \geq 0$ and $b > 0$ with $a + 2b + 2c \leq 1$. Then for an arbitrary $x_0 \in C$, the Picard iterates $\{S^n x_0\}$ converges to the unique fixed point of $T$.

**PROOF.** By Theorem 2 of Goebel, Kirk and Shimi [4] the mapping $T$ has a unique fixed point $p$, say. Setting $y = p$ in (2.3) we have

$$\begin{aligned}\|Tx-p\| &\le (a+c)\|x-p\| + b\|x-Tx\| + c\|Tx-p\| \\ &\le (a+b+c)\|x-p\| + (b+c)\|Tx-p\|,\end{aligned}$$

implying

$$\|Tx-p\| \le \frac{a+b+c}{1-b-c}\|x-p\| \le \|x-p\|, \tag{2.4}$$

since $a+2b+2c \le 1$. Thus $T$ is quasi-nonexpansive. Further, it is easy to verify that

$$\|Sx-p\| \le \|Tx-p\| \le \|x-p\|, \tag{2.5}$$

implying $S$ is also quasi-nonexpansive.

It is obvious that $p$ is also a fixed point of $S$. We now show that $S$ cannot have a fixed point other than $p$. If possible, let $q(\neq p)$ be a fixed point of $S$. Then

$$\begin{aligned}\|q-Tq\| &= \|Sq-Tq\| \\ &= \|\alpha_0 q + \alpha_1 Tq + \alpha_2 T^2 q + \ldots + \alpha_k T^k q - Tq\| \\ &= \|\alpha_0(q-Tq) + \alpha_2(T^2q - Tq) + \ldots + \alpha_k(T^k q - Tq)\| \\ &\le \alpha_0\|q-Tq\| + \alpha_2\|T^2q-Tq\| + \ldots + \alpha_k\|T^kq-Tq\| \\ &\le \alpha_0\{\|q-p\| + \|Tq-p\|\} + \alpha_2\{T^2q-p\| + \|Tq-p\|\} \\ &\qquad + \ldots + \alpha_k\{\|T^kq-p\| + \|Tq-p\|\} \\ &\le 2(\alpha_0+\alpha_2+\ldots+\alpha_k)\|q-p\| = 2(1-\alpha_1)\|q-p\|.\end{aligned} \tag{2.6}$$

Since $T$ is generalized nonexpansive, we have

$$\begin{aligned}\|Tq-p\| &= \|Tq-Tp\| \\ &\le a\|q-p\| + b\|Tq-q\| + c\{\|Tq-p\| + \|q-p\|\} \\ &\le (a+2c)\|q-p\| + b|Tq-q\|.\end{aligned} \tag{2.7}$$

Substituting from (2.6) into (2.7) and noting that $a+2c \le 1-2b$ we obtain

$$\begin{aligned}\|Tq-p\| &\le (1-2b)\|q-p\| + 2b(1-\alpha_1)\|q-p\| \\ &= \|q-p\| - 2b\alpha_1\|q-p\| \\ &< \|q-p\| - 2b\alpha_1\|Tq-p\|,\end{aligned}$$

implying

$$\|Tq-p\| \le \frac{1}{1+2b\alpha_1}\|q-p\| \tag{2.8}$$

Now from (2.8) we have

$$\|q-p\| = \|Sq-p\| \le \|Tq-p\| \le \frac{1}{1+2b\alpha_1}\|q-p\|, \tag{2.9}$$

which implies $q=p$, since $b,\ \alpha_1 > 0$. Thus $S$ and $T$ have a unique fixed point $p$.

Next, we show that $T$ satisfies Condition-B. For $x \in C$ we have

$$\|Tx-p\| = \|Tx-Tp\| \le a\|x-p\| + b\|x-Tx\| + c\{\|x-p\| + \|Tx-p\|\},$$

implying

$$\|Tx-p\| \le \frac{a+c}{1-c}\|x-p\| + \frac{b}{1-c}\|x-Tx\|. \tag{2.10}$$

Now,

$$\begin{aligned}\|Tx-x\| &\le \|Sx-Tx\| + \|Sx-x\| \\ &\le \alpha_0\|x-Tx\| + \alpha_2\|T^2x-Tx\| + \ldots + \alpha_k\|T^kx-Tx\| + \|Sx-x\| \\ &\le 2(1-\alpha_1)\|x-p\| + \|Sx-x\|. \end{aligned} \tag{2.11}$$

Also we observe that

$$\|Sx-p\| \le \alpha_0\|x-p\| + (\alpha_1+\alpha_2+\ldots+\alpha_k)\|Tx-p\| \tag{2.12}$$

and that

$$\|Sx-x\| \ge \|x-p\| - \|Sx-p\|. \tag{2.13}$$

Now, substituting from (2.12) into (2.13) we derive

$$\begin{aligned}\|Sx-x\| &\ge \|x-p\| - \alpha_0\|x-p\| - (\alpha_1+\alpha_2+\ldots+\alpha_k)\|Tx-p\| \\ &= (1-\alpha_0)\{\|x-p\| - \|Tx-p\|\},\end{aligned}$$

whence, using (2.10), we obtain

$$\begin{aligned}\|Sx-x\| &\ge (1-\alpha_0)\left\{\|x-p\| - \frac{a+c}{1-c}\|x-p\| - \frac{b}{1-c}\|x-Tx\|\right\} \\ &= (1-\alpha_0)\left\{\frac{1-a-2c}{1-c}\|x-p\| - \frac{b}{1-c}\|x-Tx\|\right\} \\ &\ge (1-\alpha_0)\left\{\frac{2b}{1-c}\|x-p\| - \frac{b}{1-c}\|x-Tx\|\right\} \\ &= \frac{b(1-\alpha_0)}{1-c}\{2\|x-p\| - \|x-Tx\|\}. \end{aligned} \tag{2.14}$$

Now, substituting from (2.11) into (2.14) we get

$$\begin{aligned}\|Sx-x\| &\ge \frac{b(1-\alpha_0)}{1-c}\{2\|x-p\| - 2(1-\alpha_1)\|x-p\| - \|Sx-x\|\} \\ &= \frac{b(1-\alpha_0)}{1-c}\{2\alpha_1\|x-p\| - \|Sx-x\|\},\end{aligned}$$

implying

$$\|Sx-x\| \ge \alpha\|x-p\|, \tag{2.15}$$

where

$$\alpha = \frac{2b\alpha_1(1-\alpha_0)}{1-c+b(1-\alpha_0)} > 0,$$

since $b,\ \alpha_1 > 0$. Thus $T$ satisfies Condition-B. Hence, by Theorem 2, the result follows.

**REMARK 4.** It may be noted that the stipulation $\alpha_1 > 0$ in $S$ is necessary to rule out the possibility that fixed point of $S$ is a point at which $T$ may be periodic.

**REMARK 5.** If we do not restrict $b > 0$ in Theorem 3, then the fixed pint set of $T$ is not a singleton, and Condition-A is to be imposed to ensure the convergence of $\{S^nx_0\}$.

The present analysis can be extended to a more general mapping $T$ which satisfies

$$\|Tx-Ty\| \le \max\{\|x-y\|, [\|x-Tx\| + \|y-Ty\|]/2, [\|y-Tx\| + \|x-Ty\|]/2\} \tag{2.16}$$

for all $x, \in C$. This mapping includes nonexpansive and generalized nonexpansive mappings (see Rhoades [8]). It is easy to verify that $T$ is quasi-nonexpansive. It has been proved by Ray and Rhoades [9] that $S$ and $T$ have the same fixed point set. Further, they have established the following theorem in this connection.

**THEOREM 4.** ([9, Theorem 2]). Let $C$ be a nonempty, closed convex and bounded subset of a uniformly convex Banach space $B$ and $T$ a self-mapping of $C$ which satisfies (2.16). If $I$-$S$ maps bounded closed subsets of $C$ into closed sets of $B$, then, for each $x_0 \in C$, the sequence $\{S^n x_0\}$ converges to a fixed point of $T$ in $C$.

However, the fact that $I$-$S$ maps bounded closed subsets of $C$ into closed sets implies Condition-A (see Senter and Dotson [11]). Thus Condition-A is more general and incorporating this condition we may obtain the following as generalizations of Theorems 2 and 3 of Ray and Rhoades [9].

**THEOREM 5.** Let $C$ be a nonempty, closed and convex subset of a uniformly convex Banach space $B$ and $T$ a self-mapping of $C$ which satisfies (2.16). If $T$ satisfies Condition-A, where $F$ is the nonempty fixed point set of $T$ in $C$, then, for an arbitrary $x_0 \in C$, the Picard iterates $\{S^n x_0\}$ converge to a member of $F$.

We may note that $C$ need not be bounded in Theorem 5. Because we have assumed the existence of nonempty fixed point set of $T$ and Condition-A (see [11]). But the boundedness of $C$ cannot be omitted from the statement of Theorem 4.

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