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ASYMPTOTICS OF REGULAR CONVOLUTION QUOTIENTS

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ABSTRACT. The asymptotic behaviour of a class of generalized functions, named regular convolution quotients, has been defined and analysed. Some properties of such asymptotics, which can be useful in applications, have been proved.

KEY WORDS AND PHRASES. S-asymptotics, convolution quotient, regular convolution quotients, distributions, generalized functions.
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1. INTRODUCTION.

T.K. Boehme in [1] defined and investigated a class of generalized functions named *regular convolution quotients*. This class is a generalization of the Schwartz distributions and also of the regular Mikusinski operators (see [2], [3] and [4]). On the other hand, for the Schwartz distributions a theory of the asymptotic behaviour, S-asymptotics, has been developed (see for example [5], [6], [7] and [8]), which can be applied in solving a lot of mathematical models. A distribution T has S-asymptotics related to a positive and measurable function c iff $\lim_{h\to\infty} (T*w)(h)/c(h) = (S*w)(0)$ for every $w \in \mathbf{D}$. We write: $T \overset{S}{\sim} c(h).S$, $h \to \infty$. In this paper we shall enlarge the definition of S-asymptotics of distributions to the regular convolution quotients having in view the application of this class of generalized functions.

2. REGULAR CONVOLUTION QUOTIENTS.

By Boehme [1], an *approximate identity* is a sequence of functions $(u_n) \subset L(\mathbf{R})$ satisfying the following conditions:

i) $\int_{\mathbf{R}} u_n(x)\,dx = 1$, $n \in \mathbf{N}$;

ii) there is an $M > 0$ such that $\int_{\mathbf{R}} |u(x)|\,dx < M$, $n \in \mathbf{N}$;

iii) there exists a sequence $(k_n) \subset \mathbf{R}_+$ such that $k_n \to 0$ as $n \to \infty$ and $\operatorname{supp} u_n \subset [-k_n, k_n]$, $n \in \mathbf{N}$.

Δ will be the set of all approximate identities and $\Delta^\infty = \{(u_n) \in \Delta,\ u_n \in C^\infty,\ n \in \mathbf{N}\}$. A *defining sequence* for a regular convolution quotient is a sequence of pairs $((f_n, u_n))$, where $(f_n) \subset L_{loc}(\mathbf{R})$, $(u_n) \in \Delta$ and for all $m, n \in \mathbf{N}$ the following convolution products are equal:

iv) $f_n * u_m = f_m * u_n$ (the asterisk is the sign of the convolution).

Two defining sequences $((f_n,u_n))$ and $((g_n,v_n))$ are said to be equivalent if:

v) $f_n*v_m = g_m*u_n$ for $n,m \in \mathbf{N}$.

By f_n/u_n we shall denote the equivalence class containing the difining sequence $((f_n,u_n))$. A r e g u l a r c o n v o l u t i o n q u o t i e n t X is an equivalence class of defining sequences . The regular convolution quotients are a vector space when the usual multiplication by scalars and addition of fractions is used; we denote it by $B(L_{loc},\Delta)$. The space $B(L_{loc},\Delta)$ contains $\mathbf{D'}$ (space of Schwartz's distributions) under the isomorphism: $\mathbf{D'} \ni T \Longleftrightarrow (T*v_n)/v_n \in B(L_{loc},\Delta)$, where $(v_n) \in \Delta^{\infty}$ Moreover, $B(L_{loc},\Delta)$ contains the class of all regular Mikusinski operators. Both of these containments are proper.

Let (h_n) be any continuously differentiable approximate identity. By $D = h'_n/h_n \in B(L_{loc},\Delta)$ is defined the differentiation operator. The derivative of an $X = f_n/h_n \in B(L_{loc},\Delta)$ is, now, defined to be $DX = (f_n*h'_n)/(u_n*h_n) \in B(L_{loc},\Delta)$.

For a distribution $T \in \mathbf{D'}$ and $w \in \mathbf{D}$ we shall write $T(w) = \langle t,w \rangle$. We shall use the following properties of elements belonging to Δ^{∞} and distributions defined by local integrable functions :

1. For $(f_n) \subset L_{loc}$ and $(v_n) \in \Delta^{\infty}$ we have $\langle f_n(x+h),\hat{v}_n(x)\rangle = (f_n*v_n)(h)$, $h \in \mathbf{R}$, where $\hat{v}_n(x) = v_n(-x)$.

2. If (u_n) and (v_n) belong to Δ^{∞}, then $(u_n*v_n) \in \Delta^{\infty}$, as well.

3. If $(f_n*v_n)(0) = 0$, $n \in \mathbf{N}$, for every $(v_n) \in \Delta^{\infty}$, then $f_n(x) = 0$ for almost all $x \in \mathbf{R}$.

3. S-ASYMPTOTICS OF REGULAR CONVOLUTION QUOTIENTS

Let Σ be the set of all real valued, positive and measurable functions: $\mathbf{R} \to \mathbf{R}_+$.

DEFINITION 1. A regular convolution quotient X has S-asymptotics at infinity, related to $c \in \Sigma$ and with the limit $U = F_n/u_n \in B(L_{loc},\Delta)$ if there exists $((f_n,u_n))$ belonging to the class X such that

$$\lim_{h\to\infty} \frac{(f_n*v_n)(h)}{c(h)} = (F_n*v_n)(0) \quad , \quad n \in \mathbf{N}$$

for every $(v_n) \in \Delta^{\infty}$. We shall write $X \overset{s}{\sim} c(h).U$, $h \to \infty$.

This definition does not depend on the defining sequence $((f_n,u_n))$ in the equivalence class X . Let $((g_n,j_n)) \in f_n/u_n$, and let $G_n/j_n \in B(L_{loc},\Delta)$ such that

$$\lim_{h\to\infty} \frac{(g_n*v_n)(h)}{c(h)} = (G_n*v_n)(0) \ , \quad n \in \mathbf{N} \text{ and } (v_n) \in \Delta^{\infty} .$$

Then $((F_n,u_n))$ and $((G_n,j_n))$ belong to the same class because of:

$$\langle (F_n*j_m),\hat{v}_n\rangle = ((F_n*j_m)*v_n)(0) =$$

$$= \lim_{h\to\infty} \frac{(f_n*(j_m*v_n))(h)}{c(h)} = \lim_{h\to\infty} \frac{(g_m*(u_n*v_n))(h)}{c(h)}$$

$$= ((G_m*u_n)*v_n)(0) = \langle G_m*u_n,\hat{v}_n\rangle$$

for every $(v_n) \in \Delta^{\infty}$ and $m,n \in \mathbf{N}$. Hence, $F_n*j_m = G_m*u_n$ for $m,n \in \mathbf{N}$.

PROPOSITION 1. If a distribution T has S-asymptotics, $T \overset{s}{\sim} c(h).S$, $h \to \infty$, $c \in \Sigma$, then the regular convolution quotient $X = (T*u_n)/u_n$ which corresponds to T , has S-asymptotics, as well and $X \overset{s}{\sim} c(h).(S*u_n)/u_n$, $h \to \infty$.

Proof. For every $(v_n) \in \Delta^\infty$ we have:

$$\lim_{h\to\infty} \frac{((T*u_n)*v_n)(h)}{c(h)} = \lim_{h\to\infty} \frac{(T*(u_n*v_n))(h)}{c(h)}$$
$$= (S*(u_n*v_n))(0) = ((S*u_n)*v_n)(0) \ , \ n\in \mathbf{N} \ .$$

$(S*u_n)/u_n$ belongs to $B(L_{loc},\Delta)$ because of $(S*u_n)*u_m = (S*u_m)*u_n$ for every $m,n\in\mathbf{N}$. Hence $X \overset{s}{\sim} c(h).(S*u_n)/u_n$, $h\to\infty$. Let us remark that $(S*u_n)/u_n$ corresponds to $S\in D'$ by the mentioned isomorphism. In such a way, S-asymptotics of regular convolution quotients, defined by Definition 1, generalizes S-asymptotics of distributions.

PROPOSITION 2. If X has S-asymptotics, $X \overset{s}{\sim} c(h).U$, $h\to\infty$, $c\in\Sigma$, then D^nX has S-asymptotics, as well and $D^nX \overset{s}{\sim} c(h).D^nU$, $h\to\infty$: D is the differentiation operator in $B(L_{loc},\Delta)$.

Proof. It is enough to prove for n=1. Let $X = f_n/u_n$ and let for every $(v_n)\in\Delta^\infty$

$$\lim_{h\to\infty} \frac{(f_n*v_n)(h)}{c(h)} = (F_n*v_n)(0) \ , \ n\in\mathbf{N} \ .$$

By definition, $DX = (f_n*h_n')/(u_n*h_n)$, where (h_n) is any continuously differentiable approximate identity. Now, the following relation is true:

$$\lim_{h\to\infty} \frac{((f_n*h_n')*v_n)(h)}{c(h)} = \lim_{h\to\infty} \frac{(f_n*(h_n'*v_n))(h)}{c(h)} = ((F_n*h_n')*v_n)(0) \ , \ n\in\mathbf{N} \ .$$

Hence, $(f_n*h_n')/(u_n*h_n) \overset{s}{\sim} c(h).(F_n*h_n')/(u_n*h_n)$ and $DX \overset{s}{\sim} c(h).DU$, $h\to\infty$, where $U = F_n/u_n$.

This proposition can be useful in applying regular convolution quotients to differential equations. The next proposition precises the analytical form of the function $c\in\Sigma$, which measures the asymptotical behaviour of a regular convolution quotient and the form of the regular convolution quotient U, the limit in Definition 1.

PROPOSITION 3. Suppose that $X\in B(L_{loc},\Delta)$ and $X \overset{s}{\sim} c(h).U$, $h\to\infty$, where $c\in\Sigma$ and $U = F_n/u_n$. If $F_n \neq 0$ for one $n\in\mathbf{N}$, then $c(h) = \exp(ah)\,L(\exp h)$, $h \geq h_0 > 0$, and $F_n(x) = C_n \exp(ax)$, where $a\in\mathbf{R}$, $C_n\in\mathbf{R}$, $C_n\neq 0$ and L is a slowly varying function.

Proof. L is a slowly varying function, by definition iff $L\in\Sigma$ and $\lim_{x\to\infty} L(ux)/L(x) = 1$, $u>0$. (For slowly varying functions see, for example [9]). By Definition 1, there exists $((f_n,u_n))\in X$ such that

$$\lim_{h\to\infty} \frac{(f_n*v_n)(h)}{c(h)} = (F_n*v_n)(0) \ , \ n\in\mathbf{N} \quad \text{for every} \quad (v_n)\in\Delta^\infty \ .$$

Now, the proof of Proposition 3 follows directly from Proposition 4.3 in [5], or propositions 9 and 10 in [7].

PROPOSITION 4. If $X\in B(L_{loc},\Delta)$, then X has a compact support if and only if : $X \overset{s}{\sim} c(h).0$, $|h|\to\infty$ for any $c\in\Sigma$.

Proof. We know (see [10]) that $X\in B(L_{loc},\Delta)$ has compact support if and only if there is a $(u_n)\in\Delta$ such that $u_nX = f_n$, $n\in\mathbf{N}$ and f_n, $n\in\mathbf{N}$, has compact support. Moreover, if X has compact support, then this is true for every $g_n = Xj_n$, $n\in\mathbf{N}$, $((g_n,j_n))\in X$. Suppose that supp $f_n \subset [-a_n,a_n]$ and supp $v_n \subset [-k_n,k_n]$, $a_n>0$, $k_n>0$, $n\in\mathbf{N}$ and $(v_n)\in\Delta^\infty$. Then we have: $(f_n*v_n)(h)=0$ for $|h|>a_n+k_n$. Hence,

$$\lim_{h\to\infty} \frac{(f_n*v_n)(h)}{c(h)} = 0 \ , \ n\in\mathbf{N} \ , \quad \text{for any} \quad c\in\Sigma \quad \text{and any} \quad (v_n)\in\Delta^\infty.$$

Suppose ,now, that $X \overset{s}{\sim} c(h).0$, $|h| \to \infty$ for every $c \in \Sigma$, where $X = f_n/u_n$ and suppose that for every $(v_n) \in \Delta^\infty$ we have:

$$\lim_{h \to \infty} \frac{(f_n * v_n)(h)}{c(h)} = 0 \ , \quad n \in \mathbf{N} \ ,$$

then by Proposition 8.1 , p. 98 in [5] or by Proposition 12 in [7], f_n , $n \in \mathbf{N}$, has a compact support .

The S-asymptotic behaviour of a regular convolution quotient is a local property. This property precises the following proposition.

PROPOSITION 5. Suppose that X and Y belong to $B(L_{loc}, \Delta)$ and $X \overset{s}{\sim} c(h).U$, $h \to \infty$, $c \in \Sigma$. If $X = Y$ on an interval (a, ∞) , $a \in \mathbf{R}$, then $Y \overset{s}{\sim} c(h).U$, $h \to \infty$, as well.

Proof. Let $X = f_n/u_n$, $Y = g_n/j_n$ and for every $(v_n) \in \Delta^\infty$

$$\lim_{h \to \infty} \frac{(f_n * v_n)(h)}{c(h)} = (F_n * v_n)(0) \ , \quad n \in \mathbf{N} \, .$$

By properties of the convolution it follows:

$$\lim_{h \to \infty} \frac{((f_n * j_n) * v_n)(h)}{c(h)} = ((F_n * j_n) * v_n)(0) \ , \quad n \in \mathbf{N} \ , \quad (v_n) \in \Delta^\infty .$$

If $X = Y$, then $X - Y = 0$, where $X - Y = (f_n * j_n - g_n * u_n)/(j_n * u_n)$. Hence, there exists a sequence $(b_n) \in \mathbf{R}$ such that $\mathrm{supp}\,(f_n * j_n - g_n * u_n) \subset (b_n, \infty)$. Now,

$$\lim_{h \to \infty} \frac{((f_n * j_n - g_n * u_n) * v_n)(h)}{c(h)} = 0 \ , \ n \in \mathbf{N} \ , \ (v_n) \in \Delta^\infty .$$

Therefore,

$$\lim_{h \to \infty} \frac{((g_n * u_n) * v_n)(h)}{c(h)} = ((F_n * j_n) * v_n)(0) \ , \quad n \in \mathbf{N} \ , \quad (v_n) \in \Delta^\infty .$$

The equivalence class $(g_n * u_n)/(j_n * u_n)$ is just Y because of $(g_n * u_n) * j_m = g_m * (j_n * u_n)$ and $Y \overset{s}{\sim} c(h).(F_n * j_n)/(j_n * u_n)$.It remains only to see that $(F_n * j_n)/(j_n * u_n) = F_n/u_n$. This follows from the relation $(F_n * j_n) * u_m = F_m * (j_n * u_n)$, m,n $\mathbf{N}$.

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