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ON CERTAIN BAZILEVIĆ FUNCTIONS OF ORDER β

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ABSTRACT. A certain class $B(n,\alpha,\beta)$ of Bazilević functions of order β in the unit disk is introduced. The object of the present paper is to derive some properties of functions belonging to the class $B(n,\alpha,\beta)$. Our result for the class $B(n,\alpha,\beta)$ is the improvement of the theorem by N. E. Cho ([1]).

KEY WORDS AND PHRASES. Analytic function, class $B(n,\alpha,\beta)$, Bazilević function.
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1. INTRODUCTION.

Let $A(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in N = \{1,2,3,...\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. A function $f(z) \in A(n)$ is said to be a member of the class $B(n,\alpha,\beta)$ if it satisfies

$$Re\left\{\frac{f'(z)f(z)^{\alpha-1}}{z^{\alpha-1}}\right\} > \beta \tag{1.2}$$

for some $\alpha(\alpha > 0)$, $\beta(0 \leqq \beta < 1)$, and for all $z \in U$. We note that $B(n,\alpha,\beta)$ is the subclass of Bazilević functions in the unit disk U (cf. [1]). Also we say that $f(z)$ in the class $B(n,\alpha,\beta)$ is a Bazilević function of order β.

Recently, Cho [1] has studied the class $B(n,\alpha,0)$ when $\beta = 0$, and has proved

THEOREM A. If $f(z) \in B(n,2,0)$ when $\alpha = 2$ and $\beta = 0$, then

$$Re\left\{\frac{f(z)}{z}\right\} > \frac{n}{n+2} \qquad (z \in U). \tag{1.3}$$

In the present paper, we improve the above theorem by Cho [1].

2. PROPERTIES OF THE CLASS $B(n,\alpha,\beta)$.

In order to establish our main result, we have to recall here the following lemma due to Miller and Mocanu [2].

LEMMA. Let $\phi(u,v)$ be a complex valued function,

$$\phi: D \to C, D \subset C^2 \text{ (C is the complex plane)},$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -n(1+u_2^2)/2$,

$$Re\{\phi(iu_2, v_1)\} \leqq 0 .$$

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ be regular in the unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$Re\{\phi(p(z), zp'(z))\} > 0 \qquad (z \in U),$$

then

$$Re\{p(z)\} > 0 \qquad (z \in U).$$

Using the above lemma, we prove

THEOREM 1. If $f(z) \in B(n,\alpha,\beta)$, then

$$Re\left\{\frac{f(z)}{z}\right\}^{\alpha} > \frac{n+2\alpha\beta}{n+2\alpha} \qquad (z \in U). \tag{2.1}$$

PROOF. We define the function $p(z)$ by

$$\left\{\frac{f(z)}{z}\right\}^{\alpha} = \gamma + (1-\gamma)p(z) \tag{2.2}$$

with

$$\gamma = \frac{n+2\alpha\beta}{n+2\alpha}. \tag{2.3}$$

Then, we see that $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is regular in U.
It follows from (2.2) that

$$\frac{f'(z)f(z)^{\alpha-1}}{z^{\alpha-1}} = \gamma + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{\alpha}, \tag{2.4}$$

or

$$Re\left\{\frac{f'(z)f(z)^{\alpha-1}}{z^{\alpha-1}} - \beta\right\} \tag{2.5}$$

$$= Re\left\{\gamma - \beta + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{\alpha}\right\}$$

$$> 0 .$$

Defining the function $\phi(u,v)$ by

$$\phi(u,v) = \gamma - \beta + (1-\gamma)u + \frac{(1-\gamma)v}{\alpha}, \tag{2.6}$$

(note that $u = p(z)$ and $v = zp'(z)$, we have that

(i) $\phi(u,v)$ is continuous in $D = C^2$;

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} = 1 - \beta > 0$;

(iii) for all (iu_2, v_1) such that $v_1 \leqq -n(1+u_2^2)/2$,

$$\begin{aligned} Re\{\phi(iu_2, v_1)\} &= \gamma - \beta + \frac{(1-\gamma)v_1}{\alpha} \\ &\leqq \gamma - \beta - \frac{n(1-\gamma)(1+u_2^2)}{2\alpha} \\ &= -\frac{n(1-\gamma)u_2^2}{2\alpha} \\ &\leqq 0 . \end{aligned}$$

Therefore, the function $\phi(u,v)$ satisfies the conditions in Lemma. This implies that $Re\{p(z)\} > 0 (z \in U)$, which is equivalent to

$$Re\left\{\frac{f(z)}{z}\right\}^{\alpha} > \gamma = \frac{n+2\alpha\beta}{n+2\alpha} \quad (z \in U) . \tag{2.7}$$

This completes the assertion of Theorem 1.

Letting $\beta = 0$ in Theorem 1, we have

COROLLARY 1. If $f(z) \in B(n, \alpha, 0)$, then

$$Re\left\{\frac{f(z)}{z}\right\}^{\alpha} > \frac{n}{n+2\alpha} \quad (z \in U) . \tag{2.8}$$

REMARK. If we take $\alpha = 1$ in Corollary 1, then we have the inequality (1.3) by Cho [1].

Making $\alpha = 1/2$, Theorem 1 gives

COROLLARY 2. If $f(z) \in B(n, 1/2, \beta)$, then

$$Re\sqrt{\frac{f(z)}{z}} > \frac{n+\beta}{n+1} \quad (z \in U) . \tag{2.9}$$

Finally, we derive

THEOREM 2. If $f(z) \in B(n, \alpha, \beta)$, then

$$Re\left\{\frac{f(z)}{z}\right\}^{\alpha/2} > \frac{n+\sqrt{n^2+4\alpha\beta(n+\alpha)}}{2(n+\alpha)} \quad (z \in U) . \tag{2.10}$$

PROOF. Defining the function $p(z)$ by

$$\left\{\frac{f(z)}{z}\right\}^{\alpha/2} = \gamma + (1-\gamma)p(z) \tag{2.11}$$

with

$$\gamma = \frac{n+\sqrt{n^2+4\alpha\beta(n+\alpha)}}{2(n+a)} , \tag{2.12}$$

we easily see that $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is regular in U. Taking the differentiations of both sides in (2.11), we obtain that

$$\begin{aligned} &\frac{f'(z)f(z)^{\alpha-1}}{z^{\alpha-1}} \qquad (2.13) \\ &= (\gamma + (1-\gamma)p(z))^2 + \frac{2}{\alpha}(1-\gamma)(\gamma + (1-\gamma)p(z))zp'(z) , \end{aligned}$$

that is, that

$$Re\left\{\frac{f'(z)f(z)^{\alpha-1}}{z^{\alpha-1}} - \beta\right\} \tag{2.14}$$

$$= Re\left\{(\gamma + (1-\gamma)p(z))^2 + \tfrac{2}{\alpha}(1-\gamma)(\gamma + (1-\gamma)p(z))zp'(z) - \beta\right\}$$

$$> 0 .$$

Therefore, letting

$$\phi(u,v) = (\gamma + (1-\gamma)u)^2 + \tfrac{2}{\alpha}(1-\gamma)(\gamma + (1-\gamma)u)v - \beta , \tag{2.15}$$

(note that $p(z) = u = u_1 + iu_2$ and $zp'(z) = v = v_1 + iv_2$), we observe that

(i) $\phi(u,v)$ is continuous in $D = C^2$;

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} = 1 - \beta > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -n(1+u_2^2)/2$,

$$Re\{\phi(iu_2, v_1)\} = \gamma^2 - (1-\gamma)^2u_2^2 + \tfrac{2}{\alpha}\gamma(1-\gamma)v_1 - \beta$$

$$\leqq \gamma^2 - \beta - (1-\gamma)^2u_2^2 - \tfrac{n}{\alpha}\gamma(1-\gamma)(1+u_2^2)$$

$$\leqq 0 .$$

Thus, the function $\phi(u,v)$ satisfies the conditions in Lemma. Applying Lemma, we conclude that

$$Re\left\{\frac{f(z)}{z}\right\}^{\alpha/2} > \gamma = \frac{n + \sqrt{n^2 + 4\alpha\beta(n+\alpha)}}{2(n+\alpha)} \quad (z \in U) . \tag{2.16}$$

Taking $\alpha = 1$ in Theorem 2, we have

COROLLARY 3. If $f(z) \in B(n,1,\beta)$, then

$$Re\sqrt{\frac{f(z)}{z}} > \frac{n + \sqrt{n^2 + 4n\beta + 4\beta}}{2(n+1)} \quad (z \in U) . \tag{2.17}$$

REMARK. If we take $\alpha = 2$ and $\beta = 0$ in Theorem 2, then we have Theorem A by Cho [1].

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