Internat. J. Math. & Math. Sci
VOL. 15 NO. 3 (1992) 523-536

GENERALIZED GREEN'S FUNCTIONS FOR HIGHER ORDER BOUNDARY VALUE MATRIX DIFFERENTIAL SYSTEMS

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(Received May 29, 1991)

ABSTRACT. In this paper, a Green's matrix function for higher order two point boundary value differential matrix problems is constructed. By using the concept of rectangular co-solution of certain algebraic matrix equation associated to the problem, an existence condition as well as an explicit closed form expression for the solution of possibly not well-posed boundary value problems is given avoiding the increase of the problem dimension.

KEYWORDS AND PHRASES. Two point boundary value problem, Green's matrix function, co-solution, algebraic matrix equation, Moore-Penrose pseudoinverse.

CLASSIFICATION A.M.S. (1991). 34B27 15A24 34B10 34B05.

1.- INTRODUCTION.

Two-point boundary value problems for higher order matrix of differential systems of the type

$$X^{(p)} + A_{p-1} X^{(p-1)} + \dots + A_1 X^{(1)} + A_0 X = f(t); \qquad 0 \le t \le b$$
$$\sum_{j=1}^{p} \left\{ E_{ij} X^{(j-1)}(0) + F_{ij} X^{(j-1)}(b) \right\} = r_i; \qquad 1 \le i \le q. \tag{1.1}$$

where $f(t)$, $X(t)$, r_i are matrices in $\mathbb{C}^{n\times m}$ for $1\le i\le q$ and A_k, E_{ij}, F_{ij} are matrices in $\mathbb{C}^{n\times n}$ for $1\le i\le q$, $1\le j\le p$, $0\le k\le p-1$, appear in different physical problems [1,chap 1].

The standard approach to study such problems is based on the consideration of an extended first order problem

$$Y'(t) = C\,Y(t) + F(t);\; B_a Y(a) + B_b Y(b) = R. \tag{1.2}$$

where $Y = (X, X', \dots, X^{(p-1)})^T$, $F = (0,\dots,f)^T$ are matrices in $\mathbb{C}^{np\times m}$ B_a and B_b are appropriate matrices in $\mathbb{C}^{nq\times np}$, R is a matrix in $\mathbb{C}^{nq\times n}$, and C is the companion matrix defined by

$$C = \begin{bmatrix} 0 & I & \cdots & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & I \\ -A_0 & -A_1 & \cdots & -A_{p-1} \end{bmatrix} \tag{1.3}$$

See [1][3][14].

This classical approach has the inconvenience of the lack of explicitness due to the relationship $X(t) = [I,0,\ldots,0]\,Y(t)$, as well as the computational cost due to the increase of the problem dimension. In particular it needs the computation of the matrix exponential exp(tC) and it is well known that it is not an easy task [11].

These inconveniences motivates the study of some alternative approach that avoids the increase of the problem dimension. In [4], a solution for a very particular second order problem of the type (1.1) is proposed avoiding the increase of the problem dimension, however, the method is not applicable to more general problems. In a recent paper [7] a method for solving problems of the type (1.1) for the case p=2, without considering the extended system (1.2) have been proposed. Results of [7] are based on the existence of an appropriate pair of solutions of the characteristic algebraic matrix equation

$$Z^2 + A_1 Z + A_0 = 0. \tag{1.4}$$

Unfortunately, equation (1.4) may be unsolvable [6] and in such case, the method given in [7] is not available.

The aim of this paper is to study an existence condition for the solution of problem (1.1) as well as an explicit expression of a solution of the problem in terms of a generalized Green's matrix function G(t,s), taking advantage of the ideas developed in [7] but without the restriction of the existence of solutions of the associated algebraic matrix equation

$$Z^p + A_{p-1} Z^{p-1} + \ldots + A_1 Z + A_0 = 0. \tag{1.5}$$

The paper is organized as follows. In section 2, we introduce the concept of rectangular co-solution for the equation (1.5) and we state some results recently given [8], that will be used in the following sections. In section 3, we construct a generalized Green's matrix function of problem (1.1) by using an appropriate set of co-solutions of equation (1.5) and a procedure analogous to the one developed in [5] for the scalar case. Finally, in section 4 an explicit closed form solution of problem (1.1) in terms of a generalized Green's matrix function is given.

If S is a matrix in $\mathbb{C}^{m\times n}$, we denote by S^+ its Moore-Penrose pseudoinverse. We recall that an account of uses and properties of this concept may be found in [2] and that the computation of S^+ is an easy matter using MATLAB [10].

2.- RECTANGULAR CO-SOLUTIONS OF POLYNOMIAL MATRIX EQUATIONS AND APPLICATIONS.

We begin by introducing the concept of rectangular co-solution of equation (1.5), recently given in [8].

DEFINITION 2.1. We say that (X,T) is a (n,q) co-solution of equation (1.5) if $X \in \mathbb{C}^{n\times q}$, $T \in \mathbb{C}^{q\times q}$, $X \neq 0$ and

$$XT^{p} + A_{p-1}XT^{p-1} + \ldots + A_{0}X = 0. \tag{2.1}$$

DEFINITION 2.2. Let (X_i,T_i) be a (n,m_i) co-solution for $1\le i\le k$. We say that $\{(X_i,T_i),\ 1\le i\le k\}$ is a k-complete set of co-solutions of (1.5) if the block matrix $W=(W_{ij})$, with $W_{ij}= X_jT_j^{i-1}$ for $1\le i\le p$, $1\le j\le k$, is invertible.

THEOREM 1. ([8]) Let C be the companion matrix. If $M = (M_{ij})$ with $M_{ij} \in \mathbb{C}^{n\times m_j}$, is a nonsingular matrix in $\mathbb{C}^{np\times np}$, $1\le i\le p$, $1\le j\le k$, and if the Jordan canonical form J of C is $J = \operatorname{diag}(J_1,\ldots,J_k)$, with $J_j \in \mathbb{C}^{m_j\times m_j}$, $m_1 +\ldots+ m_k = np$, such that

$$M\operatorname{diag}(J_1,\ldots,J_k) = C\,M \tag{2.2}$$

then $\{(M_{1s},J_s),\ 1\le s\le k\}$ is a k-complete set of co-solutions of (1.5).

COROLLARY 1.([8]) Let us suppose the notation of theorem 1, and let $\{(M_{1s},J_s),\ 1\le s\le k\}$ be a k-complete set of co-solutions of equation (1.5). Then, the general solution of the matrix differential equation (1.1) is given by

$$X(t) = \sum_{s=1}^{k} M_{1s}\exp(tJ_s)D_s \tag{2.3}$$

where D_s, is an arbitrary matrix in $\mathbb{C}^{m_s\times m}$. If W is the block partitioned matrix associated to the set $\{(M_{1s},J_s),\ 1\le s\le k\}$ by definition 2.2, the only solution of (1.1) satisfying the Cauchy conditions $X^{(j)}(0) = C_j$, $0\le j\le p-1$, is given by (2.3), where the matrices D_s, for $1\le i\le k$, are uniquely determined by the expression

$$\begin{bmatrix} D_1 \\ \vdots \\ D_k \end{bmatrix} = W^{-1}\begin{bmatrix} C_0 \\ \vdots \\ C_{p-1} \end{bmatrix}. \tag{2.4}$$

For the sake of clarity in the presentation, we recall a result about the solutions of rectangular systems of equations, that will be used in the following sections.

THEOREM 2. ([13,p.24]) The matrix system SP=Q, where S, P, Q are matrices in $\mathbb{C}^{m\times n}$, $\mathbb{C}^{n\times r}$ and $\mathbb{C}^{m\times r}$ respectively, is compatible if and only if $S\,S^{+}Q = Q$ and in this case, the solution of the system is given by

$$P = S^{+}Q + (I - S^{+}S)Z,$$

where Z is an arbitrary matrix in $\mathbb{C}^{n\times r}$.

Note that under the conditions of theorem 2, a particular solution of system $SP=Q$ is given by $P=S^{+}Q$.

3.- CONSTRUCTION OF GREEN'S MATRIX FUNCTIONS.

Let us consider the homogeneous problem

$$X^{(p)}+A_{p-1}X^{(p-1)}+\dots+A_1X^{(1)}+A_0X=0;\quad 0\le t\le b \tag{3.1}$$

$$\sum_{j=1}^{p}\left\{E_{ij}X^{(j-1)}(0)+F_{ij}X^{(j-1)}(b)\right\}=0;\qquad 1\le i\le q. \tag{3.2}$$

and let $\{(M_{1i},J_i),\ 1\le i\le k\}$ be the k-complete set of co-solutions of equation (1.5) provided by corollary 1. Then, the general solution of equation (3.1) is given by

$$X(t)=\sum_{i=1}^{k}U_i(t)D_i \tag{3.3}$$

where D_i is an arbitrary matrix in $\mathbb{C}^{m_i\times m}$ and

$$U_i(t)=M_{1i}\exp(tJ_i).$$
$$i=1,\dots,k.$$

Let us consider the matrix function $G(t,s)$ defined by

$$G(t,s)=\begin{cases}\sum_{i=1}^{k}U_i(t)P_i(s), & 0\le t\le s\\ \sum_{i=1}^{k}U_i(t)Q_i(s), & s\le t\le b\end{cases} \tag{3.5}$$

where the $\mathbb{C}^{m_i\times m}$ valued matrix functions $P_i(s)$, $Q_i(s)$ have to be determined so that

1.- $G(t,s)$ is a continuous matrix function in $[0,b]\times[0,b]$ and moreover, $\partial^{(j)}G/\partial t^{(j)}$ is a continuous function in (t,s), for (t,s) in the triangles $0\le t<s\le b$ and $0\le s<t\le b$ for $j=1,\dots,p-2$.

2.- If I is the identity matrix in $\mathbb{C}^{n\times n}$, one gets the jump discontinuity

$$\frac{\partial^{(p-1)}G}{\partial t^{(p-1)}}(s+0,s)-\frac{\partial^{(p-1)}G}{\partial t^{(p-1)}}(s-0,s)=I. \tag{3.6}$$

3.- As a function of t, $G(t,s)$ satisfies (3.1) and (3.2) in $[0,b]$, if $t\neq s$.

From (3.5) the continuity condition at $t=s$ of Green's function gives us that

$$\sum_{i=1}^{k}U_i(s)P_i(s)=\sum_{i=1}^{k}U_i(s)Q_i(s),$$

or

$$\sum_{i=1}^{k} U_i(s)(P_i(s) - Q_i(s)) = 0. \tag{3.7}$$

On the other hand, by the continuity condition of the partial derivatives of the Green's function until order p-2 at t=s, we obtain

$$\sum_{i=1}^{k} U_i^{(j)}(s)P_i(s) = \sum_{i=1}^{k} U_i^{(j)}(s)Q_i(s) \ , \ j = 1,\dots,p-2$$

and then

$$\sum_{i=1}^{k} U_i^{(j)}(s)(P_i(s) - Q_i(s)) = 0 \ , \ j= 1,\dots,p-2. \tag{3.8}$$

From (3.5) and (3.6) it follows that

$$\sum_{i=1}^{k} U_i^{(p-1)}(s)(P_i(s) - Q_i(s)) = I. \tag{3.9}$$

Let us write

$$R_i(s) = P_i(s) - Q_i(s) \ , \quad i = 1,\dots,k \tag{3.10}$$

then, conditions (3.7) - (3.10) may be written in the compact form

$$\mathbf{U(s)} \begin{bmatrix} R_1(s) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ R_k(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ I \end{bmatrix} \tag{3.11}$$

where

$$\mathbf{U(s)} = \begin{bmatrix} U_1(s) & \dots\dots\dots & U_k(s) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ U_1^{(p-1)}(s) & \dots & U_k^{(p-1)}(s) \end{bmatrix} \tag{3.12}$$

Note that the matrix function $\mathbf{U(s)}$ defined by (3.12) is invertible for all s, because we may decompose $\mathbf{U(s)}$ in the form

$$\mathbf{U(s)} = W \text{ diag } \{\exp(sJ_i), \ 1 \le i \le k\} \tag{3.13}$$

where

$$W = \begin{bmatrix} M_{11} & \dots\dots\dots & M_{1k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ M_{11}J_1^{p-1} & \dots & M_{1k}J_k^{p-1} \end{bmatrix} \tag{3.14}$$

is invertible since $\{(M_{1i}, J_i),\ 1 \le i \le k\}$ is a k-complete set of co-solution of equation (1.5).

Let us denote $Y = [Y_{ij}]_{1 \le i \le p,\, 1 \le j \le k}$ with $Y_{ij} \in \mathbb{C}^{m_i \times n}$ the inverse of the matrix W. Then, from (3.11) and (3.13), it follows that

$$R_i(s) = \exp(-sJ_i)\, Y_{ik}\ ;\ 1 \le i \le k \tag{3.15}$$

If we impose that G(t,s) defined by (3.5) satisfies the initial conditions (3.2), we obtain

$$\sum_{j=1}^{p} \left\{ E_{ij} \sum_{m=1}^{k} U_m^{(j-1)}(0) P_m(s) + F_{ij} \sum_{m=1}^{k} U_m^{(j-1)}(b) Q_m(s) \right\} = 0, \tag{3.16}$$

$$i = 1, \ldots, q.$$

From (3.10) we have

$$Q_i(s) = P_i(s) - R_i(s), \qquad i = 1, \ldots, k. \tag{3.17}$$

Substituting (3.17) into (3.16), it follows that

$$\sum_{j=1}^{p} \left\{ E_{ij} \sum_{m=1}^{k} U_m^{(j-1)}(0) P_m(s) + F_{ij} \sum_{m=1}^{k} U_m^{(j-1)}(b) \Big[P_m(s) - R_m(s) \Big] \right\} = 0$$

$$i = 1, \ldots, q,$$

and

$$\sum_{m=1}^{k} \sum_{j=1}^{p} \left(E_{ij}\, U_m^{(j-1)}(0) + F_{ij}\, U_m^{(j-1)}(b) \right) P_m(s) =$$

$$= \sum_{m=1}^{k} \sum_{j=1}^{p} F_{ij}\, U_m^{(j-1)}(b)\, R_m(s), \qquad i = 1, \ldots, q. \tag{3.18}$$

Let S be the block matrix

$$S = \left[\sum_{j=1}^{p} \left(E_{ij}\, U_m^{(j-1)}(0) + F_{ij}\, U_m^{(j-1)}(b) \right) \right]_{1 \le i \le q,\, 1 \le m \le k} \tag{3.19}$$

and let S^+ be the Moore-Penrose pseudoinverse matrix

$$S^+ = [T_{mi}]_{1 \le m \le k,\, 1 \le i \le q}, \text{ with } T_{mi} \in \mathbb{C}^{m_i \times n}. \tag{3.20}$$

Note that the conditions (3.18) may be written in the form

$$S \begin{bmatrix} P_1(s) \\ \vdots \\ P_k(s) \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^{k}\sum_{j=1}^{p} F_{1j} U_m^{(j-1)}(b) R_m(s) \\ \vdots \\ \sum_{m=1}^{k}\sum_{j=1}^{p} F_{qj} U_m^{(j-1)}(b) R_m(s) \end{bmatrix} \tag{3.21}$$

From theorem 2, the equation (3.21) is solvable if and only if

$$S\,S^{+} \begin{bmatrix} \sum_{m=1}^{k}\sum_{j=1}^{p} F_{1j} U_m^{(j-1)}(b) R_m(s) \\ \vdots \\ \sum_{m=1}^{k}\sum_{j=1}^{p} F_{qj} U_m^{(j-1)}(b) R_m(s) \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^{k}\sum_{j=1}^{p} F_{1j} U_m^{(j-1)}(b) R_m(s) \\ \vdots \\ \sum_{m=1}^{k}\sum_{j=1}^{p} F_{qj} U_m^{(j-1)}(b) R_m(s) \end{bmatrix} \tag{3.22}$$

Let us suppose the algebraic equation (3.21) is compatible. Then, from theorem 2 and (3.20) a solution of (3.21) is given by

$$\begin{bmatrix} P_1(s) \\ \vdots \\ P_k(s) \end{bmatrix} = \begin{bmatrix} T_{11} & \cdots & T_{1q} \\ \vdots & & \vdots \\ T_{k1} & \cdots & T_{kq} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{p}\sum_{m=1}^{k} F_{1j} U_m^{(j-1)}(b) \exp(-sJ_m) Y_{mp} \\ \vdots \\ \sum_{j=1}^{p}\sum_{m=1}^{k} F_{qj} U_m^{(j-1)}(b) \exp(-sJ_m) Y_{mp} \end{bmatrix}$$

and then,

$$P_i(s) = \sum_{l=1}^{p} \sum_{j=1}^{p} \sum_{m=1}^{k} T_{il}\, F_{lj}\, U_m^{(j-1)}(b)\, \exp(-sJ_m)\, Y_{mp} \tag{3.23}$$

$$i = 1, \ldots, k.$$

Hence and from (3.15),(3.17) it follows that

$$Q_i(s) = P_i(s) - R_i(s) =$$

$$= \left(\sum_{l=1}^{p} \sum_{j=1}^{p} \sum_{m=1}^{k} T_{il}\, F_{lj}\, U_m^{(j-1)}(b)\, \exp(-sJ_m)\, Y_{mp} \right) - \exp(-sJ_i)\, Y_{ip} \tag{3.24}$$

$$i = 1, \ldots, k.$$

Thus the following result has been established

THEOREM 3. Let $\{(M_{ii}, J_i),\ 1 \le i \le k\}$ be the k-complete set of co-solutions of equation (1.5) given by theorem 1 and let $\{U_i(t),\ 1 \le i \le k\}$ be defined by (3.4). If condition (3.22) is given, then the boundary value matrix problem (3.1) - (3.2) has a generalized Green's matrix function defined by (3.5), where $P_i(s)$ and $Q_i(s)$ are given by (3.23) and (3.24).

REMARK. If the matrix S has full rank, then, from [2,p.12] $S^+S=I$, (3.21) has only one solution and there exists a unique Green's matrix function.

4.- SOLUTION OF THE NON-HOMOGENEOUS BOUNDARY PROBLEM.

Let us consider the intermediate boundary value problem,

$$X^{(p)} + A_{p-1} X^{(p-1)} + \dots + A_1 X^{(1)} + A_0 X = f(t)$$
$$\sum_{j=1}^{p} E_{ij} X^{(j-1)}(0) + F_{ij} X^{(j-1)}(b) = 0, \quad i = 1, \dots, q \tag{4.1}$$

where f(t) is a $\mathbb{C}^{n\times m}$ valued continuous matrix function in [0,b].

Let X(t) be defined by

$$X(t) = \int_0^b G(t,s) f(s)\, ds =$$
$$= \int_0^t G(t,s) f(s)\, ds + \int_t^b G(t,s) f(s)\, ds. \tag{4.2}$$

Taking derivatives and using the Leibniz' rule, we have

$$X'(t) = \int_0^t \frac{\partial G(t,s)}{\partial t} f(s)\, ds + G(t,t) f(t) + \int_t^b \frac{\partial G(t,s)}{\partial t} f(s)\, ds -$$
$$- G(t,t) f(t) = \int_0^b \frac{\partial G(t,s)}{\partial t} f(s)\, ds.$$

$$X''(t) = \int_0^t \frac{\partial^2 G(t,s)}{\partial t^2} f(s)ds + \frac{\partial G(t,t)}{\partial t} f(t) + \int_t^b \frac{\partial^2 G(t,s)}{\partial t^2} f(s)ds -$$
$$- \frac{\partial G(t,t)}{\partial t} f(t) = \int_0^b \frac{\partial^2 G(t,s)}{\partial t^2} f(s)\, ds.$$

$$X^{(p-1)}(t) = \int_0^b \frac{\partial^{(p-1)} G(t,s)}{\partial t^{(p-1)}} f(s)\, ds$$

$$X^{(p)}(t) = \int_0^t \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s)\, ds + \frac{\partial^{(p-1)} G(t,t-0)}{\partial t^{(p-1)}} f(t) +$$
$$+ \int_t^b \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s)\, ds - \frac{\partial^{(p-1)} G(t,t+0)}{\partial t^{(p-1)}} f(t) =$$
$$= + \int_0^b \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s)\, ds + \left(\frac{\partial^{(p-1)} G(t,t-0)}{\partial t^{(p-1)}} - \frac{\partial^{(p-1)} G(t,t+0)}{\partial t^{(p-1)}}\right) f(t) =$$
$$= \int_0^b \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s)\, ds + f(t).$$

Hence and from the properties of G(t,s) it follows that

$$X^{(p)}+ A_{p-1}X^{(p-1)}+ \dots + A_1X^{(1)}+ A_0X =$$

$$= \int_0^b \frac{\partial^{(p)}G(t,s)}{\partial t^{(p)}} f(s)ds + f(t) +$$

$$+ A_{p-1} \int_0^b \frac{\partial^{(p-1)}G(t,s)}{\partial t^{(p-1)}} f(s)\, ds + \dots + A_0 \int_0^b G(t,s)\, f(s)\, ds =$$

$$\int_0^b \left(\frac{\partial^{(p)}G(t,s)}{\partial t^{(p)}} + A_{p-1} \frac{\partial^{(p-1)}G(t,s)}{\partial t^{(p-1)}} + \dots A_0G(t,s) \right) f(s)ds + f(t) =$$

$$= f(t).$$

and

$$\sum_{j=1}^{p} \left\{E_{ij}\, X^{(j-1)}(0) + F_{ij}\, X^{(j-1)}(b)\right\} =$$

$$\int_0^b \left(\sum_{j=1}^{p} \left\{ E_{ij} \frac{\partial^{(j-1)}G(0,s)}{\partial t^{(j-1)}} + F_{ij} \frac{\partial^{(j-1)}G(b,s)}{\partial t^{(j-1)}} \right\}\right) f(s)\, ds = 0,$$

$$i= 1,\dots,q.$$

Now let us consider the auxiliary problem

$$X^{(p)}+ A_{p-1}\, X^{(p-1)}+ \dots + A_1X^{(1)} + A_0X = 0 \tag{4.3}$$

$$\sum_{j=1}^{p} E_{ij}\, X^{(j-1)}(0) + F_{ij}\, X^{(j-1)}(b) = r_i, \tag{4.4}$$

$$i = 1,\dots,q.$$

Then, from the corollary 1, the form of the solutions of (4.3) is

$$X(t) = \sum_{m=1}^{k} U_m(t)Q_m.$$

The boundary value conditions of (4.4) give us the next expression

$$\sum_{j=1}^{p} \left\{E_{ij} \sum_{m=1}^{k} U_m^{(j-1)}(0)\, Q_m + F_{ij} \sum_{m=1}^{k} U_m^{(j-1)}(b)\, Q_m\right\} = r_i$$

$$i = 1,\dots,q.$$

or

$$\sum_{j=1}^{p} \sum_{m=1}^{k} \left\{E_{ij}\, U_m^{(j-1)}(0) + F_{ij}\, U_m^{(j-1)}(b)\right\} Q_m= r_i.$$

$$i = 1,\dots,q.$$

If we set the last expression in matrix form

$$S \begin{bmatrix} Q_1 \\ \cdot \\ \vdots \\ \cdot \\ Q_k \end{bmatrix} = \begin{bmatrix} r_1 \\ \cdot \\ r_i \\ \cdot \\ r_q \end{bmatrix} \tag{4.5}$$

From theorem 2 of section 1, under the compatibility condition

$$S\, S^{+} \begin{bmatrix} r_1 \\ \cdot \\ r_i \\ \cdot \\ r_q \end{bmatrix} = \begin{bmatrix} r_1 \\ \cdot \\ r_i \\ \cdot \\ r_q \end{bmatrix}, \tag{4.6}$$

and taking into account (3.20), a solution of (4.5) is given by

$$\begin{bmatrix} Q_1 \\ \cdot \\ \vdots \\ \cdot \\ Q_k \end{bmatrix} = \begin{bmatrix} T_{11} \cdots\cdots T_{1q} \\ \vdots \qquad \vdots \\ T_{k1} \cdots\cdots T_{kq} \end{bmatrix} \begin{bmatrix} r_1 \\ \cdot \\ r_i \\ \cdot \\ r_q \end{bmatrix}.$$

Thus, $Q_m = \sum_{l=1}^{q} T_{ml}\, r_l$, $1 \le m \le k$, and a solution G(t) of (4.3) - (4.4) is given by the next expression

$$G(t) = \sum_{m=1}^{k} U_m(t)\left(\sum_{l=1}^{q} T_{ml} r_l\right). \tag{4.7}$$

Thus, the following result has been proved:

THEOREM 4. Let $\{(M_{1i}, J_i),\ 1 \le i \le k\}$ be a k-complete set of co-solutions of equation (1.5) and let $\{U_i(t),\ 1 \le i \le k\}$ be defined by (3.4). If the conditions (3.22) and (4.6) are satisfied, i.e., the algebraic equations (3.21) and (4.5) are compatible, S is defined by (3.19), $S^{+} = [T_{ml}]_{1 \le m \le k,\, 1 \le l \le q}$ is the Moore-Penrose pseudo-inverse and f(t) is continuous, then the boundary value problem (1.1) has a solution given by

$$X(t) = \int_0^b G(t,s)\, f(s)\, ds + G(t),$$

where G(t), is given by (4.6) and G(t,s) is the generalized Green's matrix function constructed by theorem 3.

REMARK. It is interesting to recall that the Jordan canonical form of a matrix may be efficiently computed with MACSYMA [9] and the matrix exponential of a Jordan block has a well known expression [12,p.66].

In the next example, we construct a generalized Green's matrix function for a not well-posed boundary value matrix problem.

EXAMPLE. Let us consider the second order differential equation,

$$X''(t) + A_1 X'(t) + A_0 X(t) = 0 \qquad t\in[0,1] \tag{4.8}$$

$$\begin{aligned} E_{11} X(0) + F_{11} X(1) = 0 \\ E_{22} X'(0) + F_{22} X'(1) = 0 \end{aligned} \tag{4.9}$$

and

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix},\ A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ E_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ F_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$E_{22} = \begin{bmatrix} 0 & 0 \\ -e & e \end{bmatrix},\ F_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.10}$$

Then, the matrices M, J and M^{-1} are given by

$$M = \left[\begin{array}{c|ccc} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ \hline 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],\quad J = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right],\quad M^{-1} = \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{array}\right]$$

and

$$\mathrm{diag}[\exp(sJ_1)] = e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, a complete set of co-solutions is

$$\left\{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0\right), \left(\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)\right\},$$

and we can compute the expressions $U_1(t)$, $U_2(t)$, theirs derivatives and $R_1(t)$, $R_2(t)$.

$$U_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ U_1'(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$U_2(t) = e^t\begin{bmatrix} -1 & 1-t & -t^2/2+t-1 \\ 0 & 1 & t-1 \end{bmatrix},\ U_2'(t) = e^t\begin{bmatrix} -1 & -t & -t^2/2 \\ 0 & 1 & t \end{bmatrix},$$

$$R_1(s) = \exp(-sJ_1)Y_{12} = [-1,0],$$

$$R_2(s) = \exp(-sJ_2)Y_{22} = e^{-s} \begin{bmatrix} 1 & -s+\frac{s^2}{2} \\ 0 & 1-s \\ 0 & 1 \end{bmatrix}.$$

From the boundary value conditions (4.9) the corresponding matrix S defined by (3.19) takes the form

$$S = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1+e & -1 \\ 0 & 0 & 0 & 0 \\ 0 & e & 0 & e \end{bmatrix}$$

that clearly is not invertible. Thus problem (4.8)-(4.10) is not well posed, however, the equality

$$S \begin{bmatrix} a & b \\ 0 & e^{-s} \\ 0 & e^{-s}(1-s) \\ 0 & e^{-s}(1-s) \end{bmatrix} = e^{1-s} \begin{bmatrix} 0 & 0 \\ 0 & 1-s \\ 0 & 0 \\ 0 & 2-s \end{bmatrix}$$

means that the corresponding algebraic equation (3.21) is compatible. Then, the Moore-Penrose pseudo-inverse is given by

$$S^{+} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1+e^{-1} & -e^{-1} & 0 & e^{-1} \\ -e^{-1} & e^{-1} & 0 & 0 \\ -1-e^{-1} & e^{-1} & 0 & 0 \end{bmatrix}$$

Therefore, we can obtain $P_1(s)$ and $P_2(s)$,

$$P_1(s) = [0,0],$$

$$P_2(s) = e^{-s} \begin{bmatrix} 0 & 1 \\ 0 & 1-s \\ 0 & 1-s \end{bmatrix}$$

and

$$Q_1(s) = [1,0].$$

$$Q_2(s) = e^{-s} \begin{bmatrix} 1 & 1+s-\frac{s^2}{2} \\ 0 & 0 \\ 0 & -s \end{bmatrix}.$$

Finally, a generalized Green's matrix function of problem (4.8)-(4.10) is given by

$$G(t,s) = \begin{cases} e^{t-s} \begin{bmatrix} 0 & \frac{st^2}{2}+st-\frac{t^2}{2}-t-1 \\ 0 & -st-s+t+1 \end{bmatrix} & 0 \le t < s \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{t-s} \begin{bmatrix} -1 & \frac{st^2}{2}+\frac{s^2}{2}-s-1 \\ 0 & -st \end{bmatrix} & s < t \le 1. \end{cases}$$

ACKNOWLEDGEMENTS.- This paper has been supported by the NATO grant CRG 900040.

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