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ON PAIRWISE SUPER CONTINUOUS MAPPINGS IN BITOPOLOGICAL SPACES

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ABSTRACT. The aim of this paper is to define and study super-continuous mappings and some other forms of continuity such as strong continuity, perfect continuity and complete continuity in bitopological spaces and investigate the relations between these kinds of continuity and their effects on some kinds of spaces.

KEY WORDS AND PHRASES: ij-Super-continuous mapping, ij-almost open mapping, ij-almost closed mapping, ij-almost continuous mapping, ij-δ-continuous mapping.
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1. INTRODUCTION.

The study of bitopological spaces was first initiated by J.C. Kelly [1] and thereafter a large number of papers have been done to generalize the topological concepts to bitopological setting. Munshi and Bassan [2] introduced a strong form of continuity, called super-continuous mappings and studied some of their properties in topological spaces. Noiri [3] obtained further properties of super-continuous mappings and investigated the relation between super-continuity and several strong forms of continuity such as δ-continuity and strongly θ-continuity which was introduced by Noiri [4], and completely continuity which was given by Arya and Gupta [5]. Bose and Sinha [6] defined almost continuity in bitopological spaces. Banerjee [7] defined δ-continuous and strong θ-continuous mapping in these spaces. They study these mappings and some of their results on different kinds of spaces such as nearly compact, regular, almost regular and semi-regular spaces. The purpose of this paper is to define super-continuous mappings in bitopological spaces and investigate some of their properties and relations with other forms of continuity and its effects on some kinds of spaces.

Throughout this paper, by a space X we mean a bitopological space (X, τ_1, τ_2). By $i - int\ A$ and $i - cl\ A$ we shall mean the interior and the closure of a subset A of X with respect to τ_i, respectively, where $i, j = 1$ or 2 and $i \neq j$.

A subset S of X is said to be $\underline{ij\text{-regular open}}$ (resp. $\underline{ij\text{-regular closed}}$) if $i - int(j - cl\ S) = S$ (resp. $i - cl(j - int\ S) = S$). S is said to be pariwise regular open (resp. pairwise regular closed) if it is both ij-regular open and ji-regular open (resp. ij-regular closed and ji -regular closed), denoted by $p - r.o$ (resp. $p - r.c$) [8]. A point x of X is said to be $\underline{ij - \delta\text{-cluster}}$ point of S if $S \cap U \neq \phi$ for every

$ij - r.o$ set U containing x. The set of all ij-δ cluster points of S is called $ij - \delta$-closure of S and is denoted by $ij - cl_\delta(S)$. A subset S of X is said to be $ij - \delta$-closed if $ij - \delta$-cluster points of $S \subset S$. The complement of $ij - \delta$-closed set is $ij - \delta$-open. So a set is $ij - \delta$-open if it is expressible as a union of $ij - r.o$ sets. S is said to be pairwise-δ-closed (resp. pairwise-δ-open) if it is both $ij - \delta$-closed and $ji - \delta$-closed (resp. $ij - \delta$-open, and $ji - \delta$-open) and will be denoted by $p - \delta$-closed (resp. $p - \delta$-open).

A bitopological space X is said to be ij-semi regular [8] (resp. ij-regular [1], ij- almost regular [9]) iff for each $x \in X$ and for each i-open set V of X, there is an i-open set U containing x such that $x \in U \subset i - int(j - cl\ U) \subset V$ (resp. $x \in U \subset j - cl\ U \subset V, x \in U \subset j - cl\ U \subset i - int(j - cl\ V)$). X is pairwise-semi regular (resp. pairwise-regular, pariwise almost regular) if it is both ij-semi regular and ji-semi regular) (resp. ij-regular and ji- regular, ij-almost regular and ji-almost regular).

A subset S of a bitopological space (X, τ_1, τ_2) is said to be ij-nearly compact relative to X [7] iff each i-open cover $\mathcal{U}$ of S has a finite subcollection $\mathcal{U}_0$ such that $S \subset \cup_{U \in \mathcal{U}_0} i - int(j - clU)$.

2. SUPER-CONTINUOUS MAPPINGS

DEFINITION 2.1. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be ij-super continuous at a point $x \in X$ if every i-neighborhood V of $f(x)$ there exists an i-neighborhood U of X such that $f(i - int(j - cl\ U)) \subset V$. A mapping f is said to be ij-super continuous [denoted by $ij - SC$] if it is ij-super continuous at each point of X and it is said to be pairwise SC if it is both $ij - SC$ and $ji - SC$.

REMARK 2.1. It is clear that if f is ij-super continuous at $x \in X$ then it is i-continuous at x. But the converse is not true as seen from the following example.

EXAMPLE 2.1. Let $X = Y = \{a, b, c\}$ and $\tau_1 = \{\phi, X, \{a\}, \{b, c\}, \{a, b\}, \{b\}\}$, $\tau_2 = \{\phi, X, \{b\}, \{b, c\}\}$, $\sigma_1 = \{\phi, Y, \{a\}, \{b, c\}\}, \sigma_2 = \{\phi, Y, \{b\}, \{b, c\}\}$, and let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2\}$ be given by $f(x) = x$ for each $x \in X$, then f is 1-continuous and 2-continuous but it is not 12-super continuous at $x = c$ since if $x = c$ then $f(x) = c$, let $V = \{b, c\}$, then there exists no 1-neighborhood U of c such that $f(1 - int(2 - cl\ U)) \subset V$.

THEOREM 2.1. For a mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ the following are equivalent:

(a) f is $ij - SC$.

(b) Inverse image of every i-open subset of Y is an $ij - \delta$-open subset of X.

(c) Inverse image of every i-closed subset of Y is an $ij - \delta$-closed subset of X.

(d) For each point x of X and each i-neighborhood V of $f(x)$ there exists an $ij - \delta$-open neighborhood U of x such that $f(U) \subset V$.

PROOF. (a) $\longrightarrow$ (b). Let A be any i-open subset of Y and $x \in f^{-1}(A)$. Then $f(x) \in A$ and so there exists an i-neighborhood U of x such that $x \in U$ and $f(i - int(j - cl\ U)) \subset A$. So, $x \in i - int(j - cl\ U) = V \subset f^{-1}(A)$. But V is $ij - r.o$, so $f^{-1}(A)$ is expressible as an arbitrary union of ij-regular open sets, hence $f^{-1}(A)$ is $ij - \delta$-open.

(b) $\longrightarrow$ (c). Obvious.

(c) $\longrightarrow$ (d). Let V be an i-neighborhood of $f(x)$ so $Y \sim V$ is i-closed by (c), $f^{-1}(Y \sim V)$ is $ij - \delta$-closed, hence $f^{-1}(V)$ is $ij - \delta$-open; so we have $x \in f^{-1}(V) = U$ and $f(U) \subset V$, where U is $ij - \delta$-open.

(d) $\longrightarrow$ (a). For each $x \in X$ and each i-open neighborhood V of $f(x)$ there exists an $ij - \delta$-open neighborhood U of x such that $f(U) \subset V$. But U is $ij - \delta$-open, so there exists an $ij - r.o$ set 0 such that $x \in 0 \subset U$, hence $f(x) \subset f(0) = f(i - int(j - cl\ 0)) \subset f(U) \subset V$. So f is $ij - SC$.

THEOREM 2.2. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be an i-continuous mapping of an ij-semi regular space X into Y, then f is $ij - SC$.

PROOF. Let G be an i-neighborhood of $f(x)$, so $f^{-1}(G)$ is an i-neighborhood of x. Since X is ij-semi regular, there exists an i-open set V such that $x \in V \subset i - int(j - cl\ V) \subset f^{-1}(G)$. So $f(i - int(j - cl\ V)) \subset G$ and hence f is $ij - SC$.

THEOREM 2.3. Let X and Y be bitopological spaces, then a mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $ij - SC$ iff the inverse image under f of every member of i-base for Y is $ij - \delta$-open.

PROOF. From Theorem 2.1 parts (a) and (b).

THEOREM 2.4. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $ij - SC$ iff $f(ij - cl_\delta(A)) \subset i - cl(f(A))$, for each subset A of X.

PROOF. Let f be $ij - SC$, since $i - cl\ f(A)$ is an i-closed subset of Y, then $f^{-1}(i - cl\ f(A))$ is $ij - \delta$-closed in X. But since $f(A) \subset i - cl\ f(A)$, then $A \subset f^{-1}(i - cl\ f(A))$ and so $ij - cl_\delta A \subset ij - cl_\delta(f^{-1}(i - cl\ f(A))) = f^{-1}(i - cl\ f(A))$. Hence $f(ij - cl_\delta A) \subset i - cl\ f(A)$.

Conversely, let $f(ij - cl_\delta A) \subset i - cl\ f(A)$ for each $A \subset X$ and let F be any i-closed subset of Y, so $i - cl\ F = F$. Since $f^{-1}(F) \subset X$, so $f(ij - cl_\delta f^{-1}(F)) \subset i - cl\ f\ f^{-1}(F) \subset i - cl\ F = F$ and $ij - cl_\delta f^{-1}(F) \subset f^{-1}(F)$. Then $f^{-1}(F)$ is $ij - \delta$-closed and by Theorem 2.1 f is $ij - SC$.

THEOREM 2.5. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $ij - SC$ iff $ij - cl_\delta f^{-1}(B) \subset f^{-1}(i - cl\ B)$ for each $B \subset Y$.

PROOF: Let f be $ij - SC$. Since $i - cl\ B$ is i-closed subset of Y, then $f^{-1}(i - cl\ B)$ is $ij - \delta$ closed in X and since $B \subset i - cl\ B$; then $ij - cl_\delta f^{-1}(B) \subset ij - cl_\delta f^{-1}(i - cl\ B) = f^{-1}(i - cl\ B)$ and so $ij - cl_\delta f^{-1}(B) \subset f^{-1}(i - cl\ B)$.

Conversely, let $ij - cl_\delta f^{-1}(B) \subset f^{-1}(i - cl\ B)$ for each $B \subset Y$ and let F be an i-closed subset of Y. Then $ij - cl_\delta f^{-1}(F) \subset f^{-1}(i - cl\ F) = f^{-1}(F)$, but since $f^{-1}(F) \subset ij - cl_\delta f^{-1}(F)$, so $f^{-1}(F) = ij - cl_\delta f^{-1}(F)$ and hence $f^{-1}(F)$ is $ij - \delta$-closed. Therefore f is $ij - SC$.

DEFINITION 2.2. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be ij-almost open if the image of every ij-regular open subset of X is i-open in Y. f is said to be ij-almost closed if the image of every ij-regular closed subset of X is i-closed in Y.

DEFINITION 2.3. [6] A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be ij-almost continuous at a point $x \in X$ if every i-neighborhood V of $f(x)$ there exists an i-neighborhood U of x such that $f(U) \subset i - int(j - cl\ V)$.

THEOREM 2.6. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is ij-almost open and $ij - SC$ mapping of X onto Y and if $g : (Y, \sigma_1, \sigma_2) \longrightarrow (Z, \gamma_1, \gamma_2)$ is a mapping of Y into Z, then $g\ o\ f$ is $ij - SC$ iff g is i-continuous.

PROOF. Let f be an ij-almost open $ij - SC$ mapping, and let g be i-continuous. Let U be an i-closed subset of Z, consider $f^{-1}(g^{-1}(U))$, since U is i-closed in Z and g is i-continuous, so $g^{-1}(U)$ is i-closed in Y. Also, since f is $ij - SC$, so $f^{-1}(g^{-1}(U))$ is $ij - \delta$-closed subset of X, hence $f\ o\ g$ is $ij - SC$.

Conversely, let $g\ o\ f$ be $ij - SC$, then for each i-open subset G of Z $(g\ o\ f)^{-1}(G)$ is $ij - \delta$-open subset of X. Since f is ij-almost open and onto, then $f[f^{-1}(g^{-1}(G)] = (g\ o\ f)^{-1}(G)] = g^{-1}(G)$ is i-open subset of Y, hence g is i-continuous.

THEOREM 2.7. Let X, Y, and Z be bitopological spaces and let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be ij-almost continuous and $g : (Y, \sigma_1, \sigma_2) \longrightarrow (Z, \gamma_1, \gamma_2)$ be $ij - SC$, then $g o f : (X, \tau_1, \tau_2) \longrightarrow (Z, \gamma_1, \gamma_2)$ is i-continuous.

PROOF: Let $x \in X$, so $f(x) \in Y$, consider $(g\ o\ f)(x)$. Let U be an i-neighborhood of $(g\ o\ f)(x)$, since g is $ij - SC$ so $g^{-1}(U)$ is $ij - \delta$-open subset of Y, so there exists an $ij - r.o$ subset V of Y such that $f(x) \in V \subset g^{-1}(U)$, but since f is ij-almost continuous, then there exists an i-neighborhood N of x such that $f(N) \subset V \subset g^{-1}(U)$. Then $f^{-1}(f(N)) \subset f^{-1}(g^{-1}(U))$ and $N \subset f^{-1}(g^{-1}(U))$; so

$(g\ o\ f)^{-1}(U)$ is an i-open subset f X, hence $g\ o\ f$ is i-continuous.

REMARK 2.2. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is ij-almost continuous and $g\ o\ f : (X, \tau_1, \tau_2) \longrightarrow (Z, \gamma_1, \gamma_2)$ is i-continuous, then $g : (Y, \sigma_1, \sigma_2) \longrightarrow (Z, \gamma_1, \gamma_2)$ need not be $ij - SC$as seen from the following exmaple.

EXAMPLE 2.2. Let $X = R$ with $\tau_1 = \{\phi, R$, complement of countable subsets of $R\}$. $\tau_2 = \{\phi, R\}$, and $Y = \{a, b\}$ with $\sigma_1 = \{Y, \phi, \{a\}\}, \sigma_2 = \{Y, \phi\}$, and $Z = \{1, 2\}$ with $\gamma_1 = \{Z, \phi, \{2\}\}$, $\gamma_2 = \{Z, \phi, \{1\}\}$, let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ defined by

$$\begin{aligned} f(x) &= a && \text{if } x \text{ is irrational} \\ &= b && \text{if } x \text{ is rational} \end{aligned}$$

and $g : (Y, \sigma_1, \sigma_2) \longrightarrow (Z, \gamma_1, \gamma_2)$ defined by $g(a) = 2,\ g(b) = 1$, then f is 12-almost continuous and $g\ o\ f$ is 1-continuous but g is not $12 - SC$, since if $x = a$, then $g(a) = 2$, let $V = \{2\}$ then there is no 1-neighborhood U of $\{a\}$ such that $g(1 - int(2 - cl\, U)) \subset V$.

DEFINITION 2.4. [7] A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be <u>$ij - \delta$-continuous</u> if for each x in X and each i-neighborhood V of $f(x)$ there exists an i-neighborhood U of x such that $f(i - int(j - cl\ U)) \subset i - int(j - cl\ V)$.

REMARK 2.3. For a mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, we have the following implications:

$$\begin{tikzcd} & i\text{-continuous} \arrow[dr] & \\ ij - SC \arrow[ur] \arrow[dr] & & ij\text{-almost continuous} \\ & ij - \delta\text{-continuous} \arrow[ur] & \end{tikzcd}$$

The converse may not be true as seen from Example 2.1 in [6] and Example 2.1.

THEOREM 2.8. Let X and Y be ij-semi regular spaces, then for a mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ the following properties are equivalent:

(a) $ij - SC$

(b) i-continuous

(c) $ij - \delta$-continuous

(d) ij-almost continuous

PROOF. We shall only prove that (d) $\longrightarrow$ (a). Let f be ij-almost continuous and let $x \in X$ and $y = f(x)$, since Y ij-semi regular, for every i-open neighborhood V of y there exists an i-open neighborhood V' of y such that $f(x) = y \in V' \subset (i - int(j - cl\ V')) \subset V$. Since f is ij-almost continuous, there exists an i-open neighborhood U of x such that $f(U) \subset i - int(j - cl\ V') \subset V$. By ij-semi regularity of X there exists an i-open neighborhood U' of x such that $x \in U' \subset i - int(j - cl\ U') \subset U$. Hence we have $f(i - int(j - cl\ U')) \subset V$ and so f is $ij - SC$.

THEOREM 2.9. Let $f : (A, P_1, P_2) \longrightarrow (X \times Y, Q_1, Q_2)$ be given by the equation $f(a) = (f_1(a), f_2(a))$ for every a in A, then f is $ij - SC$ iff $f_1 : (A, P_1, P_2) \longrightarrow (X, \tau_1, \tau_2)$ and $f_2 : (A, P_1, P_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ are $ij - SC$.

PROOF. Let f_1, f_2 be $ij - SC$, let $a \in A$ and $Q = U_1 \times U_2$ be an i-open subset of $X \times Y$ such that $f(a) = (f_1(a), f_2(a)) \in U_1 \times U_2$, so $f_1(a) \in U_1, f_2(a) \in U_2$, where U_1 is an i-open subset of X and U_2 is an i-open subset of Y, but since f_1, f_2 are $ij - SC$ so there exists $ij - r.o$ sets U_1' and U_2' in A such that $f_1(U_1') \subset U_1, f_2(U_2') \subset U_2$. Put $U' = U_1' \cap U_2', U'$ is $ij - r.o$ and $f(U') \subset (f_1(U_1'), f_2(U_2')) \subset U_1 \times U_2$. Hence f is $ij - SC$.

Conversely, let f be $ij-SC$ and let $a \in A$ and U_l be an i-open subset of X containing $f_1(a)$, then $U_l \times Y$ is an i-open subset of $X \times Y$ containing $f(a)$.

Since f is $ij-SC$, there exists an $ij-r.o$ subset V of A containing a such that $f(V) \subset U_l \times Y$ and so $(f_1(V), f_2(V)) \subset U_l \times Y$.

Then $f_1 \subset U_l$ and therefore f_1 is $ij-SC$. In a similar way we can prove that f_2 is $ij-SC$.

THEOREM 2.10. Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be a mapping and $g:(X,\tau_1,\tau_2) \longrightarrow (X \times Y, P_1, P_2)$ given by $g(x)=(x,f(x))$ for all x in X, be the graph mapping, then g is $ij-SC$ iff f is $ij-SC$ and X is ij-semi regular.

PROOF. Let g be $ij-SC$ and U be an i-open subset of X containing x, so $U \times Y$ is an i-open subset of $X \times Y$ containing $g(x)$. Since g is $ij-SC$, there exists an $ij-r.o$ subset W of X containing x such that $g(W) \subset U \times Y$. Then $x \in W \subset g^{-1}(U \times Y) \subset U$ and therefore X is ij-semi regular from Theorem 2.9 since g is $ij-SC$ and $g(x)=(x,f(x))$, so f is $ij-SC$.

Conversely, let f be $ij-SC$ and X be ij-semi regular. For each x in X, and each i-neighborhood W of $g(x)$, there exists an i-neighborhood U of x and an i-neighborhood V of $f(x)$ such that $(x,f(x)) \in U \times V \subset W$.

Since X is ij-semi regular, there exists an $ij-r.o$ subset G_1 of X such that $x \in G_1 \subset U$. Since f is $ij-SC$, there exists an $ij-r.o$ subset G_2 of X such that $x \in G_2$ and $f(G_2) \subset V$.

Let $G = G_1 \cap G_2$. Then G is an $ij-r.o$ subset of X and $g(G) \subset U \times V \subset W$. Hence g is $ij-SC$.

3. ij-NEARLY COMPACT SPACE AND ij-SUPER CONTINUOUS MAPPINGS.

THEOREM 3.1. A bitopological space X is ij-nearly compact iff every $ij-\delta$-open cover of X has a finite subcover.

PROOF. Let X be ij-nearly compact and let $\mathcal{U} = \cup\{U_\alpha | \alpha \in \Delta\}$ be an $ij-\delta$-open cover of X. For each $U_\alpha \in \mathcal{U}$ and $x \in U_\alpha$ there exists an i-open set V_α such that $x \in V_\alpha \subset i-int(j-cl\ V_\alpha) \subset U_\alpha$. Then $\{V_\alpha | \alpha \in \Delta\}$ is an i-open cover for X, so there is a finite subset Δ_0 such that $X \subset \cup_{\alpha\in\Delta_0} i-int(j-cl\ V) \subset \cup_{\alpha\in\Delta_0} U_\alpha$. So $\mathcal{U}$ has finite subcollection which covers X.

Coversely, let $\mathcal{U} = \cup\{U_\alpha | \alpha \in \Delta\}$ be an i-open cover of X. Since $U_\alpha \subset i-int(j-cl\ U_\alpha)$, therefore $X \subset \cup_{\alpha\in\Delta} U_\alpha \subset \cup_{\alpha\in\Delta} i-int(j-cl\ U_\alpha)$. But $\cup_{\alpha\in\Delta} i-int(j-cl\ U_\alpha)$ is an $ij-\delta$ open cover of X, so there exists a finite subset Δ_0 of Δ such that $X \subset \cup_{\alpha\in\Delta_0} i-int(j-cl\ U_\alpha)$. Hence X is ij-nearly compact.

COROLLARY 3.1. Any ij-regular closed subset of an ij-nearly compact space is ij-nearly compact.

PROOF. Obvious.

THEOREM 3.2, Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be an $ij-SC$ mapping of an ij-nearly compact space X into a bitopological space Y, then $f(X)$ is i-compact.

PROOF. Let $\{O_\alpha\}$ be any i-open cover of $f(X)$. Since f is $ij-SC$, so $f^{-1}(O_\alpha)$ is $ij-\delta$-open in X and $\cup f^{-1}(O_\alpha)$ is $ij-\delta$-open cover of X, hence $X = \cup_{i=1}^n f^{-1}(O_{\alpha_i}) = f^{-1}(\cup_{i=1}^n O_{\alpha_i})$. So, $f(X) = ff^{-1}(\cup_{i=1}^n O_\alpha) \subset \cup_{i=1}^n O_\alpha$, hence $f(X)$ is i-compact.

DEFINITION 3.1. A subset K of X is called <u>$ij-H$ set</u> if for each cover $\{U_\alpha : \alpha \in \nabla\}$ of K by i-open subsets of X there exists a finite subset ∇_0 of ∇ such that $K \subset \cup\{j-cl\ U_\alpha | \alpha \in \nabla_0\}$.

DEFINITION 3.2. [1] A bitopological space (X,τ_1,τ_2) is called <u>ij-Hausdorff</u> iff for each pair of distinct points x,y of X, there are an i-open neighborhood A of x and a j-open neighborhood B of y such that $A \cap B = \phi$.

THEOREM 3.3. For a mapping $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ let:

(a) f is $ij-SC$ and Y is ij-Hausdorff.

(b) for every $(x,y) \notin G(f)$ there exists an $ij-r.o$ subset U of X and $ji-r.o$ subset V of Y containing x and y, respectively, such that $f(U) \cap i-cl\ V = \phi$ (where $G(f)$ is the graph of f).

(c) $f^{-1}(K)$ is $ij-\delta$-closed in X for every $ji-H$ set K of Y. Then (a) $\longrightarrow$ (b) $\longrightarrow$ (c).

PROOF. (a) $\longrightarrow$ (b) Let $(x,y) \notin G(f)$, so $y \neq f(x)$ for $x \in X$. Since Y is ij-Hausdorff, there exists an $ij-r.o$ subset U and $ji-r.o$ subset V of Y such that $f(x) \in U, y \in V$ and $U \cap i-clV = \phi$. But f is $ij-SC$, so there exists an $ij-r.o$ subset U' containing x such that $f(U)' \subset U$; hence $f(U') \cap i-cl\ V = \phi$.

(b) $\longrightarrow$ (c) Let K be a $ji-H$ set of Y and $x \notin f^{-1}(K)$, for every $y \in K$, $y \neq f(x)$ and $(x,y) \notin G(f)$, there exists an $ij-r.o$ subset U_y and an $ji-r.o$ subset V_y such that $x \in U_y$ and $y \in V_y$ and $f(U_y) \cap i-cl\ V_y = \phi$. The family $\{V_y | y \in K\}$ is a j-open cover of K. Since K is a $ji-H$ set, there exists a finite subset K_0 of K such that $K \subset \cup\{i-cl\ V_y | y \in K_0\}$. Let $U = \cap\{U_y | y \in K_0\}$; so U is $ij-r.o$ subset containing x and $f(U) \cap K = \phi$. Then $U \cap f^{-1}(K) = \phi$, and $f^{-1}(K)$ is $ij-\delta$-closed in X.

4. STRONG FORMS OF CONTINUITY.

DEFINITION 4.1. A mapping $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is said to be j-strongly continuous if $f(j-cl\ A) \subset f(A)$ for every subset A of X.

COROLLARY 4.1. A function f is j-strongly continuous iff $f^{-1}(B)$ is both j-open and j-closed in X for every subset B of Y.

PROOF. Let $B \subset Y$ and let $f^{-1}(B) = F \subset X$, since f is j-strongly continuous, $f(j-cl\ F) \subset f(F)$; hence $j-clF \subset F$; so F is j-closed. Similarly $X \sim F \subset X$; hence $X \sim F$ is j-closed, so F is j-open.

DEFINITION 4.2. A mapping $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is said to be ij-clopen continuous (resp., ij-strongly θ-continuous [7]) if for every $x \in X$ and for every i-open neighborhood V of $f(x)$ there exists a j-closed and i-open neighborhood U of x (resp. i-open) such that $f(U) \subset V$ (resp. $f(j-cl\ U) \subset V$).

DEFINITION 4.3. A mapping $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is said to be ij-perfectly continuous (resp. ij-completely continuous) if $f^{-1}(V)$ is both j-closed and j-open (resp. $ij-r.o$) in X for each i-open subset V of Y.

DEFINITION 4.4. A mapping $f : (X,\tau_1,\tau_2 \longrightarrow (Y,\sigma_1,\sigma_2)$ is said to be an $ij-R$ map if $f^{-1}(V)$ is $ij-r.o$ for every $ij-r.o$ subset V of Y.

REMARK 4.1. It is obvious that ij-clopen continuous $\longrightarrow$ ij-strongly θ- continuous $\longrightarrow$ $ij-SC$ but the converse may not be true as can be seen from the follwoing example:

EXAMPLE 4.1. Let $X = Y = \{a,b,c,d\}$ and let $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,c\}\}$, $\sigma_1 = \{\phi, X, \{a\}, \{a,c\}, \{a,c,d\}\}$, $\sigma_2 = \{\phi, X, \{b\}, \{b,d\}\}$. Let $f:(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be defined by $f(a) = f(b) = a$, $f(c) = f(d) = b$. Then f is $12-SC$ but it is not 12-strongly θ-continuous.

THEOREM 4.1. If $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is $ij-SC$ and X is ij-almost regular, then f is ij-strongly θ-continuous.

PROOF. Let $x \in X$ and let V be an i-open neighborhood of $f(x)$. Then there exists an $ij-r.o$ neighborhood U of X such that $f(U) \subset V$. Since X is ij-almost regular, there exists an i-open neighborhood U' of x such that $x \in U' \subset j-cl\ U' \subset U$. Then $f(j-cl\ U') \subset f(U) \subset V$ and so f is ij-strongly θ-continuous.

COROLLARY 4.2. If X is ij-regular, then for a mapping $f : (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ the following properties are equivalent:

(a) f is ij-strongly θ-continuous.

(b) f is $ij-SC$.

(c) f is i-continuous.

PROOF. Follows from Theorems 2.2 and 4.1.

COROLLARY 4.3. If X and Y are ij-regular, then for a mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the properties: ij-strongly θ-continuous, $ij - SC$, $ij - \delta$-continuous, $i-$continuous and ij-almost continuous are equivalent.

PROOF. From Theorem 2.8 and Corollary 4.2.

THEOREM 4.2. For a mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ the following implications hold: j-strongly continuous $\longrightarrow$ ij-perfectly continuous and ij-completely continuous $\longrightarrow$ $ij - R$ map $\longrightarrow$ $ij - \delta$-continuous.

PROOF. If f is j-strongly continuous, then from Corlllary 4.1 $f^{-1}(B)$ is both j-open and j-closed for every $B \subset Y$, hence f is ij-perfectly continuous. Also, if f is ij-completely continuous, let V be $ij - r.o$ subset of Y, then V is i-open and $f^{-1}(V)$ is $ij - r.o$. So f is an $ij - R$ map.

Finally, if f is an $ij - R$ map, then $f^{-1}(V)$ is $ij - r.o$ in X for every $ij - r.o$ subset V of Y, let $f^{-1}(V) = U$, since $f(f^{-1}(V)) \subset V$ so $f(U) \subset V$. Hence f is $ij - \delta$-continuous.

REMARK 4.2. The converse of the above implications may not be true as seen from the following example:

Let $X = Y = \{a, b, c\}$ and let $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\phi, X\}$, $\sigma_1 = \{\phi, X\}$, $\sigma_2 = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be the identity map. Then f is 12-perfectly continuous but it is not 2-strongly continuous.

THEOREM 4.3. If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is ij-completely continuous, then it is $ij - SC$.

PROOF. Let V be an i-open subset of of Y and let $U = f^{-1}(V)$. Since f is ij-completely continuous, so $f^{-1}(V)$ is $ij - r.o$ and $f(f^{-1}(V)) \subset V$. Hence $f(U) = f(i - int(j - cl\, U)) \subset V$. So f is $ij - SC$.

REMARK 4.3. The converse of the above theorem may not be true as shown in the following example.

EXAMPLE 4.3. Consider Exmaple 4.1 , f is $ij - SC$ but it is not ij-completely continuous.

The following diagram gives us the relations between these kinds of continuity.

$$\begin{tikzcd}
ij - COC \arrow[r] & ij - S\theta C \arrow[r] & ij - SC \arrow[r] \arrow[d] & i - C \arrow[d] \\
ij - CC \arrow[r] \arrow[urr] & ij - R \arrow[r] & ij - \delta - C \arrow[r] & ij - aC
\end{tikzcd}$$

also: $j - StC \longrightarrow ij - PC$.

where

$ij - COC = ij$-clopen continuous, $ij - S\theta C = ij$-strongly θ-continuous

$i- C=i$-continuous, $ij-CC=ij$-completely continuous, $ij-R= ij-R$ map

$ij - \delta C = ij - \delta$-continuous, $ij - aC = ij$-almost continuous,

$j - StC = j$-strongly continuous, $ij - PC = ij$-perfectly continuous.

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