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ON A GENERALIZATION OF u-MEANS

FRANCOIS DUBEAU

Département de mathématiques
Collège militaire royal de Saint-Jean
Saint-Jean-sur-Richelieu, Québec, Canada, JOJ 1RO

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ABSTRACT. In this paper we present an extension of Bauer's work about u-means. We consider a kind of composition of an admissible function $u(x)$ (described by Bauer) and of a compatible function $\phi(x)$. This construction allows us to define (u,ϕ)-means. When $\phi(x) = x$, the (u,ϕ)-means are the u-means introduced by Bauer. The arithmetic, geometric and harmonic means are special cases.

KEY WORDS AND PHRASES. Means, u-means, generalized u-means.
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1. INTRODUCTION.

In [2] Bauer introduced a class of admissible functions. To each function $u(x)$ in this class it was possible to associate a u-mean. The arithmetic and geometric means were special cases of u-means but not the harmonic mean.

In this paper we introduce a class of monotone compatible functions. We consider a kind of composition of an admissible function $u(x)$ and a monotone function $\phi(x)$ compatible with respect to $u(x)$ which permits the definition of (u,ϕ)-means. When $\phi(x) = x$ the (u,ϕ)-means are the u-means of Bauer. The arithmetic, geometric and harmonic means are special cases.

2. CONTRACTIVE INTERVAL.

In this paper we consider intervals $I \subset [0, +\infty[$ of the following type

(i) $]0, +\infty[$,

(ii) $]0, \alpha]$ or $]0, \alpha[$ for $0 < \alpha \leq 1$,

(iii) $[\beta, +\infty[$ or $]\beta, +\infty[$ for $1 \leq \beta < +\infty$.

Any interval I of this type is said to be *contractive* because for any $n \in N = \{1, 2, 3, \cdots\}$ and $x \in I$ we have $x^n \in I$, or equivalently, for any $x, y \in I$ we have $xy \in I$ (see [2]).

3. CLASSES OF FUNCTIONS.

The first class of functions we consider is the class of admissible functions introduced by Bauer [2].

A strictly positive continuous function $u(x)$ defined on a contractive interval I_u is said *admissible* (of type (A) or (B)) if it satisfies one of the following conditions:

(A) $x \to u(x)$ is decreasing,

(B) $x \to u(x)/x$ is strictly increasing.

EXAMPLE 1. $u(x) = x^p$ for $p \leq 0$ or $p > 1$ are admissible functions on I_u. The function $u(x) = \sqrt{1 - x^2}$ is admissible on $I_u =]0, 1[$. The function $u(x) = e^{x-1}$ is admissible on $[1, +\infty[$.

To extend the work of Bauer we introduce the following class of functions. A strictly positive strictly monotone continuous function $\phi(x)$ defined on a contractive interval I_ϕ is said *compatible* if it satisfies the following condition:

$$x \to \phi(\alpha x)/\phi(x) \text{ is monotone (as } \phi(x)) \text{ for any } \alpha \in I_\phi.$$

Let us consider the following examples.

EXAMPLE 2. $\phi(x) = x^p$ is strictly increasing for $p > 0$ and strictly decreasing for $p < 0$. Also $\phi(\alpha x)/\phi(x) = \alpha^p$, a constant for any fixed $\alpha \in I_\phi$. Note that in this case $\phi(x^n) = [\phi(x)]^n$.

EXAMPLE 3. $\phi(x) = e^x$ and $I_\phi = [1, +\infty[$. The function $\phi(x)$ is a strictly increasing continuous function such that $\phi(\alpha x)/\phi(x) = e^{(\alpha - 1)x}$ which is an increasing function for any fixed $\alpha \in I_\phi$.

EXAMPLE 4. $\phi(x) = e^{-\frac{1}{x}}$ and $I_\phi =]0, 1]$. The function $\phi(x)$ is a strictly increasing function on I_ϕ such that $\phi(\alpha x)/\phi(x) = e^{-(\frac{1-\alpha}{\alpha x})}$ which is an increasing function for any fixed $\alpha \in I_\phi$.

The following preliminary results will be useful in the next section.

LEMMA 1. Let $\phi(x)$ be a compatible function. Then

(i) $x \to \phi(x^n)/\phi(x)$ is strictly monotone (as $\phi(x)$) for any integer $n \in N$,

(ii) $x \to \phi(\alpha^n x)/\phi(x)$ is monotone (as $\phi(x)$) for any $n \in N$ and any fixed $\alpha \in I_\phi$.

PROOF. Let us assume first that $\phi(x)$ is strictly increasing. To prove (i) consider $x < y$, then $x^n < x^{n-1}y < xy^{n-1} < y^n$. Hence $\phi(x^n)/\phi(x) < \phi(xy^{n-1})/\phi(x) \le \phi(y^n)/\phi(y)$ because $\phi(x)$ is strictly increasing and compatible. To prove (ii) replace α by α^n in the definition. The proof is almost the same when $\phi(x)$ is strictly decreasing.

LEMMA 2. Let $u(x)$ be an admissible function and $\phi(x)$ be a compatible function. If $\phi(I(\phi)) \subset I_u$, then for any $n \in N$ the function $x \to \psi_{n+1}(x) = uo\phi(x^n)/\phi(x)$ is strictly monotone (here $uo\phi(x) = u(\phi(x))$). The different cases are summarized in the following table:

type of $u(x)$	$\phi(x)$ strictly monotone	$\psi_{n+1}(x)$ strictly monotone
(A)	increasing	decreasing
	decreasing	increasing
(B)	increasing	increasing
	decreasing	decreasing

PROOF. Let us assume that $\phi(x)$ is strictly increasing (decreasing). If $u(x)$ is of type (A) then $uo\phi(x)$ is decreasing (increasing). Also $1/\phi(x)$ is strictly decreasing (increasing). It follows that $uo\phi(x^n)/\phi(x)$ is strictly decreasing (increasing). If $u(x)$ is of type (B) then $uo\phi(x)/\phi(x)$ is strictly increasing (decreasing). From Lemma 1, $\phi(x^n)/\phi(x)$ is strictly increasing (decreasing). The result follows from $uo\phi(x^n)/\phi(x) = [uo\phi(x^n)/\phi(x^n)]\,[\phi(x^n)/\phi(x)]$

4. (u, ϕ)-MEANS. Let $u(x)$ be an admissible function and $\phi(x)$ be a compatible function such that $\phi(I_\phi) \subset I_u$. Let $n \ge 2$ and choose any $\vec{a} = (a_1, a_2, \cdots, a_n) \in I_\phi^n = I_\phi \times \cdots \times I_\phi$. We consider

$$S_{(u,\phi)}(\vec{a}) = \frac{\sum_{i=1}^{n} uo\phi(\pi_i(\vec{a}))}{\sum_{i=1}^{n} \phi(a_i)}$$

where

$$\pi_i(\vec{a}) = \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j = \left(\prod_{j=1}^{n} a_j\right)/a_i.$$

Using now the continuity of the functions and the strict monotonicity of $\psi_n(x) = uo\phi(x^{n-1})/\phi(x)$, we can prove the following result (which is a generalization of Theorem 2.1 of Bauer).

THEOREM 3. Let $u(x)$ be an admissible function defined on I_u and $\phi(x)$ a compatible function defined on I_ϕ such that $\phi(I_\phi) \subset I_u$. Let $n \geq 2$ and $\vec{a} = (a_1, \cdots, a_n) \in I_\phi^n$. Then the equation

$$\psi_n(x) = S_{(u,\,\phi)}(\vec{a}) \tag{4.1}$$

has exactly one solution in I_ϕ. It lies in the interval

$$]\alpha, \beta[\text{ if } \alpha = \min\{a_1, \cdots, a_n\} < \max\{a_1, \cdots, a_n\} = \beta$$

and is equal to α if $\alpha = \beta$.

NOTE. With the assumptions made on $\phi(x)$ and the preceding two lemmas, the proof of this theorem is almost identical to the proof of Theorem 2.1 of Bauer.

PROOF. If $\alpha = \beta$ the result follows from Lemma 2. Let us assume that $\alpha = \beta$ and let us consider the following two cases:

(i) *$u(x)$ is of type* (A) *and $\phi(x)$ is strictly decreasing.* In this case $uo\phi(x)$ is increasing. For any $i = 1, \cdots, n$ we have $\alpha^{n-1} \leq \pi_i(\vec{a}) \leq \beta^{n-1}$ and it follows that $uo\phi(\alpha^{n-1}) \leq uo\phi(\pi_i(\vec{a})) \leq uo\phi(\beta^{n-1})$. Also $1/\phi(x)$ is strictly increasing, then we have $\phi(a_i)/\phi(\alpha) \leq 1 \leq \phi(a_i)/\phi(\beta)$ with strict inequality for at least one i (not necessarily the same i for both inequalities). It follows that

$$\phi(a_i)\, \psi_n(\alpha) \leq uo\phi(\pi_i(\vec{a})) \leq \phi(a_i)\, \psi_n(\beta) \tag{4.2}$$

for $i = 1, \cdots, n$.

(ii) *$u(x)$ is of type* (B) *and $\phi(x)$ is strictly increasing.* In this case $uo\phi(x)/\phi(x)$ is strictly increasing and we have

$$\frac{uo\phi(\alpha^{n-1})}{\phi(\alpha^{n-1})} \leq \frac{uo\phi(\pi_i(\vec{a}))}{\phi(\pi_i(\vec{a}))} \leq \frac{uo\phi(\beta^{n-1})}{\phi(\beta^{n-1})}$$

and again with strict inequality for at least one i. We also have

$$\phi(\alpha^{n-2}\, a_{i+1}) \leq \phi(\pi_i(\vec{a})) \leq \phi(\beta^{n-2}\, a_{i+1})$$

for $i = 1, \cdots, n$ (where $a_{n+1} \equiv a_1$). From Lemma 1 we have

$$\phi(\alpha^{n-1})/\phi(\alpha) \leq \phi(\alpha^{n-2}\, a_{i+1})/\phi(a_{i+1}) \text{ and } \phi(\beta^{n-2}\, a_{i+1})/\phi(a_{i+1}) \leq \phi(\beta^{n-1})/\phi(\beta).$$

It follows that

$$\phi(a_{i+1})\, \psi_n(\alpha) \leq uo\phi(\pi_i(\vec{a})) \leq \phi(a_{i+1})\, \psi_n(\beta). \tag{4.3}$$

for $i = 1, \cdots, n$.

By adding up (4.2) or (4.3) for $i = 1, \cdots, n$ it follows that $\psi_n(\alpha) < S_{(u,\,\phi)}(\vec{a}) < \psi_n(\beta)$ and the result follows from the continuity and the strict monotonicity of $\psi_n(x)$.

For the other cases we obtain reverse inequalities and the result follows again.

Under the assumptions of Theorem 3, the (u, ϕ)-*mean of the n numbers* $a_1, \cdots, a_n$ taken in I_ϕ will be the unique solution of (4.1) and will be denoted $M_{(u,\,\phi)}(\vec{a})$. If $n = 1$ we put $M_{(u,\,\phi)}(a_1) = a_1$.

REMARK 1. The u-means introduced by Bauer, denoted $M_u(\vec{a})$, are obtained when $\phi(x)$ is the identity function $\mathrm{id}(x)$, i.e. $\phi(x) = \mathrm{id}(x) = x$ for any $x \in I_\phi$, and we have $M_{(u,\,id)}(\vec{a}) = M_u(\vec{a})$.

REMARK 2. For $u(x) = 1$ we have

$$\phi(M_{(1,\phi)}(\vec{a})) = A(\phi(\vec{a})) \tag{4.4}$$

where $\phi(\vec{a})$ denotes the vector $(\phi(a_1),\cdots,\phi(a_n))$ and $A(\vec{v})$ is the arithmetic mean of the n components of the vector $\vec{v} = (v_1,\cdots,v_n)$.

EXAMPLE 5. Consider $u(x) = 1$. For $\phi(x) = \text{id}(x) = x$ we obtain

$$M_{(1,\,\text{id})}(\vec{a}) = A(\vec{a})$$

and for $\phi(x) = 1/x = (1/\text{id}\ (x)$ since

$$H(a_1,\cdots,a_n)^{-1} = A(a_1^{-1},\cdots,a_n^{-1})$$

it follows from (4.4) that

$$M_{(1,\,1/\text{id})}(\vec{a}) = H(\vec{a})$$

where $H(\vec{a})$ denotes the harmonic mean of the n components of $\vec{a}$. Let us note that it is not possible to obtain the harmonic mean as a u-means (see [1], [2]).

REMARK 3. More generally, if the function $\phi(x)$ is such that

$$\phi\left(\prod_{i=1}^{n} a_i\right) = \prod_{i=1}^{n} \phi(a_i)$$

then we have

$$S_{(u,\,\phi)}(\vec{a}) = S_{(u,\,\text{id})}(\phi(\vec{a}))$$

and it follows that

$$\phi(M_{(u,\,\phi)}(\vec{a})) = M_{(u,\,\text{id})}(\phi(a)) = M_u(\phi(\vec{a})). \tag{4.5}$$

EXAMPLE 6. For $u(x) = 1/x = (1/\text{id})\ (x)$, if $\phi(x) = \text{id}(x)$ we obtain

$$M_{(1/\text{id},\,\text{id})}(\vec{a}) = G(\vec{a})$$

where $G(\vec{a})$ is the geometric mean of the n components of $\vec{a}$. If $\phi(x) = 1/x = (1/\text{id})\ (x)$ it follows from (4.5) that

$$M_{(1/\text{id},\,1/\text{id})}(\vec{a}) = G(\vec{a})$$

because $G(a_1^{-1},\cdots,a_n^{-1}) = G(a_1,\cdots,a_n)^{-1}$.

EXAMPLE 7. More generally if $u_p(x) = x^p$ (for $p \le 0$ or $p > 1$) and $\phi(x) = x = \text{id}(x)$ we have

$$M_{(u_p,\,\text{id})}(\vec{a}) = \left[\frac{G^{np}(a_1,\cdots,a_n)}{A(a_1,\cdots,a_n)\,H(a_1^p,\cdots,a_n^p)}\right]^{\frac{1}{pn-p-1}}$$

(see [2]). For $\phi(x) = 1/x = (1/\text{id})\ (x)$ we have

$$M_{(u_p,\,1/\text{id})}(\vec{a}) = M_{(u_p,\,\text{id}\,)}(\tfrac{1}{\text{id}}\,(\vec{a}))^{-1}$$

$$= \left[\frac{G^{np}(a_1,\cdots,a_n)}{A(a_1^p,\cdots,a_n^p)\,H(a_1,\cdots,a_n)}\right]^{\frac{1}{pn-p-1}}$$

EXAMPLE 8. Consider $\phi(x) = e^x$ on $I_\phi = [1, +\infty[$. If $u(x) = 1$ then

$$M_{(1,\phi)}(\vec{a}) = ln\left(\frac{1}{n}\sum_{i=1}^{n} e^{a_i}\right) = ln(A(\phi(a_1),\cdots,\phi(a_n))).$$

More generally, if $u_p(x) = x^p (p \leq 0 \text{ or } > 1)$ we have that $M_{(u_p,\phi)}(\vec{a})$ is the unique positive solution, not smaller than 1, of the polynomial equation

$$pM^{n-1} - M = ln\left[\frac{\sum_{i=1}^{n} \exp\left(\prod_{j=1}^{n} a_j/a_i\right)}{\sum_{i=1}^{n} \exp(a_i)}\right].$$

5. APPLICATIONS TO INEQUALITIES. In [2] Bauer presented inequalities between u-means (or (u, id)-means) and the arithmetic mean. Using the relation (4.5) it is possible to obtain similar inequalities for the harmonic mean. In fact we have the following results.

THEOREM 4. Let $u(x)$ be a convex admissible function of type (A) defined on the interval $I_u \supset \phi(I_\phi)$. For every choice of finitely many numbers $a_1,\cdots,a_n \in I_\phi$, if
(i) $\phi(x) = \text{id}(x)$ then $M_{(u,\text{id})}(a_1,\cdots,a_n) \leq A(a_1,\cdots,a_n)$,
(ii) $\phi(x) = (1/\text{id})(x)$ then $M_{(u,1/\text{id})}(a_1,\cdots,a_n) \geq H(a_1,\cdots,a_n)$.
Moreover if $u(x)$ is strictly convex then strict inequalities hold provided that $a_1,\cdots,a_n$ are not all equal.

THEOREM 5. Let $u(x)$ be a concave admissible function of type (B) defined on $I_u \supset \phi(I_\phi)$. For every choice of finitely many numbers $a_1,\cdots,a_n \in I_\phi$, if
(i) $\phi(x) = \text{id}(x)$ then $M_{(u,\text{id})}(a_1,\cdots,a_n) \leq A(a_1,\cdots,a_n)$,
(ii) $\phi(x) = (1/\text{id})(x)$ then $M_{(u,1/\text{id})}(a_1,\cdots,a_n) \geq H(a_1,\cdots,a_n)$.
Strict inequalities hold for $n \geq 3$ provided that $a_1,\cdots,a_n$ are not all equal.

The parts (i) of these two theorems are the results presented by Bauer in [2] because $M_u(\vec{a}) = M_{(u,\text{id})}(\vec{a})$. To prove the parts (ii) we only have to consider the relation (4.5) to obtain

$$M_{(u,1/\text{id})}(a_1,\cdots,a_n) = 1/M_u(a_1^{-1},\cdots,a_n^{-1}).$$

Then, using the parts (i) we have

$$M_u(a_1^{-1},\cdots,a_1^{-1}) \leq A(a_1^{-1},\cdots,a_n^{-1})$$

but $A(a_1^{-1},\cdots,a_n^{-1}) = H(a_1,\cdots,a_n)^{-1}$ and the results follow.

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REFERENCES

1. ACZEL, J., "Related functional equations applied to Korovkin approximation and to the characterization of Renyi entropies - links to the uniqueness theory", C.R. Math. Rep. Acad. Sci. Canada VI (1984), 319-336.

2. BAUER, H., "A class of means and related inequalities", Manuscripta Mathematica 55 (1986), 199-211.

3. HARDY, G.H., LITTLEWOOD, J.E. and POLYA, G., Inequalities, Cambridge Univ. Press, 1934.