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A SPECIAL PRIME DIVISOR OF THE SEQUENCE: $Ah+B,\ A(h+1)+B, \ldots, A(h+k-1)+B$

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1. INTRODUCTION. Schur showed [1,2,3] that for every pair of integers h, k where $h \geq k$, at least one of the integers

$$h+1, \quad h+2, \quad h+3, \ldots, \quad h+k,$$

is divisible by a prime $p > k$.

Schur also showed [1] that for $h > k > 2$, one of the odd integers

$$2h+1, \quad 2(h+1)+1, \ldots, \quad 2(h+k-1)+1$$

is divisible by a prime $p > 2k+1$. In this paper we generalize these two results by showing the following theorem.

THEOREM 1. Let A and B be two relatively prime positive integers. Then for $h > k$ and sufficiently large k, at least one of the integers

$$Ah+B, \quad A(h+1)+B, \ldots, \quad A(h+k-1)+B \tag{1.1}$$

is divisible by a prime p such that

$$p > Ak+B. \tag{1.2}$$

We need the following lemma.

LEMMA 1. Let $\beta > 1$ be given. Then for sufficiently large x, there is always a prime p such that

$$x < p \leq \beta x \quad \text{and} \quad p \equiv B \ (\text{mod A}).$$

PROOF. Define the function $\theta_A(x)$ by

$$\theta_A(x) = \sum_{\substack{p \leq x \\ p \equiv B \ (\text{mod } A)}} \log p,$$

where the sum is taken over all primes less than or equal to x and congruent to B modulo A. Then the prime number theorem for an arithmetic progressions asserts that

$$\theta_A(x) \sim \frac{x}{\varphi(A)},$$

where $\varphi(A)$ is the number of integers that are less than A and relatively prime to A. Let $\epsilon > 0$ be given, then if x is sufficiently large we have

$$(1-\epsilon)\frac{x}{\varphi(A)} < \theta_A(x) < (1+\epsilon)\frac{x}{\varphi(A)}.$$

Thus

$$\sum_{\substack{x < p \le \beta x \\ p \equiv B \pmod{A}}} \log p = \theta_A(\beta x) - \theta_A(x)$$

$$> \frac{1}{\varphi(A)}\left[(1-\epsilon)\beta x - (1+\epsilon)x\right]$$

$$= \frac{x}{\varphi(A)}\left[\beta - 1 - \epsilon(\beta+1)\right].$$

If ϵ is chosen so that $0 < \epsilon < \frac{\beta-1}{\beta+1}$, then

$$\sum_{\substack{x < p \le \beta x \\ p \equiv (B \bmod A)}} \log p > 0.$$

Thus if x is large, then there is at least one prime p such that $x < p \le \beta x$ and $p \equiv B$ (mod A), and the lemma is proved.

PROOF OF THEOREM 1. Suppose the theorem is false for a pair (h, k), then the numbers

$$Ah+B, A(h+1)+B, \ldots, A(h+k-1)+B,$$

have only prime divisors which are less than or equal to $Ak+B$. Consider

$$G = \frac{(Ah+B)(A(h+1)+B)\cdots(A(h+k-1)+B)}{B(A+B)(2A+B)\cdots(Ak-A+B)} \tag{1.3}$$

and let w_p be the integer exponent (positive, negative or zero) of p which appears in G. Then by our assumption, every prime appearing in G is less than or equal to $Ak+B$. Thus,

$$G = \prod_{p \le Ak+B} p^{w_p}. \tag{1.4}$$

We claim that

$$\begin{cases} w_p = 0 & \text{if } p \mid A \\ w_p \le \dfrac{\log(Ah+Bk)}{\log p} & \text{if } p \nmid A \end{cases}$$

For if $p \mid A$, then $p \nmid Aj+B$ for any integer j; otherwise we would have $p \mid B$ and so p divides both A and B. This is impossible, since A and B are relatively prime. Thus p does not divide any factor of either the numerator or the denominator of (1.3), hence $w_p = 0$.

Suppose now that $p \nmid A$; then it is easy to see that

$$w_p = \sum_{1 < p^r \le A(h+k-1)+B} (U(p^r) - V(p^r)), \tag{1.6}$$

where the sum is taken over all prime powers p^r between 1 and $A(h+k-1)+B$. $U(p^r)$ is the number of factors in the numerator of (1.3) that are divisible by p^r and $V(p^r)$ is the number of factors in the denominator of (1.3) that are divisible by p^r.

Since $Ax+B \equiv 0 \pmod{p^r}$ has only one solution for x modulo p^r, $Ax+B$ is divisible by p^r for only one value of x when x runs through p^r consecutive integers. Therefore,

$$\left[\frac{k}{p^r}\right] \le U(p^r) \le \left[\frac{k}{p^r}\right] + 1,$$

$$\left[\frac{k}{p^r}\right] \le V(p^r) \le \left[\frac{k}{p^r}\right] + 1.$$

Thus

$$-1 \le U(p^r) - V(p^r) \le 1.$$

This and (1.6) give

$$w_p \le \sum_{p^r \le A(h+k)} 1 \le \frac{\log(Ah + Ak)}{\log p},$$

and the claim is proved. Thus

$$p^{w_p} \le Ah + Ak, \quad \text{for all } p.$$

This and (1.4) give

$$G \le \prod_{p \le Ak + B} (Ah + Ak);$$

thus

$$G \le (Ah + Ak)^{\pi(Ak+B)}. \tag{1.7}$$

On the other hand, by (1.3) we have

$$\begin{aligned} G &= \prod_{j=1}^{k} \frac{A(h+j-1)+B}{A(j-1)+B} \\ &= \prod_{j=1}^{k} \frac{Ah + Aj - A + B}{Aj - A + B} \\ &= \prod_{j=1}^{k} \left(1 + \frac{Ah}{Aj - A + B}\right) \\ &\ge \prod_{j=1}^{k} \left(1 + \frac{Ah}{Aj}\right) \quad (\text{ since } A > B) \\ &\ge \left(1 + \frac{h}{k}\right)^k, \end{aligned}$$

or

$$G \ge \left(1 + \frac{h}{k}\right)^k. \tag{1.8}$$

Combining (1.7) and (1.8) yields

$$\left(1 + \frac{h}{k}\right)^k \le (Ah + Ak)^{\pi(Ak+B)}.$$

Taking logarithms, we get

$$k \log\left(1 + \frac{h}{k}\right) \le \pi(Ak + B) \log(Ah + Ak).$$

Writing $\log(Ah + Ak) = \log Ak + \log\left(1 + \frac{h}{k}\right)$ gives

$$\{k - \pi(Ak + B)\} \log\left(1 + \frac{h}{k}\right) \le \pi(Ak + B) \log Ak.$$

Dividing both sides of this inequality by $Ak + B$, we get

$$\begin{aligned} \left\{\frac{k}{Ak + B} - \frac{\pi(Ak + B)}{Ak + B}\right\} \log\left(1 + \frac{h}{k}\right) &\le \frac{\pi(Ak + B) \log Ak}{Ak + B} \\ &\le \frac{\pi(Ak + B) \log(Ak + B)}{Ak + B} \\ &\le \frac{3}{2}. \end{aligned}$$

Thus,

$$\left\{\frac{k}{Ak + B} - \frac{\pi(Ak + B)}{Ak + B}\right\} \log\left(1 + \frac{h}{k}\right) \le \frac{3}{2}. \tag{1.9}$$

Consider two cases.

Case I. $\frac{h}{k} \geq e^{2A} - 1$

Then $\log\left(1 + \frac{h}{k}\right) \geq 2A$. Using this in (1.9) we obtain

$$\left\{\frac{k}{Ak+B} - \frac{\pi(Ak+B)}{Ak+B}\right\}(2A) \leq \frac{3}{2}.$$

Letting $k \to \infty$ in this inequality gives

$$\frac{1}{A} \cdot 2A \leq \frac{3}{2},$$

or

$$2 \leq \frac{3}{2}.$$

This provides a contradiction that proves the theorem in this case.

Case II. $\frac{h}{k} < e^{2A} - 1$

Then

$$\frac{Ah + Ak + B}{Ah} = 1 + \frac{k}{h} + \frac{B}{Ah}$$

$$> 1 + \frac{1}{e^{2A} - 1} + \frac{B}{Ah}$$

$$> 1 + \frac{1}{e^{2A} - 1},$$

or

$$\frac{Ah + Ak + B}{Ah} \geq 1 + c,$$

where c is a positive constant (depending only on A). Thus

$$\frac{Ah + Ak + B}{Ah} > \beta, \qquad \text{where } \beta = 1 + c > 1.$$

By Lemma 1 if h is large (or k is large, since $h > k$), there exists a prime integer p such that $p \equiv B$ (mod A) and

$$Ah < p \leq \beta Ah < Ah + Ak + B.$$

Thus

$$Ah + B \leq p \leq Ah + Ak + B - A.$$

Therefore one of the integers

$$Ah + B,\ A(h+1) + B,\ \ldots,\ A(h+k-1) + B,$$

is a prime p. Since $p \geq Ah + B$ and $h > k$, then

$$p > Ak + B,$$

which is condition (1.2). This completes the proof of the theorem.

REFERENCES

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