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GENERALIZED EQUIVALENCE OF MATRICES OVER PRÜFER DOMAINS

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ABSTRACT: Two $m \times n$ matrices A, B over a commutative ring R are equivalent in case there are invertible matrices P, Q over R with $B = PAQ$. While any $m \times n$ matrix over a principle ideal domain can be diagonalized, the same is not true for Dedekind domains. The first author and T. J. Ford introduced a coarser equivalence relation on matrices called homotopy and showed any $m \times n$ matrix over a Dedekind domain is homotopic to a direct sum of 1×2 matrices. In this article we give necessary and sufficient conditions on a Prüfer domain that any $m \times n$ matrix be homotopic to a direct sum of 1×2 matrices.

Key Words and Phrases: Prüfer domain, Progenerator module, Bezout domain, matrix equivalence
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1. INTRODUCTION

Let M, N be finitely generated projective faithful modules (progenerators) over a commutative ring R. An R homomorphism $h : M \to N$ is called image split in case $h(M)$ is a faithful R-direct summand of N. If $f : M \to N$ and $g : P \to Q$ are homomorphisms of R progenerators then f and g are said to be homotopic if there are image split homomorphisms $h : A \to B$ and $k : C \to D$ and R isomorphisms ϕ, ψ making the commuting diagram of R-modules

$$\begin{array}{ccc} M \otimes A & \xrightarrow{f \otimes h} & N \otimes B \\ \downarrow \phi & & \downarrow \psi \\ P \otimes C & \xrightarrow{g \otimes k} & Q \otimes D \end{array}$$

If $f = \rho^{-1} g \gamma$ for isomorphisms ρ, γ then f and g are homotopic (where $R = A = B = C = D$ and $h = k = 1_R$). Thus equivalent homomorphisms are homotopic but not conversely. The notion of homotopy of homomorphisms was introduced in [4] to remove most of the obstruction observed by L. Levy in [13] to diagonalization of matrix transformations under equivalence over Dedekind domains. Summarizing some of the results in [4], homotopy is an equivalence relation on homomorphisms of progenerator modules and tensor product of homomorphisms induces a multiplication on homotopy classes which turns this set of classes into a monoid denoted $\mathcal{M}(R)$. Each homotopy class is represented by at least one matrix transformation, and if R is a Dedekind domain by a matrix transformation which is a direct sum of 1×2 matrices, a matrix of the form

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & & & \\ 0 & 0 & a_2 & b_2 & & & \\ & & & & \cdots & & \\ & & & & & a_m & b_m \end{bmatrix}_{m \times 2m}$$

Moreover, if $I_j = a_jR + b_jR$ then $I_1 \supset I_2 \supset \cdots \supset I_m$. If R is a discrete valuation ring then $\mathcal{M}(R) \cong \mathbf{PN}[x]$, the monoid of primitive polynomials with coefficients in $\mathbf{N} = \{0,1,2,\ldots\}$ together with the 0-polynomial. If R is a Dedekind domain then $\mathcal{M}(R)$ is naturally isomorphic to $\bigoplus_{P \in MaxSpec(R)} \mathcal{M}(R_P)$ and this isomorphism gives an isomorphism between $\mathcal{M}(R)$ and primitive polynomials over $\mathbf{N}$ in indeterminates indexed by $MaxSpec(R)$.

The purpose of this paper is to determine the extent to which these results can be generalized to arbitrary domains. In fact, they come close to characterizing Dedekind domains. We first observe that if R is a commutative ring containing a maximal ideal P such that $\dim_{R/P}(P/P^2) \geq 2$ then there is a homotopy class in $\mathcal{M}(R)$ which contains no matrix transformation which is a direct sum of 1×2 matrices. Thus, if R is a Noetherian domain and every homotopy class in $\mathcal{M}(R)$ contains a matrix which is a direct sum of 1×2 matrices then $\dim R \leq 1$. The inclusion map from a domain R to its integral closure $\bar{R}$ induces a monoid homomorphism $\mathcal{M}(R) \to \mathcal{M}(\bar{R})$ which was studied in [6]. Here we relax the Noetherian condition and study $\mathcal{M}(R)$ for Prüfer domains. If R is a Prüfer domain of Krull dimension = 1 or if R is a Prüfer domain of finite character (each nonzero element of R is contained in only finitely many maximal ideals) we show every class in $\mathcal{M}(R)$ contains a representing matrix which is a direct sum of 1×2 matrices. If R is any valuation domain with value group $G \leq (\mathbf{R}, +)$ and G^+ is the monoid of nonnegative elements of G form the monoid $\mathbf{PN}(G^+)$ of "primitive polynomials" $\sum_{g \in G^+} a_g x^g$ with $a_g \in \mathbf{N}$, almost all $a_g = 0$ and $gcd\{a_g | g \in G^+\} = 1$. We show $\mathcal{M}(R) \cong \mathbf{PN}(G^+)$. After giving a slight generalization of L. Levy's "Separated DivisorTheorem" for matrices over Dedekind domains [13], we can show for Prüfer domains that $\mathcal{M}(R)$ is naturally isomorphic to $\bigoplus_{P \in MaxSpec(R)} \mathcal{M}(R_P)$ if and only if R is of finite character and the valuation rings at the maximal ideals of R are pairwise independent. The principal examples of Prüfer domains of finite character whose valuation rings at maximal ideals are pairwise independent are Dedekind domains and valuation domains.

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2. SEPARATED DIVISOR THEOREM

PROPOSITION 1. Let R be a commutative ring containing a maximal ideal P with $\dim_{R/P}(P/P^2) > 1$. Then there is a matrix transformation over R which is not homotopic to a direct sum of 1×2 matrix transformations.

PROOF: Let $\phi : R \to S$ be a homomorphism of commutative rings, so S is an R-algebra. Then ϕ induces a monoid homomorphism $\mathcal{M}(\phi) : \mathcal{M}(R) \to \mathcal{M}(S)$ by $M(\phi)(|f|) = |1 \otimes f|$, where if $f \in Hom_R(N_1, N_2)$ then $1 \otimes f \in Hom_S(S \otimes N_1, S \otimes N_2)$ (Theorem 1 of [4]). Since each class in $\mathcal{M}(S)$ is represented by a matrix transformation, if ϕ is an epimorphism then $\mathcal{M}(\phi)$ is an epimorphism. If ϕ is an epimorphism and if every class in $\mathcal{M}(R)$ is represented by a matrix which is a direct sum of 1×2 matrices, then every class in $\mathcal{M}(S)$ is represented by a matrix which is a direct sum of 1×2 matrices. Thus, it suffices to check the conclusion of the proposition for a homomorphic image of R. Let $\{\alpha_1 + P^2, \alpha_2 + P^2\} \cup \{\alpha_i + P^2\}_{i \in I}$ be a basis for P/P^2 over R/P. Let J be the ideal in R generated by P^2 and $\{\alpha_i\}_{i \in I}$. The ring $S = R/J$ is a local ring with maximal ideal $M = P + J/J$. Moreover, $M^2 = (0)$ and $\dim_{S/M}(M/M^2) = 2$. Let $a_1, a_2 \in M$ be linearly independent over S/M. We check the matrix $\begin{bmatrix} a_1 & a_2 \\ 0 & a_1 \end{bmatrix}$ is not homotopic over S to any

matrix of the form

$$H = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & & \\ 0 & 0 & \alpha_2 & \beta_2 & & \\ & & & \cdots & & \\ & & & & \alpha_m & \beta_m \end{bmatrix} m \times 2m.$$

Since S is local, with respect to a suitable basis choice, each image split homomorphism has a matrix representation of the form $\mathrm{diag}(1,\cdots,1,0,\cdots,0)$ (Proposition 3(9) of [4]). We need to check

$$F = \begin{bmatrix} a_1 & a_2 \\ 0 & a_1 \end{bmatrix} \otimes diag(1,\ldots,1,0,\quad,0) = \left[\begin{array}{cc|cc|c|cc} a_1 & a_2 & & & & & \\ 0 & a_1 & & & & & \\ \hline & & a_1 & a_2 & & & \\ & & 0 & a_1 & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & 0 & \\ \hline & & & & & & \ddots \\ \hline & & & & & & 0 \end{array}\right] m \times 2m$$

is not equivalent over S to H. View F as the relation matrix of the factor module $S^{(2m)}/L_F$ where L_F is the submodule of $S^{(2m)}$ generated by the rows of F. Then $S^{(2m)}/L_F$ is isomorphic to a direct sum of modules of the form $A = S \oplus S/ < (a_1,a_2),(0,a_1) >$ together with 0-summands. A direct calculation shows the S-endomorphisms of $S \oplus S$ leaving $< (a_1,a_2),(0,a_1) >$ invariant are given by matrices of the form $\begin{bmatrix} \alpha & m' \\ \beta & \alpha + m \end{bmatrix}$ where $m,m' \in M; \alpha,\beta \in S$. That is, $End_S(A)$ is a homomorphic image of the ring of these matrices. A direct but messy calculation shows that if $\begin{bmatrix} \alpha & m' \\ \beta & \alpha + m \end{bmatrix}^2 = \begin{bmatrix} \alpha & m' \\ \beta & \alpha + m \end{bmatrix}$ then $\alpha = 0,1$ and $\beta = m = m' = 0$. Since idempotents can be lifted modulo a nilpotent ideal and the natural homomorphism from $\left\{ \begin{bmatrix} \alpha & m' \\ \beta & \alpha + m \end{bmatrix} \middle| \alpha,\beta \in S, m,m' \in M \right\}$ to $End_S(A)$ has kernel $\left\{ \begin{bmatrix} \alpha & m' \\ \beta & \alpha + m \end{bmatrix} \middle| \alpha,\beta,m,m' \in M \right\}$ which is nilpotent we see $End_S(A)$ has no idempotents other than 0 and 1 so A is an indecomposable S-module. In the same way view G as the relation matrix of the factor module $S^{(2m)}/L_G$ where L_G is the submodule of $S^{(2m)}$ generated by the rows of G. Then $S^{(2m)}/L_G$ is isomorphic to a direct sum of modules of the form $B_i = S \oplus S/ < (\alpha_i,\beta_i) >$. An easy calculaton shows $\dim_{S/M}(M \cdot A) = 2$, $\dim_{S/M}(M \cdot B_i) \geq 3$, and $\dim_{S/M}(A/MA) = 2 = \dim_{S/M}(B_i/MB_i)(1 \leq i \leq m)$. If $A \oplus L \cong B_i$ then $A/MA \oplus L/ML \cong B_i/MB_i$ so $L = ML$ so by Nakayama's lemma $L = (0)$. In this case $A \cong B_i$ which is impossible by the first dimension count above. Thus A is an indecomposable S-module which is not a direct summand of any B_i. By the Krull-Schmidt Theorem $S^{(2m)}/L_G \cong S^{(2m)}/L_F$ and F,G cannot be equivalent matrices over S.

REMARK: If R is noetherian then $\dim\ R \leq \sup_{P \in MaxSpec(R)} \dim_{R/P}(P/P^2)$ so if R is noetherian, Proposition 1 implies that if every matrix over R is homotopic to a direct sum of 1×2 matrices then $\dim\ R \leq 1$. This may not be the case when R is not noetherian as the next result shows.

Let K denote a field and v a valuation on K with value group $G \subset (\mathbf{R},+)$. Let R be the valuation ring corresponding to v. Since R is an elementary divisor ring [9], each $m \times n$ matrix over R is equivalent to a diagonal matrix $\mathrm{diag}\{d_1,\cdots,d_m\}$ with $v(d_i) \leq v(d_{i+1})$ when $d_{i+1} \neq 0$ and $d_j = 0$ implies $d_k = 0$ for $k \geq j$. The "elementary divisors" $d_1,\ldots,d_m$ uniquely determine the equivalence class. In this case, following [4], we can explicitly determine the monoid of homotopy classes. Let $G^+ = \{g \in G | g \geq 0\}$ and $\mathbf{N}(G^+) = \{a(x) = \sum_{i=1}^m n_i x^{g_i} | n_i \in \mathbf{N}, g_i \in G^+\}$, where $\mathbf{N}$ is the set of nonnegative integers. Then $\mathbf{N}(G^+)$ is a multiplicative monoid with multiplication induced from the equation $x^{g_i}x^{g_j} = x^{g_i+g_j}$. For $a(x),b(x) \in \mathbf{N}(G^+)$ say $a(x) \sim b(x)$ if there exists positive integers r and s with $ra(x) = sb(x)$. It is easy to check that $\sim$ is a congruence on $\mathbf{N}(G^+)$. Let $\mathbf{PN}(G^+) = \mathbf{N}(G^+)/\sim$. Then $\mathbf{PN}(G^+)$ is a monoid which can be identified with the primitive polynomials in x with exponents from G^+. Let $|a(x)|$ be the congruence class in $\mathbf{PN}(G^+)$ represented by $a(x) \in \mathbf{N}(G^+)$. To the homotopy class in $\mathcal{M}(R)$ represented

by the 1x1 matrix transformation (d) over R we can assign the congruence class $|x^{v(d)}|$ in $\mathbf{PN}[x]$. Our next result is that this assignment extends to an isomorphism.

PROPOSITION 2. If R is a valuation ring corresponding to a valuation v on a field K with value group G then $\mathcal{M}(R) \cong \mathbf{PN}(G^+)$.

PROOF: Lemma 2 and Proposition 3 (1) of [4] imply any $0 \neq |h| \in \mathcal{M}(R)$ contains an $r \times r$ matrix transformation $\mathrm{diag}\{d_1, \ldots, d_r\}$ where $d_i \neq 0$ and $v(d_i) \leq v(d_{i+1})$ for all i. Let $\phi : \mathcal{M}(R) \to \mathbf{PN}(G^+)$ by $\phi(|h|) = \sum_{i=1}^{r} x^{v(d_i)}$. We only check ϕ is well defined, then the rest of the argument is routine. If $|diag\{d_1, \ldots, d_r\}| = |diag\{f_1, \ldots, f_s\}|$ in $\mathcal{M}(R)$ with $v(f_j) \leq v(f_{j+1})$ for all j then by Proposition 3(9) of [4], $diag\{d_1, \ldots, d_r\} \otimes diag\{1, \overbrace{\ldots, 1}^{p}, 0, \ldots, 0\}$ is equivalent to $diag\{f_1, \ldots, f_s\} \otimes diag\{1, \overbrace{\ldots, 1}^{q}, 0, \ldots, 0\}$. Let $d_1', \ldots, d_k'$ be the entries in $diag\{d_1, \ldots, d_r\}$ with pairwise distinct valuations. By uniqueness of invariant factors in an elementary divisor ring, the entries in $\mathrm{diag}\{f_1, \ldots, f_s\}$ with distinct valuations are $f_1', \ldots, f_k'$ where $v(f_i') = v(d_i')$ when we order d_i', f_i' so $v(d_i') < v(d_{i+1}')$ and $v(f_i') < v(f_{i+1}')$ for all i. Let $r_i = \#\{d_j | v(d_j) = v(d_i'), 1 \leq j \leq r\}$ and $s_i = \#\{f_j | v(f_j) = v(f_i') 1 \leq j \leq s\}$. Then $\phi(|h|) = |\sum_{i=1}^{k} r_i x^{v(d_i')}|$ and $\phi(|h|) = |\sum_{i=1}^{k} s_i x^{v(d_i')}|$. Moreover, by uniqueness of invariant factors, $pr_i = qs_i$, $1 \leq i \leq k$ so $|\sum_{i=1} r_i x^{v(d_i')}| = |\sum_{i=1} s_i x^{v(f_i')}|$ in $\mathbf{PN}[x]$. Thus ϕ is well defined.

The following is needed to prove a generalized separated divisor theorem. Undefined terminology can be found in [7].

LEMMA 3. Let R denote a Prüfer domain of finite character whose valuation rings at maximal ideals are pairwise independent. If $0 \neq L$ is an ideal in R then there is a factorization $L = \prod_{i=1}^{k} L_i$ where each L_i is contained in exactly one maximal ideal P_i of R and $P_i \neq P_j$ if $i \neq j$.

PROOF: Since $0 \neq L$ and R has finite character, L is contained in only finitely many maximal ideals $P_1, \cdots, P_k$ of R. Let $L_i = LR_{P_i} \cap R (1 \leq i \leq k)$. Theorem 4.10 of [7] implies $L = \cap_{P \in MaxSpec(R)} (LR_P \cap R)$. Since $LR_P = R_P$ if $L \not\subseteq P$ we have $L = \cap_{i=1}^{k} (LR_{P_i} \cap R) = \cap_{i=1}^{k} L_i$. We always have $L_i \subset P_i$. Let v_i, v_j be valuations corresponding to P_i, P_j respectively. Since these valuations are pairwise independent, Theorem 22.9 (2) of [7] implies that for each $0 \neq x \in L$ there is an $a \in R_{P_i} \cap R_{P_j}$ with $v_i(a) = v_i(x)$ and $v_j(a) = 0$. If $S = R - P_i \cup P_j$ then an elementary exercise gives, $S^{-1}R = R_{P_i} \cap R_{P_j}$ so after clearing the denominator we can assume $a \in R$. Thus $a \in LR_{P_i} \cap R = L_i$ but $a \notin P_j$ so each L_i is contained in exactly one maximal ideal P_i of R. Moreover, $L_i + L_j = R$ whenever $i \neq j$ since $L_i + L_j$ is contained in no maximal ideals of R. Thus $L = \cap_{i=1}^{k} L_i = \prod_{i=1}^{k} L_i$. To check uniqueness assume $L = \prod_{j=1}^{w} L_j'$ where each L_j' is contained in exactly one maximal ideal P_j' of R and $P_j' \neq P_q''$ if $j \neq q$. From the above $w = k$ and after relabeling we can assume $P_j = P_j'$. Now $LR_{P_i} = (\prod_{j=1}^{k} L_j') R_{P_i} = L_i' R_{P_i}$ so $L_i' R_{P_i} = L_i R_{P_i}$. Since $L_i = LR_{P_i} \cap R$ it follows that $L_i' \subset L_i$. But $R_{P_i} L_i' = R_{P_i} L_i$ and $R_P L_i' = R_P L_i = R_P$ if P is a maximal ideal of R not equal to P_i so $R_P \otimes L_i / L_i' = 0$ for all maximal ideals P of R so $L_i = L_i'$.

Following [13], the ideals L_i in Lemma 3 are called the separated divisors of L. The separated divisors $Div\{J_i\}_{i=1}^{r}$ of a finite sequence of ideals in R is the collection, counting multiplicity, of all the separated divisors of the individual ideals J_i.

For convenience we list some definitions and a result we need from [2]. A Prüfer domain R with quotient field K is said to satisfy the Invariant Factor Theorem if for any finitely generated submodule M of $R^{(n)}$ there exist simultaneous decompositions of $R^{(n)}$ and M

$$R^{(n)} = Rx_1 \oplus \ldots \oplus Rx_{r-1} \oplus J_r x_r \oplus \ldots \oplus J_n x_n$$

$$M = E_1 x_1 \oplus \ldots \oplus E_{r-1} x_{r-1} \oplus E_r x_r$$

where $x_i \in K$, the J_i are invertible fractional ideals of R, the E_i are invertible integral ideals of R and $E_i \subset E_{i+1}$ for $i = 1, 2, \ldots, r-1$. A Prüfer domain R has the Steinitz property if for fractional ideals I

and J, $I \oplus J \cong R \oplus IJ$. A Prüfer domain R has the $1\frac{1}{2}$ generator property in case for any fractional ideal I and any $0 \neq x \in I$ there is a $y \in I$ with $I = Rx + Ry$.

PROPOSITION 4. If R is a Prüfer domain of finite character or Krull dimension = 1 then R satisfies the Invariant Factor Theorem, R has the Steinitz property and R has the $1\frac{1}{2}$ generator property

PROOF: [2]

SEPARATED DIVISOR THEOREM (Levy). Let R be a Prüfer domain and A an $m \times n$ matrix over R of rank r. Let M_A be the submodule of $R^{(n)}$ generated by the rows of A and let $S_A = R^{(n)}/M_A$.

I. If R is of finite character or $\dim R = 1$ then there exist invertible ideals $E_1, \ldots, E_r$ of R with $E_i \subset E_{i+1} (1 \leq i \leq r-1)$ and invertible fractional ideals $J_r, J_{r+1}, \ldots, J_n$ such that

$$S_A = \begin{cases} \bigoplus_{i=1}^{r} R/E_i \oplus J_{r+1} \oplus \ldots \oplus J_n & \text{if } r < n \\ \bigoplus_{i=1}^{r} R/E_i & \text{if } r = n \end{cases}$$

where $J_r^{-1} = \prod_{i=1}^{r} E_i$ if $r = m$, $\prod_{k=r}^{n} J_k \cong R$ if $r = 0$ and $J_n \cong R$ if $r = n$.

II. If R is of finite character with pairwise independent valuation rings at maximal ideals and A, A' are two $m \times n$ matrices over R of rank r then A is equivalent to A' if and only if $Div\{E_i\}_{i=1}^{r} = Div\{E_i'\}_{i=1}^{r}$ and $J_{r+1} \cdots J_n \cong J_{r+1}' \cdots J_n'$

PROOF: I. Let M be any finitely generated submodule of $R^{(n)}$. Since R satisfies the Invariant Factor Theorem there exist simultaneous decompositions of R^n and M,

$$R^{(n)} = Rx_1 \oplus \ldots \oplus Rx_{r-1} \oplus J_r x_r \oplus \ldots \oplus J_n x_n$$

$$M = E_1 x_1 \oplus \quad \oplus E_{r-1} x_{r-1} \oplus E_r J_r x_r$$

with $x_i \in K$ = quotient field of R, the J_i invertible fractional ideals of R and E_i invertible integral ideals of R such that $E_i \subset E_{i+1}$, $1 \leq i \leq r-1$. Since R satisfies the $1\frac{1}{2}$ generator property, Proposition 1 of [2] implies $J_r/E_r J_r \cong R/E_r$. Hence

$$R^n/M \cong \bigoplus_{1}^{r} R/E_i \oplus J_{r+1} \oplus \ldots \oplus J_n$$

(where $J_{r+1} \oplus \cdots \oplus J_n$ does not occur if $r = n$). Here r is the rank of M which is the rank of A if M is generated by the rows of A. This proves S_A has the decomposition given in I. If $r = m$ then $M_A \cong R^{(m)}$ so $E_1 x_1 \oplus \ldots \oplus E_{r-1} x_{r-1} \oplus E_r J_r x_r \cong R^{(m)}$. The Steinitz Property and cancellation imply $\prod_{i=1}^{r} E_i J_r \cong R$ and $J_r^{-1} \cong \prod_{i=1}^{r} E_i$. In the same way, if $r = 0$ then $R^{(n)} = J_1 x_1 \oplus \ldots \oplus J_n x_n$ and $J_1 \ldots J_n = J_{r+1} \ldots J_n \cong R$. If $r = n$ then $R^{(n)} = Rx_1 \oplus \ldots x \oplus Rx_{r-1} \oplus J_r x_r$ so $J_r \cong R$.

II. Let A, A' be two $m \times n$ matrices over R. Suppose we have the decompositions given in I for S_A and $S_{A'}$. If A is equivalent to A' then $S_A \cong S_{A'}$. The uniqueness part of the Invariant Factor Theorem (see [12]) implies $Div\{E_i\}_{i=1}^{r} = Div\{E_i'\}_{i=1}^{r}$ and $J_{r+1} \ldots J_n \cong J_{r+1}' \ldots J_n'$. Conversely, if $Div\{E_i\}_{i=1}^{r} = Div\{E_i'\}_{i=1}^{r}$ and $J_{r+1} \ldots J_n \cong J_{r+1}' \ldots J_n'$ then S_A and $S_{A'}$ have the same invariant factors and thus are isomorphic. To complete the proof of II we need to check that if $S_A \cong S_{A'}$ then A is equivalent to A'. Our problem is to find isomorphisms ϕ_1, ϕ_2, ϕ_3 making the commuting diagram

$$\begin{array}{ccccccc} R^{(M)} & \xrightarrow{A} & R^{(M)} & \xrightarrow{\eta} & S_A & \longrightarrow 0 \\ \downarrow \phi_3 & & \downarrow \phi_2 & & \downarrow \phi_1 & \\ R^{(M)} & \xrightarrow{A'} & R^{(M)} & \xrightarrow{\eta'} & S_{A'} & \longrightarrow 0 \end{array}$$

For then, if P_i is the matrix transformation representing ϕ_i, then $A' = P_2 A P_3^{-1}$ so A will be equivalent to A'. Since R is Prüfer, Theorem 1.8 of [9] implies $S_A = P \oplus T$ where P is projective and T is torsion.

If $\pi\ \ S_A \to P$ is the projection let ρ be a splitting map so $R^{(n)} = \rho(P) \oplus Q$ and $\ker\ \eta \subset Q$. In the same way $S_{A'} = P' \oplus T'$, $R^{(n)} = \rho'(P') \oplus Q'$ and $\ker\ \eta' \subset Q'$. Since $S_A \cong S_{A'}$ there are isomorphisms $\sigma : P \to P'$ and $\tau\ \ T \to T'$. By Proposition 1 of [12], T is a direct sum of cyclic modules R/L for ideals $0 \neq L \subset R$. By lemma 3 we can let $\{L_i\}$ be the set of separated divisors of L. The Chinese Remainder Theorem implies $R/L \cong \oplus R/L_i$. Since L_i is contained in only one maximal ideal of R, R/L_i is local for all i. Since $Q/ker\ \eta \cong T$, Theorem 1.6 of [11] implies there is a simultaneous decomposition of Q and $\ker\ \eta$. In the same way there is a simultaneous decomposition of Q' and $\ker\ \eta'$. Thus the given isomorphism $\tau : T \to T'$ extends to an isomorphism $\bar{\tau} : Q \to Q'$ such that $\bar{\tau}(\ker \eta) = \ker \eta'$. This gives isomorphisms $\phi_1 = \sigma \oplus \tau$ and $\phi_2 = \rho' \sigma\ \pi\ \eta \oplus \bar{\tau}$ making the commutative diagram

$$\begin{array}{ccccccc} R^{(M)} & \xrightarrow{A} & R^{(M)} & \xrightarrow{\eta} & S_A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \\ R^{(M)} & \xrightarrow{A'} & R^{(M)} & \xrightarrow{\eta'} & S_{A'} & \longrightarrow & 0 \end{array}$$

Since $\phi_2(\ker \eta) = \ker \eta'$ we have the exact

$$\begin{array}{ccccc} & & & & R^{(m)} \\ & & & \overset{\phi_3}{\swarrow} & \downarrow{\scriptstyle \phi_2 \circ A} \\ R^{(m)} & \xrightarrow{A'} & Image A' & \longrightarrow & 0 \end{array}$$

and ϕ_3 is the R-homomorphism given since $R^{(m)}$ is free. This completes the proof of the Separated Divisor Theorem.

3. THE MONOID OF HOMOTOPY CLASSES

LEMMA 5. Let R be a Prüfer domain of finite character or Krull dimension ≤ 1.

(1) If $0 \neq |f| \in \mathcal{M}(R)$ then there exists g such that $|f| = |g|$ in $\mathcal{M}(R)$ and coker(g) is a torsion R-module.

(2) Let $|f|, |g| \in \mathcal{M}(R)$ with coker(g) and coker(f) torsion R-modules. If $f : M \to N$, $g : P \to Q$ then $|f| = |g|$ in $\mathcal{M}(R)$ if and only if there exists R-progenerators K, L such that $N \otimes K \cong Q \otimes L$ and $\text{coker}(f) \otimes K \cong coker(g) \otimes L$.

PROOF: The proof now follows "mutatis mundantis" as the proof of lemma 6 of [4].

Following page 393 of [4] a description of $\mathcal{M}(R)$ in terms of ideals of R can be given now. Consider the set

$$A = \{(M, R^{(m)}) | m \geq 1, M \text{ is a finitely generated } R \text{ submodule of } R^{(m)} \text{ such that } R^{(m)}/M \text{ is torsion } R \text{ module.}$$

Define multiplication in A as follows: $(M, R^{(m)})(N, R^{(n)}) = (M \otimes N, R^{(m)} \otimes R^{(n)})$, then A is a commutative semigroup. Define a semigroup homomorphism

$$\rho : A \to \mathcal{M}^*(R) = \mathcal{M}(R) - \{0\}$$

by $\rho(M, R^{(m)}) = |i|$ where $i : M \to R^{(m)}$ is the inclusion map. Note that by Lemma 5 $\rho(M, R^{(m)}) = \rho(N, R^{(n)})$ if and only if there exist R-progenerators K and L such that

$$R^{(m)} \otimes K \cong R^{(n)} \otimes L \quad \text{and} \quad (R^{(m)}/M) \otimes K \cong (R^{(n)}/N) \otimes L.$$

Thus ρ induces an equivalence relation $\sim$ *on* A as follows:

$$(M, R^{(m)}) \sim (N, R^{(n)}) \quad \text{if and only if} \quad \rho(M, R^{(m)}) = \rho(N, R^{(n)})$$

Let $\mathcal{A}$ be the set of equivalence classes of A. Then the product on A induces the multiplication on $\mathcal{A}$, turning it into a commutative monoid with identity the class containing (R, R).

LEMMA 6. Let R be a Prüfer domain of finite character or Krull dimension ≤ 1, then the map $\bar{\rho} : \mathcal{A} \to \mathcal{M}*(R)$ induced by ρ is an isomorphism.

PROOF: Same as Proposition 7 of [4].

THEOREM 7. If R is a Prüfer domain of finite character or Krull dimension ≤ 1 then every homomorphism of R-progenerators is homotopic to a matrix transformation of the form

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & & \\ 0 & 0 & a_2 & b_2 & & \\ & & & & \cdots & \\ & & & & a_m & b_m \end{bmatrix} m \times 2m$$

where if $I_j = (a_j, b_j)$ then $I_j \supset I_{j+1} (1 \leq j \leq n-1)$.

PROOF: The proof of Theorem 7 now follows exactly as on pages 391-394 of [4].

LEMMA 8. Let R denote an integral domain. Then the following are equivalent.

1) Each nonzero element of R is contained in only finitely many maximal ideals of R.

2) If N is a finitely generated submodule of $R^{(n)}$ then $R_P \otimes N$ is a direct summand of $R_P^{(n)}$ for almost all $P \epsilon MaxSpec(R)$.

(3) For each exact sequence $0 \to N \to Q \to M \to 0$ of finitely generated R-modules with Q projective the associated sequence $0 \to R_P \otimes N \to R_P \otimes Q \to R_P \otimes M \to 0$ is split exact for almost all $P \epsilon MaxSpec(R)$

PROOF: The equivalence of 2 and 3 follows easily since every finitely generated projective module is a direct summand of a free module of finite rank. To see $2 \to 1$ let $0 \neq a \epsilon R$. Then (a) is a submodule of R so $1 \otimes a$ is either 0 or a unit in R_P for almost all $P \epsilon MaxSpec(R)$. If R is an integral domain then $1 \otimes a \neq 0$ in R_P so $a \notin P$ for almost all $P \epsilon MaxSpec(R)$. To show $1 \to 2$ let $R^{(n)} = Rx_1 \oplus \ldots \oplus Rx_n$ and let $N = Rn_1 + \ldots + Rn_k$ with $0 \neq n_i \epsilon R^{(n)} (1 \leq i \leq k)$. Let $n_1 = a_1 x_1 + \ldots + a_n x_n$ where we can assume $a_1 \neq 0$. Over $a_1^{-1}R$ we have $a_1^{-1}n_1, x_2, \ldots, x_n$ is a basis for $(a_1^{-1}R)^{(n)}$ and $a_1^{-1}n_1, n_2, \ldots, n_k$ generates $a_1^{-1}N$ so replace x_1 and n_1 by $a_1^{-1}n_1$ over $a_1^{-1}R$. Then $x_1, x_2, \ldots, x_n$ is a basis for $(a_1^{-1}R)^{(n)}$ and $x_1, n_2, \ldots, n_k$ generates $a_1^{-1}N$. Write $n_2 = b_1 x_1 + b_2 x_2 + \ldots + b_k x_k$. Since $x_1 \epsilon\ a_1^{-1}N$ we can replace n_2 by $b_2 x_2 + \ldots + b_k x_k$ where $b_2 \neq 0$ or $n_2 \epsilon\ a_1^{-1}Rn_1 = a_1^{-1}Rx_1$. If $b_2 \neq 0$ then over $b_2^{-1}a_1^{-1}R$ we have $x_1, b_2^{-1}n_2, x_3, \ldots, x_n$ is a basis for $(b_2^{-1}a_1^{-1}R)^{(n)}$ and $x_1, b_2^{-1}n_2, n_3, \ldots, n_k$ generates $b_2^{-1}a_1 N$ so replace x_2 and n_2 by $b_2^{-1}n_2$ over $b_2^{-1}a_1^{-1}R$. Then $N = (b_2^{-1}a_1^{-1}R)x_1 + (b_2^{-1}a_1^{-1}R)x_2 + (b_2^{-1}a_1^{-1}R)n_3 + \cdots + (b_2^{-1}a_1^{-1}R)n_k$. After finitely many steps we can find an element $c \epsilon R$ with $c^{-1}N$ a direct summand of $R^{(n)}$. Since c is in only finitely many maximal ideals of R by 1, $R_P \otimes N$ is a direct summand of $R_P^{(n)}$ for almost all $P \epsilon MaxSpec(R)$.

If R is an integral domain of finite character then a nonzero homomorphism $f : M \to N$ of R-progenerators induces an image split homomorphism $R_P \otimes M \xrightarrow{1 \otimes f} R_P \otimes N$ for almost all $P \epsilon MaxSpec(R)$ by 3 of Lemma 8. In this case the inclusions $R \to R_P$ induce a natural map $\phi : \mathcal{M}(R) \to \oplus_{P \epsilon MaxSpec(R)} \mathcal{M}(R_P)$

LEMMA 9. If R is a Prüfer domain of finite character and $\phi : \mathcal{M}(R) \to \oplus_{P \epsilon MaxSpec(R)} \mathcal{M}(R_P)$ is induced from the inclusions $R \to R_P$ then ϕ is a monomorphism.

PROOF: Let $0 \neq |f|$, $0 \neq |g|$ be in $\mathcal{M}(R)$ and assume $\phi(|f|) = \phi(|g|)$. Then $|1 \otimes f| = |1 \otimes g|$ in $\mathcal{M}(R_P)$ for all $P \epsilon MaxSpec(R)$. By Theorem 7, $0 \neq |f|$ determines a descending sequence $I_1 \supset I_2 \supset \ldots \supset I_n$ of ideals in R with $0 \neq I_j = (a_j, b_j)$ and $0 \neq |g|$ determines $J_1 \supset \ldots \supset J_m$ with $0 \neq J_k = (c_k, d_k)$. Under the isomorphism $\bar{\rho}$ of Lemma 6, $(\oplus_{j=1}^{n} I_j, R^{(n)})$ corresponds to $|f|$ and $(\oplus_{j=1}^{m} J_k, R^{(m)})$ corresponds to $|g|$. We show $(\oplus_{j=1}^{n} I_j, R^{(n)}) = (\oplus_{k=1}^{m} J_k, R^{(m)})$ in $\mathcal{A}$. Let P be a maximal ideal for which $|1 \otimes f| = |1 \otimes g|$

is split. In $\mathcal{A}(R_P)$ the split class is represented by (R_P, R_P) and projective modules are free over R_P so $(R^{(n)}/\oplus_{k=1}^n I_j)^{(\ell)} \otimes R_P \cong (0)$ which implies $I_j \subset P$ for $1 \leq j \leq n$. In the same way $J_k \subset P$ so $I_j R_P = R_P = J_k R_P$ for all i, j whenever $|1 \otimes f| = |1 \otimes g|$ is split over R_P. Consider the finite set (since R has finite character) $S = \{P \epsilon MaxSpec(R) || 1 \otimes f| \neq |1_{R_P}|\} = \{P \epsilon MaxSpec(R) || 1 \otimes g| \neq |1_{R_P}|\}$. Since $|\oplus_{j=1}^n R_P I_j, R_P^{(n)}| = |\oplus_{k=1}^m R_P J_k, R_P^{(m)}|$ in $\mathcal{A}(R_P)$ for each $P \epsilon S$ there are positive integers s_P, t_P with $s_P n = t_P m$ and $[\oplus_{j=1}^n R_P/R_P I_j]^{(s_P)} \cong [\oplus_{k=1}^m R_P/R_P J_k]^{(t_P)}$. Let $s = \Pi_{P \epsilon S} s_P$ and $t = \Pi_{P \epsilon S} t_P$. Then for each $P \epsilon S$, $\oplus_{j=1}^n [R_P/R_P I_j]^{(s)} = [\oplus_{j=1}^n R_P/R_P I_j]^{(s)} \cong [\oplus_{k=1}^m R_P/R_P J_k]^{(t)} = \oplus_{k=1}^m [R_P/R_P J_k]^{(t)}$. The ideals $R_P I_j$ and $R_P J_k$ in this decomposition are uniquely determined as sets (ii pg 260 of [12]). This means the list of ideals $\ldots R_P I_{j\ell}, \ldots, R_P I_{js}, \ldots$ and $\ldots R_P J_{k1}, \ldots, R_P J_{kt} \ldots$ where $I_{j\ell} = I_j$ and $J_{kw} = J_k$ are the same for each $P \epsilon S$. Therefore, for each j, ℓ,

$$\begin{aligned} I_{j\ell} &= \cap_{P \epsilon MaxSpec(R)} (I_{j\ell} R_P \cap R) \\ &= \cap_{P \epsilon MaxSpec(R)} (J_{kw} R_P \cap R) \qquad \text{for corresponding } k, w \\ &= J_{kw}. \end{aligned}$$

We have shown $[\oplus_{j=1}^m (R/I_j)]^{(s)} \cong [\oplus_{k=1}^n (R/J_k)]^{(t)}$ so $(R^{(m)}, \oplus_{j=1}^m I_j) = (R^{(n)}, \oplus_{k=1}^n J_k)$ in $\mathcal{A}$ and ϕ is a monomorphism.

We need two easy lemmas about valuations on Prüfer domains.

LEMMA 10. Let R be a Prüfer domain of finite character whose valuations at maximal ideals are pairwise independent. Let $P_1, \ldots, P_n$ be a finite set of maximal ideals of R, let G_i be the value group of R_{P_i} and let $0 < g_i \epsilon G_i (1 \leq i \leq n)$. Then there exists a finitely generated ideal I of R such that $IR_{P_i} = \{\alpha \epsilon R_{P_i} | v_{P_i}(\alpha) \geq g_i\}$ and the only maximal ideals of R containing I are $P_1, \ldots, P_n$.

PROOF: Since the valuations at the maximal ideals of R are pairwise independent, Theorem 22.9 of 7 implies there exists an $x \epsilon R_{P_1} \cap \ldots \cap R_{P_n}$ such that $v_{P_i}(x) = g_i (1 \leq i \leq n)$. As we observed in the proof of Lemma 3, we can choose $x \epsilon R$. Since R has finite character we can let $Q_1, \ldots, Q_m$ be all the maximal ideals of R distinct from $\{P_i\}_{i=1}^n$ such that $x \epsilon Q_j (1 \leq j \leq m)$. Again, Theorem 22.9 of [7] implies there are $y_i \epsilon R$ with $v_{P_i}(y_i) \geq g_i$ and $v_{Q_j}(y_i) = 0$ for all $j \neq i, k$. Let $I = (x, y_1, \ldots, y_m)$. Then I is finitely generated, $IR_{P_i} = xR_{P_i} = \{\alpha \epsilon R_{P_i} | v_{P_i}(\alpha) \geq g_i\}$ and the only maximal ideals of R containing I are $P_1, \cdots, P_n$.

LEMMA 11. Let v_1, v_2 be valuations on a field K with value groups G_1, G_2 and valuation rings V_1, V_2 respectively. If for each pair $(g_1, g_2) \epsilon G_1 \times G_2$ with $0 \leq g_1$ and $0 \leq g_2$ there is an $x \epsilon K$ with $v_1(x) = g_1$ and $v_2(x) = g_2$ then the valuation rings V_1, V_2 are independent.

PROOF: (See 9, pg 289 of [7]).

THEOREM 12. Let R be a Prüfer domain. The inclusion maps $R \to R_P$ induce an isomorphism $\phi : \mathcal{M}(R) \to \oplus_{P \epsilon MaxSpec(R)} \mathcal{M}(R_P)$ if and only if R is of finite character and the valuation rings at the maximal ideals of R are pairwise independent.

PROOF: Assume R is a Prüfer domain of finite character. Lemma 9 gives ϕ is a monomorphism. We check that if in addition the valuation rings at the maximal ideals of R are pairwise independent then ϕ is an epimorphism. Let $(|g_P|)_{P \epsilon MaxSpec(R)}$ be an element of $\oplus_{P \epsilon MaxSpec(R)} \mathcal{M}(R_P)$. Then g_P is an image split map for all but finitely many maximal ideals $P_1, \ldots P_k$ of R. Each $|g_{P_i}|$ can be represented by a diagonal matrix (Proposition 2). By tensoring these matrices with identity matrices of appropriate sizes we can assume each $|g_{P_i}|$ is represented by a diagonal $m \times n$ matrix. Let $|g_{P_i}|$ be represented by $diag(a_{i1}, \ldots, a_{im}) (1 \leq i \leq k)$. Let v_i be a valuation determined by the valuation ring R_{P_i} with value group G_i and let $g_{ij} = v_i(a_{ij}) (1 \leq i \leq k, 1 \leq j \leq m)$. Lemma 10 gives finitely generated ideals I_j contained

in exactly the maximal ideals $P_1, \ldots, P_k$ and $I_j R_{P_i} = \{\alpha \in R_{P_i} | v_i(\alpha) \geq g_{ij}\}$ for $1 \leq j \leq m$. Let $|f| \in \mathcal{M}(R)$ such that $|f|$ corresponds to the element $(\oplus_{j=1}^{m} I_j, R^{(m)})$ of $\mathcal{A} \cup \{0\}$ under the isomorphism $\bar{\rho}$ of Lemma 6. Then $\phi(|f|) = (|g_P|)_{P \in MaxSpec(R)}$ so ϕ is an epimorphism.

Conversely, assume the inclusion maps $R \to R_P$ for $P \in MaxSpec(R)$ induce the isomorphism ϕ. Let $0 \neq a \in R$ and let $\ell_a : R \to R$ by the homomorphism given by left multiplication by a. Then $|\ell_a| = |1_R|$ in $\mathcal{M}(R)$ if and only if a is a unit in R (Proposition 3(5) of [4]). The image of ϕ will lie in $\oplus_{P \in MaxSpec(R)} \mathcal{M}(R_P)$ only if the image of $|\ell_a|$ in $\mathcal{M}(R_P)$ is $|1_{R_P}|$ for almost all $P \in MaxSpec(R)$. This means $a \notin P$ for almost all $P \in MaxSpec(R)$ so R must have finite character. Let P, Q be maximal ideals of R and let $S = R_P \cap R_Q$. Since ϕ is an epimorphism, the induced map $\psi : \mathcal{M}(S) \to \mathcal{M}(R_P) \oplus \mathcal{M}(R_Q)$ is an epimorphism. To see the valuation rings R_P and R_Q are independent we check the condition of Lemma 11. Let v_P and v_Q be the valuations corresponding to valuation rings R_P and R_Q with value groups G_P, G_Q. Let $0 \leq g \in G_P$ and $0 \leq h \in G_Q$ and let $a \in R_P$, $b \in R_Q$ with $v_P(a) = g$, $v_Q(b) = h$. Since ψ is onto, there is an $|h| \in \mathcal{M}(S)$ with $|1 \otimes h| = |\ell_a|$ in $\mathcal{M}(R_P)$ and $|1 \otimes h| = |\ell_b|$ in $\mathcal{M}(R_Q)$. Since S is a semilocal Bezout domain, S is an elementary divisor domain [9] so we can represent h by the diagonal matrix $\mathrm{diag}(c_1, \ldots, c_k)$. As we saw in the proof of Proposition 2, we can find units $u_j \in R_P$ and $w_j \in R_Q$ such that $c_j u_j = a$ in R_P and $c_j = w_j b$ in $R_Q (1 \leq j \leq k)$. Thus $v_P(c_1) = v_P(u_1 a) = v_P(a)$ and $v_P(c_1) = v_Q(w_1 b) = v_Q(b)$ which shows the valuation rings R_P and R_Q are pairwise independent.

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