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ON THE IDEALS OF EXTENDED QUASI-NILPOTENT BANACH ALGEBRAS

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ABSTRACT. Given a quasi-nilpotent Banach algebra A, we will use the results of Seddighin [2], to study the properties of elements which belong to a proper closed two sided ideal of $\bar{A}$ and $\bar{\bar{A}}$. Here $\bar{A}$ is the extension of A to a Banach Algebra with identity.

KEY WORDS AND PHRASES. Banach Algebra, Spectrum, Quasi-Nilpotent Banach algebras, nonperturbing element.
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1. INTRODUCTION.

Let A be a Banach Algebra and I be a closed two sided ideal in A. If b is any element in I and a is any element in A, it is easy to see that $\sigma(a+b) \cap \sigma(a) \neq \theta$. In fact if $\sigma(\bar{a})$ is the spectrum of the image of a in the quotient A/I, then we have

$$\sigma(\bar{a}+\bar{b}) = \sigma(\bar{a}),\ \sigma(\bar{a}) \subseteq \sigma(a) \text{ and } \sigma(\bar{a}+\bar{b}) \subseteq \sigma(a+b).$$

An element b for which

$$\sigma(a+b) \cap \sigma(a) \neq \theta$$

for all elements a in A, is called a nonperturbing element of the Banach algebra A. Thus any element of any proper closed two sided ideal in a Banach algebra A is nonperturbing. Aiken [1] has proved that the converse is not true. i.e., there exist Banach algebras in which nonperturbing elements are not necessarily in proper closed two sided ideals. In the following we will show however, that for certain Banach algebras A, the nonperturbing elements of $\bar{A}$ must belong to a proper closed two sided ideal of $\bar{A}$. Note that for a Banach algebra A, the extended Banach algebra $\bar{A}$ is definded to be the set $\bar{A} = \{(x,\alpha) : x \varepsilon A, \alpha \text{ complex}\}$ together with the operations

$$(x,\alpha) + (y,\beta) = (x + y, \alpha + \beta) \text{ and}$$

$$(x,\alpha)(y,\beta) = (xy + \beta x + \alpha y, \alpha\beta) \text{ and norm } ||(x,\alpha)|| = ||x|| + |\alpha|.$$

2. MAIN RESULTS.

THEOREM 2.1. If $y = (x,\alpha) \varepsilon \bar{A}$ is nonperturbing and $r(x) < |\alpha|$, then (x,α) is in a proper two sided closed ideal. Here r(x) is the spectral radius of x in A.

PROOF. Choose the element $z = (0,\beta)$, where β is any complex number. Suppose $|\alpha| > 0$; then it follows from Seddighin [2] Theorem 2.1, that $\sigma(z) = \{\beta\}$ and $\sigma(z + y) = \sigma((x,\alpha + \beta))$ is a subset of the circle with center at $\alpha + \beta$ and radius r(x). Now since

$$r(x) < |\alpha| \text{ and } |\alpha + \beta - \beta| = |\alpha|,$$

β lies outside the circle with center at $\alpha + \beta$ and radius r(x). This shows that $\sigma(z+y) \cap \sigma(z) \neq \theta$, a contradiction to the fact that y is nonperturbing. Therefore, we must have $|\alpha| = 0$; i.e., $\alpha = 0$ (and hence $r(x) = 0$, also). Thus, y has the form y = (x,0), and so it belongs to the ideal

$$I = \{(x,0) : x \varepsilon A\}$$

COROLLARY 2.2. Let A be a Banach algebra such that each of its elements is quasi-nilpotent, then $\bar{A}$ has the property that each of its nonperturbing elements belongs to a proper closed two sided ideal.

REMARK. For an example of a Banach algebra in which every element is quasi-nilpotent, see Bonsall and Duncan [3].

THEOREM. 2.3. If each element in a Banach algebra A is qûasi-nilpotent, then the sets

$$I_1 = \{((a,\alpha),0) : a \varepsilon A, \alpha \text{ complex}\} \text{ and}$$

$$I_2 = \{((a,\alpha),\beta) : a \varepsilon A \text{ and } \beta = -\alpha\}$$

are proper two sided ideals in $\bar{\bar{A}}$, and each nonperturbing element of $\bar{\bar{A}}$ is either in I_1 or I_2.

PROOF. Let $x = ((f,\alpha_1), \alpha_2)$ and $y = ((g,\beta_1), \beta_2)$. Since each element of A quasi-nilpotent, by Seddighin [2], theorem 2.1 we have

$$\sigma_{\bar{A}}((f,\alpha_1)) = \alpha_1 \text{ and } \sigma_{\bar{A}}((g,\beta_1)) = \beta_1.$$

$\bar{A}$ has a unit. Thus again by Seddighin [2], Theorem 2.1, we have $\sigma(x) \subseteq \{a_2\} \cup \{\alpha_1 + \alpha_2\}$ and $\sigma(y) \subseteq \{\beta_2\} \cup \{\beta_1 + \beta_2\}$ and $\sigma(x + y) \subseteq \{\alpha_2 + \beta_2\} \cup \{\alpha_1 + \beta_1 + \alpha_2 + \beta_2\}$. Now $\sigma(x) \cap \sigma(x + y) \neq \theta$

implies that $\{\alpha_2 + \beta_2\} \cup \{\alpha_1 + \beta_1 + \alpha_2 + \beta_2\} \cap \{\beta_2\} \cup \{\beta_1 + \beta_2\} \neq \theta$,

which implies

$$\alpha_2 + \beta_2 = \beta_2 \text{ or } \alpha_2 + \beta_2 = \beta_1 + \beta_2 \text{ or } \alpha_1 + \beta_1 + \alpha_2 + \beta_2 = \beta_2$$

or

$$\alpha_1 + \beta_1 + \alpha_2 + \beta_2 = \beta_1 + \beta_2$$

Hence, $\sigma(y) \cap \sigma(x + y) \neq \theta$ implies $\alpha_2 = 0$ or $\beta_1 = \alpha_2$ or $\alpha_1 = -\beta_2 - \alpha_2$ or $\alpha_1 = -\alpha_2$. If $((f, \alpha_1), \alpha_2)$ is neither in I_1 nor I_2, then $\alpha_1 \neq -\alpha_2$ and $\alpha_2 \neq 0$. Therefore, $\sigma(y) \cap \sigma(x + y) \neq \theta$ for all $y = ((g, \beta_1), \beta_2)$ in $\bar{A}$ implies $\beta_1 = \alpha_2$ or $\beta_1 = -\alpha_1 - \alpha_2$ for all β. This is a contradiction, since for any element $y = ((g, \beta_1), \beta_2)$ with $\beta_1 \neq \alpha_2$ and $\beta_1 \neq -\alpha_1 - \alpha_2$ we have $\sigma(y) \cap \sigma(x + y) \neq \theta$. Hence $x \in I_1$ or $x \in I_2$. I_1 is trivially an ideal. To show that I_2 is an ideal, let $((f, \alpha_1), \alpha_2)$ be an element of $I_2$and $((g, \beta_1), \beta_2)$ be any element of $\bar{A}$, then $\alpha_1 = -\alpha_2$ and $((g, \beta_1), \beta_2)\ ((f, \alpha_1), \alpha_2) = ((g, \beta_1)(f, \alpha_1) + \alpha_2(g, \beta_1) + \beta_2(f, \alpha_1), \alpha_2\beta_2) = ((gf + \beta_1 f + \alpha_1 g + \alpha_2 g + \beta_2 f,\ \beta_1\alpha_1 + \alpha_2\beta_1 + \beta_2\alpha_1), \alpha_2\beta_2)$. Now $\alpha_1 = -\alpha_2$ implies that $\beta_1\alpha_1 + \alpha_2\beta_1 + \beta_2\alpha_1 = -\alpha_2\beta_1 + \alpha_2\beta_1 - \alpha_2\beta_2 = -\alpha_2\beta_2$, which shows that $((g, \beta_1), \beta_2)\ ((f, \alpha_1), \alpha_2)$ is in I_2. Similarly, $((f, \alpha_1), \alpha_2)\ ((g, \beta_1), \beta_2)$ is in I_2.

REMARK. In most Banach algebras, the class of all nonpertubing elements actually forms an ideal. i.e. the set

$$L = \{y : \sigma(x + y) \cap \sigma(x) \neq \text{ for all } x \text{ in } A\}$$

is an ideal. A fine example is the algebra B(H) of bounded operators on a separable Hilbert space, in which the set of nonperturbing elements is exactly the ideal consisting of compact operators (see Dyer, Parcelli, and Rosenfeld [4]).

However, the proof of theorem 2.3 shows that although each nonperturbing element of the algebra $\bar{A}$ in that theorem belongs to a proper closed two sided ideal, the set of nonperturbing elements is not an ideal. To see this consider the elements $u = ((x,1),0)$ and $v = ((x,1),-1)$. u and v are nonperturbing since $u \in I_1$ and $v \in I_2$, but $u-v = ((0,0),1)$ which is the identity element of $\bar{A}$.

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