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ON STABILITY OF ADDITIVE MAPPINGS

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ABSTRACT. In this paper we answer a question of Th. M. Rassias concerning an extension of validity of his result proved in [3].

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1. INTRODUCTION.

In connection with a problem posed by Ulam (cf. [5]; see also [2]) Th. M. Rassias [3] proved the following theorem on stability of linear mappings in Banach spaces.

THEOREM 1. (see [3]) Let E_1 and E_2 be two (real) Banach spaces and let $f: E_1 \to E_2$ be a mapping such that for each fixed $x \varepsilon E_1$ the transformation $\mathbb{R} \ni t \to f(tx)$ is continuous. Moreover, assume that there exist $\varepsilon \in [0, \infty)$ and $p \varepsilon [0, 1\}$ such that

$$\| f(x+y) - f(x) - f(x) \| \leq \varepsilon(\| x \|^p + \| y \|^p) \tag{1.1}$$

for all $x, y \varepsilon E_1$. Then there exists a unique linear mapping $T: E_1 \to E_2$ such that

$$\| f(x) - T(x) \| \leq \delta \| x \|^p \tag{1.2}$$

for all $x \varepsilon E_1$, where $\delta := \frac{2\varepsilon}{2 - 2^p}$.

As was mentioned by Th. M. Rassias [4], the proof presented in [3] reveals that, in fact, it works for every p from the interval $(-\infty, 1)$ and, therefore, the theorem holds true for all such p's. It is also readily seen that the only purpose of assuming that all the transformations of the form $t \to f(tx)$ are continuous is to guarantee the real homogeneity of the mapping T. Without this assumption one can show that f is approximated by an additive mapping T which means that T satisfies the following equation

$$T(x+y) = T(x) + T(y) \tag{1.3}$$

for all $x, y \varepsilon E_1$. Finally, it should be noticed that the completeness of the space E_1 may be removed from the assumptions of Theorem 1. However, there is still one non-trivial (as it seems) question concerning a possible extension of the range of validity of Theorem 1. Namely, one can ask whether the same result holds true under the hypothesis that p is taken from the interval $[1, \infty)$ (obviously in this case the constant δ should have been defined in a different manner). Such a problem was raised by Th. M. Rassias during the 27th International Symposium on Functional Equations which was held in Bielsko-Biala, Katowice and Krokow in August 1989. The goal of the present note is to give a complete solution to this problem.

2. MAIN RESULTS.

First, let us realize why the proof of Theorem 1 in its original form (see [3]) does not work for $p \geq 1$. The fundamental role in this proof is played by the sequence

$$\left\{\frac{1}{2^n} f(2^n x) \colon n \varepsilon \mathbb{N}\right\} \tag{2.1}$$

which, under the assumptions of Theorem 1 (in fact as long as $p \varepsilon (-\infty, 1)$) is convergent for each fixed $x \varepsilon E_1$. Then $T \colon E_1 \to E_2$ defined by the formula

$$T(x) \colon = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x), \qquad x \varepsilon E_1 \tag{2.2}$$

is the desired linear mapping approximating f. The argument ensuring the convergence of sequence (2.1) is no longer valid when p becomes greater or equal to 1, so in order to carry the proof over to this case, one has to change the argument itself or the definition of the mapping T. It turns out that, for $p > 1$, the latter modification of the proof is possible. As a result we obtain the following extension of Theorem 1:

THEOREM 2. Let E_1 and E_2 be two (real) normed linear spaces and assume that E_2 is complete. Let $f \colon E_1 \to E_2$ be a mapping for which there exist two constants $\varepsilon \in [0, \infty)$ and $p \in \mathbb{R} \backslash \{1\}$ such that

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon(\| x \|^p + \| y \|^p) \tag{2.3}$$

for all $x, y \varepsilon E_1$. Then there exists a unique additive mapping $T \colon E_1 \to E_2$ such that

$$\| f(x) - T(x) \| \leq \delta \| x \|^p \tag{2.4}$$

for all $x \varepsilon E_1$, where

$$\delta = \begin{cases} \dfrac{2\varepsilon}{2 - 2^p} & \text{for } p < 1, \\ \dfrac{2\varepsilon}{2^p - 2} & \text{for } p > 1. \end{cases}$$

Moreover, is for each $x \in E_1$ the transformation $\mathbb{R} \ni t \to f(tx)$ is continuous, then the mapping T is linear.

PROOF. In view of what has been said so far, it remains to consider the case $p > 1$. The main innovation in comparison with the case $p < 1$ consists in defining the mapping T by the formula

$$T(x) \colon = \lim_{n \to \infty} 2^n f(\tfrac{x}{2^n}), \qquad x \varepsilon E_1 \tag{2.5}$$

instead of (2.2). Obviously, one has to verify the convergence of the sequence occurring on the right-hand side of (2.5).

Putting $\frac{x}{2}$ in place of x and y in inequality (2.3), we obtain

$$\| f(x) - 2\, f(\tfrac{x}{2}) \| \leq 2\varepsilon \| \tfrac{x}{2} \|^p = 2^{1-p} \varepsilon \| x \|^p$$

for all $x \in E_1$. Hence for each $n \varepsilon \mathbb{N}$ and every $x \in E_1$, we have

$$\| f(x) - 2^n f(\tfrac{x}{2^n}) \| \leq \| f(x) - 2f(\tfrac{x}{2}) \| + 2 \| f(\tfrac{x}{2}) - 2f(\tfrac{x}{2^2}) \| + \dots + 2^{n-1} \| f(\tfrac{x}{2^{n-1}}) - 2f(\tfrac{x}{2^n}) \|$$

$$\leq 2^{1-p} \varepsilon \| x \|^p + 2 \cdot 2^{1-p} \varepsilon \| \tfrac{x}{2} \|^p + \dots + 2^{n-1} . 2^{1-p} \varepsilon \| \tfrac{x}{2^{n-1}} \|^p$$

$$= (2^{1-p} + 2^{2(1-p)} + \dots + 2^{n(1-p)}) \varepsilon \| x \|^p$$

$$\leq \delta \| x \|^p, \tag{2.6}$$

where δ is the sum of the following convergent series:

$$\sum_{n=1}^{\infty} 2^{n(1-p)}\varepsilon = \frac{2\varepsilon}{2^p - 2}.$$

Now, fix an $x \varepsilon E_1$ and choose arbitrary $m, n \varepsilon \mathbb{N}$ such that $m > n$. Then

$$\| 2^m f(\tfrac{x}{2^m}) - 2^n f(\tfrac{x}{2^n}) \| = 2^n \| 2^{m-n} f(\tfrac{1}{2^{m-n}} \cdot \tfrac{x}{2^n}) - f(\tfrac{x}{2^n}) \|$$

$$\leq 2^n \delta \| \tfrac{x}{2^n} \|^p = 2^{n(1-p)} \delta \| x \|^p,$$

which becomes arbitrarily small as $n \to \infty$. On account of the completeness of the space E_2, this implies that the sequence $\{2^n f(\frac{x}{2^n}) : n \varepsilon \mathbb{N}\}$ is convergent for each $x \in E_1$. Thus T is correctly defined by (2.5). Moreover, it satisfies condition (2.4) which results on letting $n \to \infty$ in (2.6).

Finally, replacing x by $\frac{x}{2^n}$ and y by $\frac{y}{2^n}$ in (2.3) and then multiplying both sides of the resulting inequality by 2^n, we get

$$\| 2^n f(\tfrac{x+y}{2^n}) - 2^n f(\tfrac{x}{2^n}) - 2^n f(\tfrac{y}{2^n}) \| \leq 2^{n(1-p)} \varepsilon (\| x \|^p + \| y \|^p),$$

for $x, y \in E_1$. Since the right-hand side of this inequality tends to zero as $n \to \infty$, it becomes apparent that the mapping T defined by (2.5) is additive.

The proof of the homogeneity of T (under the supplementary assumption that $t \to f(tx)$ is continuous for each $x \in E_1$) needs no essential alterations in comparison with the case $p < 1$. It is also clear what has to be changed in the proof of the uniqueness of T.

Theorem 2 leaves the case $p = 1$ undecided. This is not a mere coincidence. It turns out that 1 is the only critical value of p to which Theorem 2 can not be extended. In fact, we shall show that $\varepsilon > 0$ one can find a function $f: \mathbf{R} \to \mathbf{R}$ such that

$$| f(x+y) - f(x) - f(y) | \leq \varepsilon(| x | + | y |) \tag{2.7}$$

for all $x, y \varepsilon \mathbf{R}$, but, at the same time, there is no constant $\delta \in [0, \infty)$ and no additive function $T: \mathbf{R} \to \mathbf{R}$ satisfying the condition

$$| f(x) - T(x) | \leq \delta | x | \qquad \text{for all } x \varepsilon \mathbf{R}. \tag{2.8}$$

This singularity is illustrated by the following:

EXAMPLE. Fix $\varepsilon > 0$ and put $\mu := \frac{\varepsilon}{6}$. First we define a function $\phi: \mathbf{R} \to \mathbf{R}$ by

$$\phi(x) := \begin{cases} \mu & \text{for } x \varepsilon [1, \infty), \\ \mu x & \text{for } x \varepsilon (-1, 1), \\ -\mu & \text{for } x \varepsilon (-\infty, -1]. \end{cases}$$

Evidently, ϕ is continuous and $| \phi(x) | \leq \mu$ for all $x \varepsilon \mathbf{R}$. Therefore, a function $f: \mathbf{R} \to \mathbf{R}$ is correctly defined by the formula

$$f(x) := \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \qquad x \in \mathbf{R}.$$

Since f is defined by means of a uniformly convergent series of continuous functions, f itself is continuous. Moreover,

$$| f(x) | \leq \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\mu, \qquad x \in \mathbf{R}.$$

We are going to show that f satisfies (2.7).

If $x = y = 0$, then (2.7) is trivially fulfilled. Next assume that $0 < |x| + |y| < 1$. Then there exists an $N \varepsilon \mathbb{N}$ such that

$$2^N \leq |x| + |y| < \frac{1}{2^{N-1}}.$$

Hence, $|2^{N-1}x| < 1$, $|2^{N-1}y| < 1$ and $|2^{N-1}(x+y)| \leq 2^{N-1}(|x| + |y|) < 1$, which implies that for each $n \in \{0,1,..,N-1\}$ the numbers $2^n x$, $2^n y$ and $2^n(x+y)$ remain in the interval $(-1,1)$. Since ϕ is linear on this interval, we infer that

$$\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y) = 0$$

for $n = 0,1,...,N-1$. As a result, we get

$$\frac{|f(x+y) - f(x) - f(x)|}{(|x| + |y|)} \leq \sum_{n=N}^{\infty} \frac{|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|)}$$
$$\leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k 2^N(|x| + |y|)} \leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k} = 6\mu = \varepsilon.$$

Finally, assume that $|x| + |y| \geq 1$. Then merely by virtue of the boundedness of f we have

$$\frac{|f(x+y) - f(x) - f(Y)|}{|x| + |y|} \leq 6\mu = \varepsilon.$$

Thus we conclude that f satisfies (2.7) for all real x and y.

Now, contrary to what we claim, suppose that there exist a $\delta \in [0,\infty)$ and an additive function $T: \mathbb{R} \to \mathbb{R}$ such that (2.8) holds true. Hence, from the continuity of f it follows that T is bounded on some neighbourhood of zero. Then, by a classical result (see e.g. [1], 2.1.1., Theorem 1) there exists a real constant c such that

$$T(x) = cx, \qquad x \in \mathbb{R}$$

Hence,

$$|f(x) - cx| \leq \delta |x|, \qquad x \in \mathbb{R},$$

which implies that

$$\left|\frac{f(x)}{x}\right| \leq \delta + |c|, \qquad x \in \mathbb{R}.$$

On the other hand, we can choose an $N \varepsilon \mathbb{N}$ so large that $N\mu > \delta + |x|$.' Then picking out an x from the interval $(0, \frac{1}{2^{N-1}})$, we have $2^n x \in (0,1)$ for each $n \in \{0,1,...,N-1\}$. Consequently, for such an x we have

$$\frac{f(x)}{x} \geq \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n x} = \sum_{n=0}^{\infty} \frac{\mu 2^n x}{2^n x} = N\mu > \delta + |x|,$$

which yields a contradiction. Thus the function f provides a good example to the effect that Theorem 2 fails to hold for $p = 1$.

REFERENCES

1. ACZEL, J. Lectures on Functional Equations and their Applications, Academic Press, New York - San Francisco - London, 1966.
2. HYERS, D.H. On the stability of the linear functional equation, Proc. Nat. Acad. Sci., U.S.A., 27 (1941), 222-224.
3. RASSIAS, TH. M. On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.
4. RASSIAS, TH. M. Communication, 27th International Symposium on Functional Equations, Bielsko-Biala, Katowice, Krokow, Poland, 1989.
5. ULAM, S.M. Problems in modern mathematics, Chapter VI, Science Editions, Wiley, New York, 1960.