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MEASURE CHARACTERIZATIONS AND PROPERTIES OF NORMAL AND REGULAR LATTICES

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ABSTRACT. Various equivalent characterizations of normality are considered and a measure theoretic definition is given for strongly normal lattices. Measure conditions related to the space of σ-smooth, lattice-regular, 0-1 measures are noted which imply, or are equivalent to, the space being Hausdorff, regular, or prime complete.

KEY WORDS AND PHRASES. Lattice, zero-one measures, sigma-smooth, normal, regular, Hausdorff, prime complete, strongly normal, filter, disjunctive.
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1. INTRODUCTION AND BACKGROUND.

Our concerns in this paper are two-fold. First, we wish to consider in detail and from a measure theoretic point of view, characterizations of normal and strongly normal lattices. This is desirable since it is more natural when pairs of lattices are considered, one of which is contained in the other, to consider measure extensions and restrictions. Also, many results hold equally well for a more general measure than 0-1 valued measures.

Second, we initiate in some detail a study of reflections of lattice properties to the Wallman replete space $I_R^\sigma(\mathbf{L})$ (see below for definitions), and conversely how properties of this space reflect back to the underlying lattice. This space, except in special topological cases, has not been throughly investigated and is not as well-understood as the compact Wallman space $I_R(\mathbf{L})$.

We begin with a brief review of the relevant Lattice definitions, and the related topological spaces involved. Our notation and terminology is consistent with those in the literature (see e.g. Grassi [1], Huerta [2], Nöbeling [3], and Szeto [4,5]. Further details concerning $I_R^\sigma(\mathbf{L})$ can be found in (Grassi [1,7] and Frolik [6], [4]) but these will not be necessary for reading this paper.

2. NOTATION AND TERMINOLOGY.

We shall let $\mathbf{L}$ denote a lattice of subsets of a set X and shall assume that the empty set and X are in $\mathbf{L}$. $\mathcal{A}(\mathbf{L})$ and $\sigma(\mathbf{L})$ denote the algebra and σ-algebra respectively generated by $\mathbf{L}$. If $\mathbf{L}$ is closed under countable intersections then $\mathbf{L}$ is said to be a δ-lattice. $\tau\mathbf{L}$ will denote the lattice consisting of arbitrary intersections of elements of $\mathbf{L}$. $\mathbf{L}$ is said to be normal if whenever $A,B \in \mathbf{L}$ such that $A \cap B = \phi$, there exist $C,D \in \mathbf{L}$ such that $A \subset C'$, $B \subset D'$ and $C' \cap D' = \phi$. Equivalently, $\mathbf{L}$ is normal if for all $L \in \mathbf{L}$, if $L \subset L_1' \cup L_2'$ where L_1 and $L_2 \in \mathbf{L}$ then there exist A_1, $A_2 \in \mathbf{L}$ such that $L = A_1 \cup A_2$ and $A_1 \subset L_1'$, $A_2 \subset L_2'$. $\mathbf{L}$ is regular if for each $x \in X$ and $A \in \mathbf{L}$ such that $x \notin A$, there exist $B,C \in \mathbf{L}$ with $x \in B'$, $A \subset C'$ and $B' \cap C' = \phi$. Similar definitions apply to $\mathbf{L}$ being separating and T_2, which are analogous to the topological definitions, replacing closed sets with lattice sets. $\mathbf{L}$ is complement generated if for all $L \in \mathbf{L}$, $L = \bigcap_{n=1}^{+\infty} A_n'$, $A_n \in \mathbf{L}$. $\mathbf{L}$ is disjunctive if for all $x \in X$ and $A \in \mathbf{L}$ with $x \notin A$, there exists $B \in \mathbf{L}$ such that $x \in B$ and $A \cap B = \Phi$. $\mathbf{L}$ is countably paracompact if whenever $\{A_n\}$ is a decreasing sequence of lattice sets in which $\bigcap_{n=1}^{+\infty} A_n = \phi$, there exists a decreasing sequence of $\mathbf{L}'$ sets $\{B_n'\}$ such that $A_n \subset B_n'$ for all n and $\bigcap_{n=1}^{+\infty} B_n' = \phi$.

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are lattices of subsets of X and $\mathbf{L}_1 \subset \mathbf{L}_2$ then $\mathbf{L}_1$ separates $\mathbf{L}_2$ if whenever $A,B \in \mathbf{L}_2$ such that $A \cap B = \phi$, there exist $C,D \in \mathbf{L}_1$ such that $A \subset C$, $B \subset D$ and $C \cap D = \phi$; $\mathbf{L}_1$ semiseparates $\mathbf{L}_2$ if whenever $A \in \mathbf{L}_1$ and $B \in \mathbf{L}_2$ such that $A \cap B = \phi$, there exists $C \in \mathbf{L}_1$ such that $B \subset C$ and $A \cap C = \phi$. If $\mathbf{L}_1$ separates $\mathbf{L}_2$ then $\mathbf{L}_1$ is normal if and only if $\mathbf{L}_2$ is normal.

$\Pi(\mathbf{L})$ will denote the set of all premeasures on $\mathbf{L}$, and $I(\mathbf{L})$ the set of all 0-1 finitely additive measures defined on $A(\mathbf{L})$. $I_R(\mathbf{L})$ will denote the subset of $I(\mathbf{L})$ consisting of all 0-1 $\mathbf{L}$-regular measures and $I_R^\sigma(\mathbf{L})$ that subset of $I_R(\mathbf{L})$ consisting of σ-smooth, 0-1, $\mathbf{L}$-regular measures. $I_\sigma(\mathbf{L})$ denotes those measures in $I(\mathbf{L})$ which are σ-smooth on $\mathbf{L}$. $M(\mathbf{L})$ denotes the set of all finitely additive measures defined on $A(\mathbf{L})$. Without loss of generality, we assume that these measures are non-negative. We note that there is a one-to-one correspondence between filters on $\mathbf{L}$ and premeasures on $\mathbf{L}$, between prime filters on $\mathbf{L}$ and measures in $I(\mathbf{L})$, and between $\mathbf{L}$-ultrafilters and measures in $I_R(\mathbf{L})$. Furthermore, a prime filter on $\mathbf{L}$ has the countable intersection property if and only if the corresponding measure is in $I_\sigma(\mathbf{L})$. If $\mu \in M(\mathbf{L})$, $S(\mu)$ denotes the support of μ. $\mathbf{L}$ is said to be replete (prime complete) if $S(\mu) \neq \phi$ for all $\mu \in I_R^\sigma(\mathbf{L})$ ($\mu \in I_\sigma(\mathbf{L})$). If μ, $\nu \in M(\mathbf{L})$ (or $\Pi(\mathbf{L})$ we will write $\mu \leq \nu(\mathbf{L})$ whenever $\mu(L) \leq \nu(L)$ for all $L \in \mathbf{L}$. It is well-known that if $\mu \in I(\mathbf{L})$ then there exists a $\nu \in I_R(\mathbf{L})$ such that $\mu \leq \nu(\mathbf{L})$. Also, $\mathbf{L}$ is T_2 if and only if for every $\mu \in I(\mathbf{L})$, $S(\mu) = \phi$ or a singleton; $\mathbf{L}$ is regular if and only if whenever $\mu \leq \nu(\mathbf{L})$, μ, $\nu \in I(\mathbf{L})$, $S(\mu) = S(\nu)$.

A mapping $T:X \to R$, where R is the real numbers with the usual topology, is L-continuous if $T^{-1}(C) \in \mathbf{L}$ for all closed sets C of R. $Z(\mathbf{L})$ denotes the lattice of all zero-sets of L-continuous functions (i.e. $Z \in Z(\mathbf{L})$ if and only if there exists an L-continuous function T such that $Z = T^{-1}(\{0\})$). If X is a topological space, F will denote the closed sets, $\mathbf{O}$ the open sets, and Z the zero sets of X. When discussing the zero sets of a topological space X, we will always assume, without explicit mention, that X is completely regular.

For $L \in \mathbf{L}$, $W(L) = \{\mu \in I_R(\mathbf{L}) \mid \mu(L) = 1\}$, and $W_\sigma(L) = \{\mu \in I_R^\sigma(\mathbf{L}) \mid \mu(L) = 1\}$. $W(\mathbf{L}) = \{W(L) \mid L \in \mathbf{L}\}$, and $W_\sigma(\mathbf{L}) = \{W_\sigma(L) \mid L \in \mathbf{L}\}$. The space $I_R(\mathbf{L})$, with $\tau W(\mathbf{L})$ as the closed sets (i.e. the Wallman topology) is a compact, T_1 topological space. If $\mathbf{L}$ is disjunctive then $I_R(\mathbf{L})$ is T_2 if and only if $\mathbf{L}$ is normal. Also, if $\mathbf{L}$ is disjunctive then $I_R^\sigma(\mathbf{L})$ is $W_\sigma(\mathbf{L})$-replete and is $W_\sigma(\mathbf{L})$-disjunctive.

3. ON NORMAL LATTICES.

In this section we consider a variety of new characterizations of normal lattices from a measure theoretic point of view, as well as consequences of a lattice being normal. Also, the concept of a strongly normal lattice is introduced.

We begin with the following equivalent characterizations of normality. A. Koltun (unpublished) proved that statements 2 and 3 below are equivalent. We will essentially use his proof to show that 2 implies 3.

THEOREM 3.1. Let $\nu \in I_R(\mathbf{L})$ and $\rho \in I_R(\mathbf{L}')$ where $\nu \le \rho(\mathbf{L}')$. For any $E \subset X$ define $\nu'(E) = \inf \nu(L')$ and $\rho'(E) = \inf \rho(L')$ where $E \subset L'$, $L \in \mathbf{L}$. The following are then equivalent:

1. $\nu' = \rho'$ on $\mathbf{L}$.
2. For each $\mu \in I(\mathbf{L})$, there exists a unique $\mu_1 \in I_R(\mathbf{L})$ such that $\mu \le \mu_1(\mathbf{L})$.
3. $\mathbf{L}$ is normal.

PROOF. (1 implies 2.) Suppose $\nu' = \rho'$ on $\mathbf{L}$. Let $\mu \le \mu_1(\mathbf{L})$ and $\mu \le \mu_2(\mathbf{L})$ where $\mu \in I(\mathbf{L})$ and $\mu_1, \mu_2 \in I_R(\mathbf{L})$. Then there exists a $\rho \in I_R(\mathbf{L}')$ such that $\mu \le \rho(\mathbf{L}')$. Therefore $\rho \le \mu \le \mu_1(\mathbf{L})$ and $\rho \le \mu \le \mu_2(\mathbf{L})$. Therefore $\mu_1 = \mu_2$ on $\mathbf{L}$ and hence $\mu_1 = \mu_2$. (2 implies 3.) Suppose $\mathbf{L}$ is not normal. Then there exists $A_1, A_2 \in \mathbf{L}$ such that $A_1 \cap A_2 = \phi$ and $\mathbf{H} = \{B' \in L' \mid A_1 \subset B' \text{ or } A_2 \subset B'\}$ has the finite intersection property. Therefore, there exists $\mu \in I(\mathbf{L})$ such that $\mu(B') = 1$ for all $B' \in \mathbf{H}$. If $B \in \mathbf{L}$ such that $\mu(B) = 1$ then it follows that $A_1 \cap B \ne \phi$ and $A_2 \cap B \ne \phi$. Therefore, there exist $\mu_1, \mu_2 \in I_R(\mathbf{L})$ such that $\mu_1(A_1) = 1$, $\mu \le \mu_1(\mathbf{L})$ and $\mu_2(A_2) = 1$, $\mu \le \mu_2(\mathbf{L})$. Clearly $\mu_1 \ne \mu_2$. (3 implies 1.) Suppose $\nu'(A) = \nu(A) = 0$, $A \in \mathbf{L}$. Then $\nu(A') = 1$. Since $\nu \in I_R(\mathbf{L})$, there exists $L \subset A'$, $L \in \mathbf{L}$, such that $\nu(L) = 1$. Since $\mathbf{L}$ is normal, there exist $C, D \in \mathbf{L}$ such that $A \subset C' \subset D \subset L'$. Therefore, $\rho(C') \le \rho(D) \le \nu(D) = 0$ and hence $\rho'(A) = 0$.

We will now define what appears to be a condition weaker than $\mathbf{L}_1$ separates $\mathbf{L}_2$, but we shall show they are equivalent.

DEFINITION 3.1. $\underline{\mathbf{L}_1}$ <u>partly separates</u> $\underline{\mathbf{L}_2}$ if whenever $A, B \in \mathbf{L}_2$ and $A \subset B'$, there exists $C \in \mathbf{L}_1$ such that $A \subset C \subset B'$.

LEMMA 3.1. $\mathbf{L}_1$ partly separates $\mathbf{L}_2$ if and only if L_1 separates L_2.

PROOF. Clearly, if $\mathbf{L}_1$ separates $\mathbf{L}_2$ then $\mathbf{L}_1$ partly separates $\mathbf{L}_2$. Suppose $\mathbf{L}_1$ partly separates $\mathbf{L}_2$. Then $\mathbf{L}_1$ semiseparates $\mathbf{L}_2$. Therefore the restriction map $\Psi: I_R(\mathbf{L}_2) \to I_R(\mathbf{L}_1)$ is well defined. If $\mathbf{L}_1$ does not separate $\mathbf{L}_2$ then Ψ is not one-to-one. Let $\nu_1, \nu_2 \in I_R(\mathbf{L}_2)$, be such that $\nu_1 \ne \nu_2$ and $\Psi(\nu_1) = \Psi(\nu_2) = \mu$. Since $\nu_1 \ne \nu_2$ there exist $A, B \in \mathbf{L}_2$ with $A \cap B = \phi$ such that

$v_1(A) = 1$, $v_2(A) = 0$, $v_1(B) = 0$, and $v_2(B) = 1$. Since $\mathbf{L}_1$ partly separates $\mathbf{L}_2$, there exists $C \in \mathbf{L}_1$ such that $A \subset C \subset B'$. Therefore, $v_1(C) = 1$ so $\mu(C) = 1$. But $\mu(C) = v_2(C) = 0$, a contradiction.

We now present a characterization of normal lattices if the lattice is a δ-lattice.

THEOREM 3.2. Suppose $\mathbf{L}$ is a δ-lattice. Then $\mathbf{L}$ is normal if and only if the following two conditions are satisfied:

a) If $L \in \mathbf{L}$ and if $L = \bigcap_{n=1}^{+\infty} L_n'$, $L_n \in \mathbf{L}$ then $L \in Z(\mathbf{L})$.

b) $Z(\mathbf{L})$ partly separates $\mathbf{L}$.

PROOF. 1. If $\mathbf{L}$ is normal then since $\mathbf{L}$ is δ, a) follows from Alexandroff [8] (cf. Lemma 7 p. 320.) Also, since $Z(\mathbf{L})$ separates $\mathbf{L}$, $Z(\mathbf{L})$ partly separates $\mathbf{L}$. 2. $Z(\mathbf{L})$ is normal and since $Z(\mathbf{L})$ partly separates $\mathbf{L}$, $Z(\mathbf{L})$ separates $\mathbf{L}$. Therefore, $\mathbf{L}$ is normal.

COROLLARY 3.1. A topological space is normal if and only if the following two condtions are satisfied:

a) Every closed set which is a G_δ-set is a zero set.

b) For every closed set F and every open set G such that $F \subset G$, there exists a zero set Z such that $F \subset Z \subset G$.

PROOF. Follows immediately from Theorem 3.2, letting $\mathbf{L} = \mathcal{F}$.

Let $J(\mathbf{L}) = \{v \in I(\mathbf{L}) \mid \text{if } L = \bigcap_{n=1}^{+\infty} L_n;\ L, L_n \in \mathbf{L} \text{ then } v(L) = \inf v(L_n)\}$.

We now state a consequence of a normal lattice being complement generated which generalizes a well-known theorem.

THEOREM 3.3. If $\mathbf{L}$ is normal and complement generated then $\mu \in J(\mathbf{L})$ implies $\mu \in I_R(\mathbf{L})$.

PROOF. Suppose $\mathbf{L}$ is normal and complement generated. Let $\mu \in J(\mathbf{L})$ and let

$L \in \mathbf{L}$. Then $L = \bigcap_{n=1}^{+\infty} L_n'$ where we may assume that $L_n \downarrow$, $L_n \in \mathbf{L}$ for all n. Since $\mathbf{L}$ is normal there exist A_n, $B_n \in \mathbf{L}$ such that $L \subset A_n' \subset B_n \subset L_n'$ for all n. Since

$\mu \in J(\mathbf{L})$, $\mu(L) = \inf \mu(B_n) = \inf(A_n')$. Therefore $\mu \in I_R(\mathbf{L})$.

COROLLARY 3.2. If a Z-filter is prime and closed under countable intersections then this Z-filter is a Z-ultrafilter with the countable intersection property.

PROOF. Follows immediately from the above theorem and the well-known correspondence between filters and measures.

The following lemma will be useful in our development of strongly normal lattices:

LEMMA 3.2. Let $\mu_1, \mu_2 \in I(\mathbf{L})$. If $\mu_1 \wedge \mu_2 \in I(\mathbf{L})$ then either $\mu_1 \leq \mu_2(\mathbf{L})$ or $\mu_2 \leq \mu_1(\mathbf{L})$.

PROOF. Suppose $\mu_1 \not\leq \mu_2(\mathbf{L})$ and $\mu_2 \not\leq \mu_1(\mathbf{L})$. Then there exist $A, B \in \mathbf{L}$ such that

$\mu_1(A) = \mu_2(B) = 1$ and $\mu_1(B) = \mu_2(A) = 0$. Therefore $\mu_1 \wedge \mu_2 (A \cup B) = 1$. But $\mu_1 \wedge \mu_2(A) = \mu_1 \wedge \mu_2(B) = 0$ and hence $\mu_1 \wedge \mu_2 \notin I(\mathbf{L})$.

Let $\tilde{I}(\mathbf{L}) = \{\pi \in \Pi(\mathbf{L}) \mid \text{if } A \cup B = X;\ A, B \in \mathbf{L} \text{ then } \pi(A) = 1 \text{ or } \pi(B) = 1\}$.

THEOREM 3.4. Suppose $I(\mathbf{L}) = \tilde{I}(\mathbf{L})$. Then each of the following is true:

a) If $\mu \le \pi(\mathbf{L})$, $\mu \in I(\mathbf{L})$, $\pi \in \Pi(\mathbf{L})$ then $\pi \in I(\mathbf{L})$.

b) If $\mu \le \mu_1(\mathbf{L})$ and $\mu \le \mu_2(\mathbf{L})$, $\mu, \mu_1, \mu_2 \in I(\mathbf{L})$ then $\mu_1 \le \mu_2(\mathbf{L})$ or $\mu_2 \le \mu_1(\mathbf{L})$.

c) $\mathbf{L}$ is normal.

Note: We say that $\mathbf{L}$ is <u>strongly normal</u> if L satisfies b).

PROOF. a) Suppose $\mu \le \pi(\mathbf{L})$ where $\mu \in I(\mathbf{L})$ and $\pi \in \Pi(\mathbf{L})$. Let A B = X, $A, B \in \mathbf{L}$. Then $\pi(A \cup B) = 1$ and $\mu(A \cup B) = 1$. Therefore $\mu(A) = 1$ or $\mu(B) = 1$ and hence $\pi(A) = 1$ or $\pi(B) = 1$. It follows that $\pi \in \tilde{I}(\mathbf{L}) = I(\mathbf{L})$. b) Let $\mu \le \mu_1(\mathbf{L})$ and $\mu \le \mu_2(\mathbf{L})$, where $\mu, \mu_1, \mu_2 \in I(\mathbf{L})$. Now $\mu_1 \wedge \mu_2 \in \tilde{I}(\mathbf{L}) = I(\mathbf{L})$. Applying Lemma 3.2 completes the proof. c) Follows immediately from b) and Theorem 3.1 noting that if $v_1 \le v_2(\mathbf{L})$, where $v_1, v_2 \in I_R(\mathbf{L})$, then $v_1 = v_2$.

REMARK 3.1. If X is a topological space and $\mathbf{L} = Z$, then $I(\mathbf{L}) = \tilde{I}(\mathbf{L})$. Therefore, the following well-known corollary is immediate.

COROLLARY 3.3. a) A Z-filter $\mathbf{F}$ is prime if and only if $\mathbf{F}$ contains a prime Z-filter. b) Z is strongly normal.

We now show that $I(\mathbf{L}) = \tilde{I}(\mathbf{L})$ if $\mathbf{L}$ is an algebra.

THEOREM 3.5. If $\mathbf{L} = \mathbf{L}'$ then $I(\mathbf{L}) = \tilde{I}(\mathbf{L})$.

PROOF. Let $\pi \in \tilde{I}(\mathbf{L})$ and $\mathbf{F}$ the corresponding filter. For each $A \in \mathbf{L}$ either A or $A' \in \mathbf{F}$ since $\mathbf{L}$ is an algebra. Let $\mathbf{H}$ be an $\mathbf{L}$-filter containing $\mathbf{F}$. Suppose $L \in \mathbf{H}$ and $L \notin \mathbf{F}$. Then $L' \notin \mathbf{H}$. But $L' \in \mathbf{F} \subset \mathbf{H}$ and this is a contradiction. Threfore, $\mathbf{F}$ is an L-ultrafilter and hence a prime filter.

REMARK 3.2. If $\mathbf{L}_1 \subset \mathbf{L}_2$ and $\mathbf{L}_1'$ separates $\mathbf{L}_2'$ then it can be shown that any $\pi_1 \in \tilde{I}(\mathbf{L}_1)$ can be extended to $\pi_2 \in \tilde{I}(\mathbf{L}_2)$. Such is the case for example, if $\mathbf{L}_1$ equals the lattice generated by the regular open sets and $\mathbf{L}_2$ equals the open sets of a topological space X.

4. ON $W_\sigma(\mathbf{L})$ AND REGULAR LATTICES.

$I_R(\mathbf{L})$ with the Wallman topology has been investigated by many writers. In this section we investigate measure conditions on $I_R^{\sigma}(\mathbf{L})$ which are equivalent to $W_\sigma(\mathbf{L})$, or possibly $\tau W_\sigma(\mathbf{L})$, being T_2, regular or prime complete. We begin with a result concerning $I_R(\mathbf{L})$ which is not generally known.

THEOREM 4.1. Suppose $\mathbf{L}$ is disjunctive. Then the following are equivalent:

1. $W(\mathbf{L})$ is normal.
2. $W(\mathbf{L})$ is regular.
3. $W(\mathbf{L})$ is T_2.

PROOF. (1. implies 2.) $W(\mathbf{L})$ is normal and disjunctive and therefore regular.

(2. implies 3.) $W(\mathbf{L})$ is regular and T_1 and therefore T_2. (3. implies 1.) If $W(\mathbf{L})$ is T_2 then $\mathbf{L}$ is normal. Therefore $W(\mathbf{L})$ is normal.

Suppose $\mathbf{L}$ is disjunctive. If $\mu \varepsilon I(\mathbf{L})$, let $\tilde{\mu}(W_\sigma(A)) = \mu(A)$ for all $A \varepsilon A(\mathbf{L})$. Then $\tilde{\mu} \varepsilon I(W_\sigma(\mathbf{L}))$. Conversely, for any $\tilde{\mu} \varepsilon I(W_\sigma(\mathbf{L}))$ we generate a $\mu \varepsilon I(\mathbf{L})$.

We now present a measure theoretic characterization with respect to $I_R^\sigma(\mathbf{L})$ being T_2. We begin with a lemma.

LEMMA 4.1. Assume $\mathbf{L}$ is disjunctive. Let $\nu \varepsilon I(\mathbf{L})$. If $\lambda \varepsilon I_R^\sigma(\mathbf{L})$ then $\nu \varepsilon S(\tilde{\mu})$ if and only if $\mu \leq \nu(\mathbf{L})$.

PROOF. Suppose $\nu \varepsilon S(\tilde{\mu})$. Then $\mu(L) = 1 \rightarrow \tilde{\mu}(W_\sigma(L)) = 1 \rightarrow \nu \varepsilon W_\sigma(L) \rightarrow \nu(L) = 1$, $L \varepsilon \mathbf{L}$.

Conversely, suppose $\mu \leq \nu(\mathbf{L})$. $\tilde{\mu}(W_\sigma(L)) = 1 \rightarrow \mu(L) = 1 \rightarrow \nu(L) = 1 \rightarrow \nu \varepsilon W_\sigma(L) \rightarrow \nu \varepsilon S(\tilde{\mu})$, $L \varepsilon \mathbf{L}$.

THEOREM 4.2. Suppose $\mathbf{L}$ is disjunctive. Then $I_R^\sigma(\mathbf{L})$ with the Wallman topology is T_2 if and only if for all $\mu \varepsilon I(\mathbf{L})$, if $\mu \leq \nu_1(\mathbf{L})$ and $\mu \leq \nu_2(\mathbf{L})$, where $\nu_1,\nu_2 \varepsilon I_R^\sigma(\mathbf{L})$, then $\nu_1 = \nu_2$.

PROOF. Suppose $I_R^\sigma(\mathbf{L})$ is T_2. Then $\tau W_\sigma(\mathbf{L})$ is T_2 and hence $W_\sigma(\mathbf{L})$ is T_2. Let μ, ν_1 and ν_2 be as above. By Lemma 4.1, $\nu_1,\nu_2 \varepsilon S(\tilde{\mu})$. Since $W_\sigma(\mathbf{L})$ is T_2, $\nu_1 = \nu_2$.

Conversely, assume $\mu \leq \nu_1(\mathbf{L})$ and $\mu \leq \nu_2(\mathbf{L})$, where $\nu_1,\nu_2 \varepsilon I_R^\sigma(\mathbf{L})$, implies that $\nu_1 = \nu_2$. Suppose $S(\tilde{\mu}) \neq \phi$. If $\nu_1,\nu_2 \varepsilon S(\tilde{\mu})$ then $\mu \leq \nu_1(\mathbf{L})$ and $\mu \leq \nu_2(\mathbf{L})$. Therefore $\nu_1 = \nu_2$. Thus $W_\sigma(\mathbf{L})$ and hence $\tau W_\sigma(\mathbf{L})$ is T_2.

Consider the following condition which we call <u>condition A</u>:

For all $\mu_1,\mu_2 \varepsilon I(\mathbf{L})$ such that $\mu_1 \leq \mu_2(\mathbf{L})$, if $\nu \varepsilon I_R^\sigma(\mathbf{L})$ and $\mu_1 \leq \nu(\mathbf{L})$ then $\mu_2 \leq \nu(\mathbf{L})$.

We now show that condition A is equivalent to $W_\sigma(\mathbf{L})$ being regular if $\mathbf{L}$ is disjunctive.

THEOREM 4.3 Suppose $\mathbf{L}$ is disjunctive. Then $W_\sigma(\mathbf{L})$ is regular if and only if condition A is true.

PROOF. Let $\mu \varepsilon I(\mathbf{L})$ and $\tilde{\mu}(W_\sigma(A)) = \mu(A)$, $A \varepsilon A(\mathbf{L})$. Assume $W_\sigma(\mathbf{L})$ is regular. If $\mu_1 \leq \mu_2(L)$ where $\mu_1,\mu_2 \varepsilon I(\mathbf{L})$ then $\tilde{\mu}_1 \leq \tilde{\mu}_2(W_\sigma(L))$. Therefore $S(\tilde{\mu}_1) = S(\tilde{\mu}_2)$. The condition follows by applying Lemma 4.1. Conversely, one may easily show by applying Lemma 4.1 that if condition A holds then $\tilde{\mu}_1 \leq \tilde{\mu}_2(W_\sigma(\mathbf{L}))$ implies that $S(\tilde{\mu}_1) = S(\tilde{\mu}_2)$ and hence $W_\sigma(\mathbf{L})$ is regular.

We observe that condition A implies that if $\mu \varepsilon I(\mathbf{L})$ then there exists at most one $\nu \varepsilon I_R^\sigma(\mathbf{L})$ such that $\mu \leq \nu(\mathbf{L})$. Therefore, applying Theorems 4.2 and 4.3 we have:

COROLLARY 4.1. Suppose $\mathbf{L}$ is disjunctive. If $W_\sigma(\mathbf{L})$ is regular then $W_\sigma(\mathbf{L})$ is T_2.

The following two theorems give conditions which guarantee that condition A is true.

THEOREM 4.4. If $I(\mathbf{L}) = \tilde{I}(\mathbf{L})$ then condition A is true.

PROOF. Since $I(\mathbf{L}) = \tilde{I}(\mathbf{L})$, $\mathbf{L}$ is strongly normal. Therefore, if $\mu_1 \leq \mu_2(\mathbf{L})$, $\mu_1,\mu_2 \varepsilon I(\mathbf{L})$, and if $\mu_1 \leq \nu(\mathbf{L})$, $\nu \varepsilon I_R^\sigma(\mathbf{L})$, then either $\mu_2 \leq \nu(\mathbf{L})$ or $\nu \leq \mu_2(\mathbf{L})$. Since $\nu \varepsilon I_R^\sigma(\mathbf{L})$, $\mu_2 \leq \nu(\mathbf{L})$.

COROLLARY 4.2. If $\mathbf{L}$ is disjunctive and $I(\mathbf{L}) = \tilde{I}(\mathbf{L})$ then $I_R^\sigma(\mathbf{L})$, with the Wallman topology, is regular.

PROOF. If $I(\mathbf{L}) = \tilde{I}(\mathbf{L})$ then condition A is true. Therefore, $W_\sigma(\mathbf{L})$ is regular and hence $\tau W_\sigma(\mathbf{L})$ is regular.

THEOREM 4.5. If $\mathbf{L}$ is regular and replete then $\mathbf{L}$ satisfies condition A.

PROOF. Suppose $\mu_1 \le \mu_2(\mathbf{L})$, $\mu_1, \mu_2 \in I(\mathbf{L})$ and $\mu_1 \le \nu(\mathbf{L})$, $\nu \in I_R^\sigma(\mathbf{L})$. Since $\mathbf{L}$ is regular, $S(\mu_1) = S(\mu_2) = S(\nu)$ and since $\mathbf{L}$ is replete, $S(\nu) \ne \phi$. Let $x \in S(\nu)$. Then $\nu \le \mu_x(\mathbf{L})$ where μ_x is the measure concentrated at x. But $\mu_x \in I_R^\sigma(\mathbf{L})$ and therefore, $\nu = \mu_x$. Since $x \in (\mu_2)$, $\mu_2 \le \nu(L)$.

Our next goal is to find a condition which guarantees that condition A on $\mathbf{L}_1$ implies condition A on $\mathbf{L}_2$ where $\mathbf{L}_1 \subset \mathbf{L}_2$. We begin with a lemma.

LEMMA 4.2. Suppose $\mathbf{L}_1 \subset \mathbf{L}_2$ and $\mathbf{L}_1$ separates $\mathbf{L}_2$. Let $\mu \in I(\mathbf{L}_1)$, $\nu \in I_R(\mathbf{L})_1$ and let ρ and τ respectively be extensions of μ and ν to $(\mathbf{L}_2)$ where $\tau \in I_R(\mathbf{L}_2)$. If $\mu \le \nu(\mathbf{L}_1)$ then $\rho \le \tau(\mathbf{L}_2)$.

PROOF. Suppose $\tau(A_2) = 0$, $A_2 \in \mathbf{L}_2$Since $\tau \in I_R(\mathbf{L}_2)$, there exists $B_2 \in \mathbf{L}_2$ such that $A_2 \subset B_2$ and $\tau(B_2) = 0$. Since $\mathbf{L}_1$ separates $\mathbf{L}_2$, there exist A_1, $B_1 \in L_1$ such that $B_2 \subset A_1$, $A_2 \subset B_1$ and $A_1 \cap B_1 = \phi$. It follows that $\rho(A_2) = 0$. .

THEOREM 4.6. Suppose $\mathbf{L}_1 \subset \mathbf{L}_2$ and $\mathbf{L}_1$ separates $\mathbf{L}_2$. If condition A holds on $\mathbf{L}_1$ then condition A holds on $\mathbf{L}_2$.

PROOF. Let $v_1 \le v_2(\mathbf{L}_2)$ where $v_1, v_2 \in I(\mathbf{L}_2)$ and $v_1 \le v(\mathbf{L}_2)$ where $v \in I_R^\sigma(\mathbf{L}_2)$. Let μ_1, μ_2 and μ respectively be the restrictions of v_1, v_2 and v to $A(\mathbf{L}_1)$. Then by condition A on $\mathbf{L}_1$, $\mu_2 \le \mu(\mathbf{L}_1)$. The proof now follows from Lemma 4.2.

Clearly, if $\mathbf{L}$ is regular then $\tau\mathbf{L}$ is regular. The following two theorems consider conditions which guarantee the converse.

THEOREM 4.7. Assume $\tau\mathbf{L}$ is regular. Let $\mu_1 \le v_1(\mathbf{L})$, $\mu_1, v_1 \in I(\mathbf{L})$. If there exist $\mu_2, v_2 \in I(\tau\mathbf{L})$ such that μ_2 and v_2 restricted equal μ_1 and v_1 respectively and if $\mu_2 \le v_2(\tau\mathbf{L})$ then $\mathbf{L}$ is regular.

PROOF. Clearly if $\mu_1 \le v_1(\mathbf{L})$ then $S(\mu_1) = S(\mu_2) = S(v_2) = S(v_1)$ under the above hypotheses.

THEOREM 4.8. Assume $\mathbf{L}$ is normal and $\mathbf{L}$ semiseparates $\tau\mathbf{L}$. Then if $\tau\mathbf{L}$ is regular, $\mathbf{L}$ is regular.

PROOF. Let μ_1, μ_2 and v_1 be as in Theorem 4.7. Let $\rho \in I_R(\mathbf{L})$ and $v_1 \le \rho(\mathbf{L})$. Similarly, let $v \in I_R(\tau\mathbf{L})$ and $\mu_2 \le v(\tau\mathbf{L})$. Since $\mathbf{L}$ semiseparates $\tau\mathbf{L}$, v restricted to $A(\mathbf{L})$ is $\mathbf{L}$-regular and equals ρ since $\mathbf{L}$ is normal. Therefore, $S(\mu_1) = S(v_1)$.

We now find a measure theoretic condition which is equivalent to $W_\sigma(\mathbf{L})$ being prime complete.

THEOREM 4.9. Suppose $\mathbf{L}$ is disjunctive. $W_\sigma(\mathbf{L})$ is prime complete if and only if for all $\mu \in I_\sigma(\mathbf{L})$ there exists $V \in I_R{}^\sigma(\mathbf{L})$ such that $\mu \le v(\mathbf{L})$.

PROOF. Follows immediately from Lemma 4.1.

COROLLARY 4.3. Suppose $\mathbf{L}$ is disjunctive, normal and countably paracompact. Then $W_\sigma(\mathbf{L})$ is prime complete.

PROOF. Let $\mu \in I_\sigma(\mathbf{L})$ Since $\mathbf{L}$ is normal and countably paracompact, there exists $v \in I_R{}^\sigma(\mathbf{L})$ such that $\mu \le v(\mathbf{L})$, Finally, we give the following theorem which extends the known measure theoretic consequence of regular lattices to $M(\mathbf{L})$

THEOREM 4.10. Suppose $\mathbf{L}$ is regular. Let $\mu_1, \mu_2 \in M(\mathbf{L})$. If $\mu_1 \leq \mu_2(\mathbf{L})$ and $\mu_1(X) = \mu_2(X)$ then $S(\mu_1) = S(\mu_2)$.

PROOF. Clearly $S(\mu_2) \subset S(\mu_1)$.

Let $x \in S(\mu_1)$. If $x \in S(\mu_2)$ then there exists $L \in \mathbf{L}$ such that $\mu_2(L) = \mu_2(X)$ and $x \notin L$. Since $\mathbf{L}$ is regular there exist $A, B \in \mathbf{L}$ such that $x \in A'$, $L \subset B'$ and $B' \subset A$. Therefore, $\mu_2(B') = \mu_2(X) = \mu_1(X)$. Therefore, $\mu_1(A) = \mu_1(X)$ hence $x \in A$ which is a contradiction.

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