Internat. J. Math. & Math. Sci.
VOL. 14 NO. 1 (1991) 69-76

CERTAIN INVARIANT SUBSPACES FOR OPERATORS WITH RICH EIGENVALUES

KARIM SEDDIGHI

Department of Mathematics
Shiraz University
Shiraz, Iran

(Received September 29, 1987 and in revised form June 7, 1988)

ABSTRACT. For a connected open subset Ω of the plane and n a positive integer, let $B_n(\Omega)$ be the space introduced by Cowen and Douglas. In this article we study the spectrum of restrictions of T in order to obtain more information about the invariant subspaces of T. When n=1 and $T \in B_1(\Omega)$ such that $\sigma(T) = \overline{\Omega}$ is a spectral set for T we use the functional calculus we have developed for such operators to give some infinite dimensional cyclic invariant subspaces for T.

KEY WORDS AND PHRASES. Invariant subspace, generalized Bergman kernel, zero sequence.
1980 AMS SUBJECT CLASSIFICATION CODES. 47B37, 47A25.

1. INTRODUCTION.

For a connected open subset Ω of the plane and n a positive integer, let $B_n(\Omega)$ denote the operators T defined on the Hilbert space H which satisfy

(a) Ω $\sigma(T)$,

(b) $\operatorname{ran}(T-\omega) = H$ for ω in Ω,

(c) $\bigvee_{\omega\in\Omega} \ker(T-\omega) = H$, and

(d) $\dim \ker(T-\omega) = n$ for ω in Ω.

The space $B_n(\Omega)$ has been introduced by Cowen and Douglas [1]. This class of operators was further studied by Curto and Salinas [2]. In particular, they show that an operator T in $B_n(\Omega)$ can be realized as the adjoint $T_{\bar z}^*$ of the operator of multiplication by $\bar z$ acting on a Hilbert space K_K of coanalytic functions on Ω having a generalized Bergman kernel K.

The class $B_n(\Omega)$ is a rich class of operators. To give an example, let A^2 denote the Hilbert space of analytic functions on the open unit disk D such that

$$||f||_2^2 = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 | f(re^{i\theta})|^2 \, r \, dr \, d\theta < \infty.$$

The operator S on A^2 defined by $(Sf)(z) = z f(z)$ is called the Bergman shift. It is easy to see that S* is in $B_1(D)$.

Shelley Walsh [3] has obtained several results concerning the backward Bergman shift S*. In particular, the spectrum of restrictions $S^*|_m$, where m is an invariant subspace for S, is studied. It is also pointed out how this study sheds more light on the structure of the invariant subspaces for the Bergman shift.

The purpose of the present article is to extend these results to the class $B_1(\Omega)$ in hope that more insight into the invariant subspace structure for these operators can be gained from this study.

2. PRELIMINARIES.

Let H be a separable Hilbert space and let B(H) denote the algebra of all bounded linear operators on H. For $T \in B(H)$ the spectrum, point spectrum, compression spectrum, and approximate point spectrum of T are denoted by $\sigma(T), \sigma_p(T)$, $\sigma_c(T)$, and $\sigma_{ap}(T)$ respectively. For the appropriate definitions see Halmo [4].

Let T be a bounded linear operator on a Hilbert space H. If $f \in H$, then $[f]$ will be the smallest invariant subspace for T containing f. The notation $[f]_*$ is used to denote the smallest invariant subspace for T* containing f. If A H, let VA denote the closed linear span of A. It is the smallest closed linear subspace of H that contains A.

A subset S of the open unit disk D is said to be dominating (or dominating for ∂ D) if

$$\sup_{\lambda\varepsilon S} |h(\lambda)| = ||h||_\infty, \ h \in H^\infty.$$

Equivalently, a subset S D is dominating if and only if almost every point of ∂ D is a nontangential limit point of S.

The following presentation on the generalized Bergman kernels is taken from Curto and Salinas [2] and will be needed in the sequel.

For $1 \leq n \leq \infty$, let $\iota_n^2 = \{\xi = \{a_k\}_{k=1}^n : a_k \in C$ and $\Sigma_{k=1}^n |a_k|^2 < \infty\}$. Given a set Λ, a functional Hilbert space on Λ will be any Hilbert space which is a linear subspace of $F(\Lambda, \iota_n^2)$, the linear space of all ι_n^2-valued functions on Λ, for some $1 \leq n \leq \infty$.

By a kernel functions on Λ we mean a function $K: \Lambda \times \Lambda \to B(\iota_n^2)$ that satisfies the following conditions:

(a) $K(\lambda,\mu)^* = K(\mu,\lambda)$ (all $\lambda,\mu \in \Lambda$);

(b) for every integer k, $k \geq 1$, and every collection $\{\lambda_1,\ldots,\lambda_k\}$ Λ, the $k \times k$ operator matrix $\kappa_K = \{K(\lambda_i, \lambda_j)) \ (1 \leq i, j \leq k)$ is a positive operator on τ_n^2 C^k.

If in addition the $k \times k$ operator matrix K_K in (b) is injective for every collection $\{\lambda_1,\ldots,\lambda_k\}$ Λ, all $k \geq 1$, we shall say that K is strictly positive.

Let K_K be a functional Hilbert space on Λ. If for every $\lambda \in \Lambda$ and $\xi \in \iota_n^2$, $K(\lambda,.)\ \xi \in K_K$ and

$$\langle f, K(\lambda,.)\xi\rangle = \langle f(\lambda), \xi\rangle, \text{ for every } f \in K,$$

then we shall say that K_K has a reproducing kernel on Λ and K will be called the reproducing kernel for K_K. It is well known that K is unique.

Note that if K is a strictly positive kernel function on Λ, it gives rise to a functional Hilbert space on Λ with reproducing kernel K. ([2] section 3). To see this let $D(\Lambda)$ be the linear space of $F(\Lambda, \iota_n^2)$ consisting of all functions of the form

$$f = \Sigma_{i=1}^{k} K(\lambda_i,.)\xi_i, \text{ for } \lambda_i \in \Lambda, \quad \xi_i \in \iota_n^2 \ (i=1,\dots,k),\ k > 1.$$

If g is another function in $D(\Lambda)$, $g = \Sigma_{i=1}^{k} K(\lambda_j,.)\eta_j$, let $\langle f,g\rangle_K = \Sigma_{i,j=1}^{k}$ $\langle K(\lambda_i,\lambda_j)\xi_i,\eta_j\rangle$. Now $\langle .,.\rangle_K$ is an inner product on (Λ). It is easy to show that a Cauchy sequence in $D(\Lambda)$ is pointwise convergent, and hence the completion K_K of (Λ) with respect to $\langle .,.\rangle_K$ is a linear subspace of $F(\Lambda, \iota_n^2)$. Furthermore, by construction of K_K, it follows that K is the reproducing kernel for K_K. We also observe that if K is a strictly positive kernel function which is already a reproducing kernel for a functional Hilbert space K on Λ, then $K_K = K$.

Let Λ be a domain in the plane and assume that every functional Hilbert space on Λ is invariant under the map of multiplication by the conjugate of the function $f:\Lambda \to C$ defined by $f(z) = z$. The restriction of this map to such a functional Hilbert space will be denoted by $T_{\bar{z}}$.

Now let $\bar{A}(\Omega, \iota_n^2)$ denote the set of ι_n^2-valued coanalytic functions on the domain Ω in the plane. Note that $\bar{A}(\Omega, \iota_n^2)$ is invariant under multiplication by $\bar{z}$. All functional Hilbert spaces on Ω under consideration will be invariant under $T_{\bar{z}}$. A functional Hilbert space on Ω which is a linear subspace of $\bar{A}(\Omega, \iota_n^2)$ will be called a <u>conanalytic functional Hilbert space on</u> Ω.

A kernel function $K:\Omega \times \Omega \to B(\iota_n^2)$ is called sesquianalytic if $K(.,.)$ is analytic in the first variable and coanalytic in the second variable. Given a sesquianalytic kernel function K on Ω, let $H_p(\lambda; K)$ be the $(p+1) \times (p+1)$ - positive operator matrix whose m, n-entry is

$$\frac{\partial^m}{m!\,\partial z^m}\,\frac{\partial^n}{n!\,\partial \bar{z}^n}\,K(\lambda,\lambda),\ 0 < m,\ n < p.$$

We say that the kernel function K is <u>nondegenerate</u> if $H_p(\lambda; K)$ is injective for every $\lambda \in \Omega$ and every positive integer p.

Let K be a nondegenerate sesquianalytic kernel function on Ω. Then there exists a coanalytic functional Hilbert space K_K on Ω such that the nondegenerate kernel function K on Ω is a reproducing kernel for K_K ([2] section 4). Indeed, let $\lambda_0 \in \Omega$ and denote by D_{λ_0} the linear subspace of $\bar{A}(\Omega,\iota_n^2)$ spanned by functions of the form $f = \Sigma_{j=0}^{p} \dfrac{\partial^j K(\lambda_0,.)}{j!}\,\xi_j$, $\xi_j \in \iota_n^2$. If $g = \Sigma_{k=0}^{q} \dfrac{\partial^k K(\lambda_0,.)}{k!}\,\eta_k$ is another function in D_{λ_0}, we define

$$\langle f, g\rangle = \sum_{j=0}^{p}\sum_{k=0}^{q} \left\langle \frac{\partial^j\,\bar{\partial}^k\,K(\lambda_0,\lambda_0)}{j!\,k!}\,\xi_j,\ \eta_k \right\rangle .$$

It follows that the completion K_{λ_0} of D_{λ_0} with respect to this inner product is a coanalytic functional Hilbert space on Ω, which is independent of $\lambda_0 \in \Omega$ and has K as a reproducing kernel. Therefore $K_K = K_{\lambda_0}$.

A nondegenerate sesquianalytic kernel function K on Ω is called a generalized Bergman kernel (g.B.k for brevity) if $T_{\bar{z}} \in B(K_K)$ and for every $\lambda \in \Omega$, $T^*_{\bar{z}} - \lambda$ has closed range, ran $K(\lambda,.) = \ker(T^*_{\bar{z}} - \lambda)$ and $\dim \ker(T^*_{\bar{z}} - \lambda) = n$, where n is a fixed positive integer throughout the rest of this article. For a detailed treatment of generalized Bergman kernels the reader is referred to Curto and Salinas [2]. In particular, a g.B.k. K on Ω is strictly positive and hence $D(\Omega)$ is dense in K_K. Also $K(\lambda,\lambda)$ is invertible for $\lambda \in \Omega$. The operator $T^*_{\bar{z}}$ acting on K_K is called the canoncial model associated with K. For simplicity of notation set $T = T^*_{\bar{z}}$.

Finally we make a few observations which we will need later. Note that

$\partial^i K(\lambda,.)\xi$ is in K_K and it is easy to see that $(T-\lambda)\dfrac{\partial^i K(\lambda,.)}{i!} = \dfrac{\partial^{i-1}K(\lambda,.)}{(i-1)!}$ (Differentiate the relation $(T-\lambda)K(\lambda,.) = 0$ i times). We also use the fact that multiplication by $\bar{z} - \bar{\lambda}$ is an injective operator on K_K to deduce that $\sigma_p(T^*) = \phi$.

Since $\ker(T^* - \bar{\lambda}) = 0$ it follows that $\operatorname{ran}(T - \lambda)$ is dense in H. But $\operatorname{ran}(T - \lambda)$, $\lambda \quad \Omega$, is closed, so $\operatorname{ran}(T - \lambda) = H$.

3. SOME INFINITE DIMENSIONAL CYCLIC INVARIANT SUBSPACES.

In this section we will consider a special case. Let $T \in B_1(\Omega)$ such that $\sigma(T)$ is a spectral set for T and $\sigma(T) = \bar{\Omega}$. In [5,6] we have proved the following. There is a compact set L such that $\sigma(T) \quad L$, the interior of L is simply connected and L is minimal with respect to these properties. Moreover, if ϕ denotes the conformal map from L^0 onto D then $A = \phi(T)$ is in $B_1(\phi(\Omega))$ and $\sigma(A) = \phi(\Omega)^-$ is a spectral set for A. Also $\operatorname{Lat}(T) = \operatorname{Lat}(A)$.

To study the invariant subspaces for such operators we may assume, without loss of generality, that Ω is an open connected subset of the unit disk such that $L^0 = D$ and $T \in B_1(\Omega)$ is such that $\sigma(T) = \bar{\Omega}$ is a spectral set for T. In this case we also have $\partial D \quad \bar{\Omega}$ (Seddighi [5]). We assume this normalization is in effect in this section. If K is a g.B.k. on Ω then for convenience we let $H = K_K$.

In the proof of the next theorem we use the functional calculus developed in (Seddighi [6]).

THEOREM 1. If f is in H then

$$[f] = \{g \in H: g \quad h(T)f \text{ for every } h \text{ in } H^\infty\}$$

PROOF. If $g \quad h(T)f$ for every h in H^∞ then $g \quad p(T)f$ for every polynomial p. Hence $g \quad [f]$.

Conversely suppose $g \in [f]$ and $h \in H^\infty$. There is a uniformly bounded sequence $\{p_n\}$ of polynomials converging pointwise to h in D. Since p_n converges to h weak-star it follows that $p_n(T) \to h(T)$ weak-star ([6]). Hence $p_n(T)f \to h(T)f$ weakly, so $h(T)f \in [f]$. But $g \quad [f]$, so $g \quad h(T)f$ and the conclusion follows immediately.

COROLLARY 1. If $\{\lambda_m\}$ is a Blaschke sequence of distinct points in Ω which has all its limit points on ∂D and $\{c_m\}_{m=1}^\infty$ is a sequence of nonzero complex numbers such that $\Sigma_{m=1}^\infty c_m K_{\lambda_m}$ is in H then

$$[f] = \{g \in \quad : g(\lambda_m) = 0 \text{ for all } m\} .$$

PROOF. If $g(\lambda_m) = 0$ for all m then

$$\langle g, T^n f\rangle = \langle T^{*n} g, f\rangle = \langle \bar{z}^n g, f\rangle = \sum_{m=1}^{\infty} \bar{c}_m \bar{\lambda}_m g(\lambda_m) = 0. \quad \text{so } g \in [f].$$

If $g \in [f]$ and h is in H^∞ then $h(T)f = \Sigma_{m=1}^{\infty} c_m h(\lambda_m) K_{\lambda_m}$ ([6]) and $\langle g, h(t)f\rangle = \Sigma_{m=1}^{\infty} \bar{c}_m h(\lambda_m) g(\lambda_m) = 0$ by Theorem 1. Fix m and let h be a function in H^∞ such that $h(\lambda_m) = 1$ and $h(\lambda_k) = 0$ for $k \neq m$. Then $\bar{c}_m g(\lambda_m) = 0$. Since $\bar{c}_m \neq 0$, we have $g(\lambda_m) = 0$.

THEOREM 2. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence of distinct points in Ω which has all its limit points on ∂D and is not a dominating sequence, and let $\{c_k\}_{k=1}^{\infty}$ be a sequence of nonzero complex numbers such that $\sum_{k=1}^{\infty} |c_k| \, ||K_{\lambda_k}|| < \infty$. If $f = \sum_{k=1}^{\infty} c_k K_{\lambda_k}$, then $[f] = \{g \in H: g(\lambda_k) = 0 \text{ for all } k\}$.

PROOF. If $g(\lambda_k) = 0$ for all k, then for any m we have $\langle g, T^m f\rangle = 0$ just as in the proof of Corollary 1, so $g \in [f]$.

If g [f] then $\sum_{k=1}^{\infty} c_k \lambda_k^m \overline{g(\lambda_k)} = 0$, for any m. Note that for any k,

$$|g(\lambda_k)| = |\langle g, K_{\lambda_k}\rangle| \leq ||g|| \, ||K_{\lambda_k}||.$$

Since $\sum_{k=1}^{\infty} |c_k| \, ||K_{\lambda_k}|| < \infty$, the sum $\sum_{k=1}^{\infty} |c_k \overline{g(\lambda_k)}|$ is finite. By Theorem 3 of Brown et al [1] we have $c_k \overline{g(\lambda_k)} = 0$ for all k. Because $c_k \neq 0$, it follows that $g(\lambda_k) = 0$ for all k.

4. THE SPECTRUM OF RESTRICTIONS OF T.

In this section we assume that K is a g.B.k. on Ω and $T = T^*_{\bar{z}}$ is the canonical model in $B_n(\Omega)$ acting on $H = K_K$. We also assume that $\sigma_p(T) = \Omega$.

LEMMA 1. Let m be an invariant subspace for T then

$$\sigma_{ap}(T|_m) \quad \Omega = \sigma_p(T|_m).$$

PROOF. Let $\lambda \quad \sigma_{ap}(T|_m) \quad \Omega$. Then there exists a sequence of unit vectors $\{f_k\}$ in H with

$$||(T - \lambda)f_k|| \to 0. \quad \text{Write } f_k = K(\lambda,.)\xi_k + g_k,$$

where $\xi_k \in l_n^2$ and g_k ran $K(\lambda,.)$. Since $(T - \lambda)f_k = (T - \lambda)g_k$, it follows that $(T - \lambda)g_k \to 0$. Because $T - \lambda$ has closed range, it is bounded below on the orthogonal complement of $\ker(T - \lambda)$. From this observation we get that $g_k \to 0$.

Now $||f_k||^2 = ||K(\lambda,.)\xi_k||^2 + ||g_k||^2$, from which we conclude that

$$||K(\lambda,.)\xi_k|| \leq M \text{ for some } M.$$

Hence $||\xi_k|| = ||K(\lambda,\lambda)^{-1/2} K(\lambda,\lambda)^{1/2} \xi_k|| \leq ||K(\lambda,\lambda)^{-1/2}|| \quad K(\lambda,\lambda)^{-1/2}\xi_k$

$$= ||K(\lambda,\lambda)^{-1/2}|| \, ||K(\lambda,.)\xi_k|| \leq M||K(\lambda,\lambda)^{-1/2}||.$$

Therefore $\{\xi_k\}$ is uniformly bounded in norm by a constant C. If $\xi_k = \{a_{kj}\}_{j=1}^n$,

then $\sum_{j=1}^n |a_{kj}|^2 \leq C$ for all k. For each j, $1 \leq j \leq n$ there is a convergent subsequence $\{a_{k_i,j}\}_{i=1}^\infty$ such that $a_{k_i,j} \to a_j$ in C.

Let $\xi = \{a_j\}_{j=1}^n$. Then $\xi_{k_i} \to \xi$ in ι_n^2. Hence $||K(\lambda,.)\xi_{k_i} - K(\lambda,.)\xi||$

$$= ||K(\lambda,\lambda)^{1/2}\ (\xi_{k_i} - \xi)|| \leq ||K(\lambda,\lambda)^{1/2}||\ ||\xi_{k_i} - \xi|| \to 0.$$

We have shown that $K(\lambda,.)\xi_{k_i} \to K(\lambda,.)\xi$ in H. Observe that

$f_{k_i} = K(\lambda,.)\xi_{k_i} + g_{k_i} \to K(\lambda,.)\xi$. Therefore $K(\lambda,.)\xi$ is in m. Since $\{f_{k_i}\}$ is a sequence of unit vectors, $K(\lambda,.)\xi \neq 0$, so $\lambda \in \sigma_p(T|_m)$.

We now consider a cyclic invariant subspace $m = [f]_*$ for T* and discuss the spectrum of the restriction $T|_m$. Before doing this note that $\{\lambda \quad \Omega: f(\lambda) = 0\}$ $= \{\lambda \in \Omega: h(\lambda) = 0$ for every h in m $\}$ To see this let $f(\lambda) = 0$ and observe that for every polynomial p and any ξ in τ_n^2 we have

$$\langle p(T^*)f,\ K(\lambda,.)\xi\rangle = \langle p(\overline{\lambda})f(\lambda),\ \xi\rangle = 0.$$

Note let $h \in m$. Then approximating h by polynomials in T* applied to f, we have

$$\langle h,\ K(\lambda,.)\xi\rangle = \langle h(\lambda),\xi\rangle = 0 \text{ for every } \xi \text{ in } \iota_n^2. \text{ Hence } h(\lambda) = 0.$$

We also need the fact that $\{\lambda: f(\lambda) = 0\} \quad \sigma_p(T|_m)$. To see this assume $f(\lambda) = 0$, then $h(\lambda) = 0$ for every h in m. This says that $\langle h,\ K(\lambda,.)\xi\rangle = 0$, ξ in ι_n^2. Hence $\ker(T - \lambda) \quad m$, so $\lambda \in \sigma_p(T|_m)$.

THEOREM 3. If m is a cyclic invariant subspace for T* then

$$\sigma(T|_m)\quad \Omega = \sigma_p(T|_m).$$

PROOF. Let $m = [f]_*$. We first show that $\sigma_c(T|_m) \quad \Omega \quad \{\lambda: f(\lambda) = 0\}$. If we prove this then since $\{\lambda: f(\lambda) = 0\} \quad \sigma_p(T|_m)$, by the argument preceding the theorem, we have $\sigma_c(T|_m) \quad \Omega \quad \sigma_p(T|_m)$. We now apply Lemma 1 to complete the proof.

To prove $\sigma_c(T|_m) \quad \Omega \quad \{\lambda: f(\lambda) = 0\}$ it suffices to show that if $f(\lambda) \neq 0$, then λ is not in the compression spectrum of $T|_m$. If g $\quad ((T-\lambda)m\ )$ then for any h in m , $0 = \langle g,\ (T - \lambda)h\rangle = \langle (T^* - \overline{\lambda})g, h\rangle$, so $(T^* - \overline{\lambda})g \in (m\ ) = m$. Thus there is a sequence $\{p_n\}$ of polynomials such that $p_n(T^*)f \to (T^* - \overline{\lambda})g$. It follows that

$$\langle p_n(T^*)f,\ K(\lambda,.)\xi\rangle \to \langle (T^* - \overline{\lambda})g,\ K(\lambda,.)\xi\rangle = 0$$

for every ξ in ι_n^2. Hence $p_n(\overline{\lambda})f(\lambda) = (p_n(T^*)f)\ (\lambda)$ converges weakly to zero in ι_n^2. In particular, $\langle p_n(\overline{\lambda})f(\lambda),\ f(\lambda)\rangle = p_n(\lambda)\ \ ||f(\lambda)||^2 \to 0$. Because $f(\lambda) \neq 0$, we have $p_n(\bar{\lambda}) \to 0$. Let $q_n(z) = \dfrac{p_n(z) - p_n(\bar{\lambda})}{z - \bar{\lambda}}$. Then $(T^* - \lambda)q_n(T^*)f = p_n(T^*)f - p_n(\bar{\lambda})f \to (T^* - \bar{\lambda})g$. Since $\operatorname{ran}(T^* - \bar{\lambda})$ is closed, it follows that $T^* - \bar{\lambda}$ is bounded below on $\ker(T^* - \bar{\lambda}) = \operatorname{ran}(T - \lambda) = H$. Therefore $q_n(T^*)\, f \to g$, so $g \quad m = (m \ )$. Thus $((T - \lambda)m \ ) \quad (m \ )$, so λ is not in the compression spectrum of $T|_m$.

COROLLARY 2. If $n = 1$ and m is a cyclic invariant subspace for T^* then

$$\sigma(T|_m) \quad \Omega = \{\lambda\colon h(\lambda) = 0 \text{ for every } h \text{ in } m\}.$$

PROOF. Let $m = [f]_*$. If $n = 1$ then $\sigma_p(T|_m) \quad \{\lambda\colon f(\lambda) = 0\}$. Indeed if $K_\lambda \in m$ then $f(\lambda) = \langle f, K_\lambda\rangle = 0$. By the argument preceding Theorem 3 we have $\{\lambda\colon f(\lambda) = 0\} \quad \sigma_p(T|_m)$. Thus $\sigma_p(T|_m) = \{\lambda\colon f(\lambda) = 0\} = \{\lambda\colon h(\lambda) = 0 \text{ for every } h \text{ in } \ \}$.

For a sequence $\{\lambda_k\}$ in Ω and f in H the notation $f(\{\lambda_k\}) = 0$ is used to mean that if λ occurs in $\{\lambda_k\}$ N times then f has a zero at λ of order at least N. We also let $n(\lambda)$ denote the number of times λ occurs in $\{\lambda_k\}$.

DEFINITION 1. A sequence $\{\lambda_k\}$ of points in Ω is a zero sequence for if there exists f in H with $f(\{\lambda_k\}) = 0$.

Zero sequences for A^2 are studied by C. Horowitz [8]. He has shown that they are quite different from Blaschke sequences.

THEOREM 4. Let $n = 1$, and $\{\lambda_k\} \quad \Omega$ be a zero sequence for H. If $m = \{h\colon h(\{\lambda_k\}) = 0\}$ then

$$\sigma(T|_m) \quad \Omega = \{\lambda\colon h(\lambda) = 0 \text{ for every } h \in m\}.$$

PROOF. It is easy to see that

$$m = V\{\partial^i K_\lambda\colon \lambda \in \{\lambda_k\},\ i = 0,\dots,n(\lambda) - 1\}$$

Therefore the finite linear combinations of functions of the form

$$\sum_{k=1}^{N} \sum_{i=0}^{n_k} C_{k,i}\, \partial^i K_{\lambda_k}, \text{ where } n_k < n(\lambda_k) - 1 \text{ for each } k \text{ are dense in } m.$$

If $\lambda \notin \{\lambda_k\}$ then

$$(T - \lambda) \sum_{k=1}^{N} \sum_{i=0}^{n_k} C_{k,i}\, \partial^i K_{\lambda_k}$$

$$= \sum_{k=1}^{N} \sum_{i=0}^{n_k} C_{k,i} \left((T - \lambda_k)\, \partial^i K_{\lambda_k} + (\lambda_k - \lambda)\, \partial^i K_{\lambda_k}\right)$$

$$= \sum_{k=1}^{N} C_{k,0}\, (\lambda_k - \lambda)\, K_{\lambda_k} + \sum_{k=1}^{N} \sum_{i=1}^{n_k} C_{k,i} \left(i\, \partial^{i-1} K_{\lambda_k} + (\lambda_k - \lambda)\, \partial^i K_{\lambda_k}\right)$$

$$= \sum_{k=1}^{N} \sum_{i=1}^{n_k - 1} [(i+1)C_{k,i+1} + (\lambda_k - \lambda)C_{k,i}]\, \partial^i K_{\lambda_k} + \sum_{k=1}^{N} C_{k,n_k} (\lambda_k - \lambda)\, \partial^{n_k} K_{\lambda_k}.$$

These functions are dense in m , so λ is not in the compression spectrum of $T|_m$.

Hence

$$\sigma_c(T|_m) \ \Omega \ \{\lambda_k\} \ \sigma_p(T|_m).$$

Note that $\lambda \in \sigma_p(T|_m)$ if and only if $K_\lambda \in m$, so $\sigma_p(T|_m) = \{\lambda : h(\lambda) = 0$ for every $h \in m\}$. Applying Lemma 1 we obtain the result.

ACKNOWLEDGEMENT. The author would like to thank the research council, Shiraz University, for a grant (No. 65-SC-392-187) during the preparation of this article.

The content of this article is presented to the ICM 86, Berkeley, August 3, 1986.

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