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A THEOREM FOR FOURIER COEFFICIENTS OF A FUNCTION OF CLASS L^P

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Abstract: This paper deals with the Fourier coefficients of a function of class L^P. We give a necessary and sufficient condition for a function to be of class L^P for p greater than one.

Key Words and Phrases: Fourier coefficients, L^P class, and monotonically decreasing.
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1. INTRODUCTION.

A function f(x) is said to belong to the class $L(p,\alpha)$ if $\int_0^\pi |f(x)|^p (\sin x)^{\alpha p}\, dx < \infty$ [1].

If $f(x) \in L(p,\alpha)$, then we define $||f||_{p,\alpha} = \{\int_0^\pi |f(x)|^p (\sin x)^{\alpha p}\, dx\}^{\frac{1}{p}}$.

Hardy [2] gave the following theorem concerning the Fourier coefficients of a function belonging to L^P class.

THEOREM 1.1: Let $a_1, a_2, \ldots$ be Fourier cosine coefficients of a function of class L^P, $p \geq 1$, and $s_n = \sum_{k=1}^{n} a_k$. Then $\frac{s_1}{1}, \frac{s_2}{2}, \frac{s_3}{3}, \ldots$ are also Fourier coefficients of a function of class L^P.

The converse of Theorem 1.1 is not necessarily true. But Siddiqui [3] proved the following theorem.

THEOREM 1.2: Let $f(x) \approx \sum_{n=1}^{\infty} a_n \cos nx$ with $a_n \downarrow 0$. Then a necessary and sufficient condition that $\sum_{n=1}^{\infty} a_n \cos nx$ be the Fourier series of $f(x) \in L^P$ is that $\sum_{n=1}^{\infty} A_n \cos nx$ be the Fourier series of a function belonging to L^P class, where $p > 1$ and $A_n = \frac{1}{n}\sum_{k=1}^{n} a_k$.

2. MAIN RESULT. The object of this paper is to weaken the hypothesis that $a_n \downarrow 0$ of Theorem 1.2 to a condition that $n^{-\beta} a_n$ should be monotonic for some non-negative integer β and also for weighted L^P spaces. In fact we have the following theorem.

THEOREM 2.1: Let $\{a_n\}$ be a positive null sequence such that $n^{-\beta} a_n$ is monotonically decreasing for some non-negative integer β. Suppose $f(x) \approx \sum_{n=1}^{\infty} a_n \cos nx$. Then a necessary and sufficient condition that the series $\sum_{n=1}^{\infty} a_n \cos nx$ be the Fourier series of $f(x) \in L(p,\alpha)$ is that $\sum_{n=1}^{\infty} A_n \cos nx$ be the Fourie

series of a function belonging to $L(p,\alpha)$ class, where $1 \leq p < \infty$, $-1 < \alpha p < p-1$ and $A_n = \frac{1}{n}\sum_{k=1}^{n} a_k$.

We shall require the following Lemmas for the proof of our theorem.

LEMMA 2.2 [1]: Let $f(x) \approx \sum_{n=1}^{\infty} a_n \cos nx$ where the a_n are positive and tend to zero and $n^{-\beta} a_n$ is monotonically decreasing for some non-negative integer β. Then a necessary and sufficient condition that $f(x) \in L(p,\alpha)$ is that $\sum_{n=1}^{\infty} n^{p-\alpha p-2} a_n{}^p < \infty$ where $1 \leq p < \infty$ and $-1 < \alpha p < p-1$.

LEMMA 2.3: If $n^{-\beta} a_n$ for some non-negative integer β is monotonically decreasing, then

$$A_n = \frac{\frac{1}{n}\sum_{k=1}^{n} a_k}{n^{\beta}}$$

is also monotonically decreasing.

Proof: Let $\beta=0$, then we have to show that

$$A_n = \frac{1}{n}\sum_{k=1}^{n} a_k \geq A_{n+1} = \frac{1}{n+1}\sum_{k=1}^{n+1} a_k,$$

or

$$(n+1)\sum_{k=1}^{n} a_k \geq n\sum_{k=1}^{n+1} a_k$$

or

$$n\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \geq n\sum_{k=1}^{n} a_k + n\, a_{n+1},$$

or

$$n\, a_{n+1} \leq \sum_{k=1}^{n} a_k.$$

Since

$$a_{n+1} \leq a_1,$$
$$a_{n+1} \leq a_2,$$
$$\vdots$$
$$a_{n+1} \leq a_n,$$

it follows that

$$n\, a_{n+1} \leq a_1 + a_2 + \dots + a_n,$$

or

$$n\, a_{n+1} \leq \sum_{k=1}^{n} a_k.$$

Thus $$A_n \geq A_{n+1}.$$

Now let $\beta \geq 1$. Let $C_n = n^{-\beta} a_n$, then

$$A_n = \frac{\frac{1}{n}\sum_{k=1}^{n} a_k}{n^{\beta}} = n^{-(1+\beta)} \sum_{k=1}^{n} a_k$$

$$= n^{-(1+\beta)} \sum_{k=1}^{n} k^{\beta} \frac{a_k}{k^{\beta}}$$

$$= n^{-(1+\beta)} \sum_{k=1}^{n} k^{\beta} C_k,$$

and

$$A_{n+1} = (n+1)^{-(1+\beta)} \sum_{k=1}^{n+1} k^{\beta} C_k$$

$$= (n+1)^{-(1+\beta)} \{\sum_{k=1}^{n} k^{\beta} C_k + (n+1)^{\beta} C_{n+1}\}$$

$$= (n+1)^{-(1+\beta)} \sum_{k=1}^{n} k^{\beta} C_k + (n+1)^{-1} C_{n+1}.$$

Now

$$A_n - A_{n+1} = n^{-(1+\beta)} \sum_{k=1}^{n} k^{\beta} C_k - (n+1)^{-(1+\beta)} \sum_{k=1}^{n} k^{\beta} C_k - (n+1)^{-1} C_{n+1}$$

$$= \{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} \sum_{k=1}^{n} k^{\beta} C_k - (n+1)^{-1} C_{n+1}$$

therefore

$$(n+1)(A_n - A_{n+1}) = (n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} \times \sum_{k=1}^{n} k^{\beta} C_k - C_{n+1}$$

$$\geq (n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} C_n \sum_{k=1}^{n} k^{\beta} - C_{n+1}$$

$$\geq (n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} C_{n+1} \sum_{k=1}^{n} k^{\beta} - C_{n+1}$$

$$= C_{n+1}[(n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} \sum_{k=1}^{n} k^{\beta} - 1]$$

$$= C_{n+1}\{(n+1)\theta_n - 1\}, \text{ where}$$

$$(n+1)\theta_n = (n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} \sum_{k=1}^{n} k^{\beta}$$

$$= \{(n+1)n^{-(1+\beta)} - (n+1)(n+1)^{-(1+\beta)}\} \sum_{k=1}^{n} k^{\beta}$$

$$= \{(n+1)n^{-(1+\beta)} - (n+1)^{-\beta}\} \sum_{k=1}^{n} k^{\beta}$$

$$= \{(n+1)n^{-1}n^{-\beta} - n^{-\beta}(1+\tfrac{1}{n})^{-\beta}\}\sum_{k=1}^{n} k^{\beta}$$

$$= n^{-\beta}\{n(1+\tfrac{1}{n})n^{-1} - (1+\tfrac{1}{n})^{-\beta}\}\sum_{k=1}^{n} k^{\beta}$$

$$= n^{-\beta}\{(1+\tfrac{1}{n}) - (1+\tfrac{1}{n})^{-\beta}\}\sum_{k=1}^{n} k^{\beta}$$

$$= n^{-\beta}\{(1+\tfrac{1}{n}) - (1-\tfrac{\beta}{n}) + \frac{\beta(\beta-1)}{2}\cdot\frac{1}{n^2}\ldots\} \times \sum_{k=1}^{n} k^{\beta}$$

$$= n^{-\beta}\{\tfrac{1}{n} + \tfrac{\beta}{n} - \frac{\beta(\beta-1)}{2}\cdot\frac{1}{n^2} + \ldots\}\sum_{k=1}^{n} k^{\beta}$$

Now by the following formula in [4],

$$\sum_{k=1}^{n} k^{\beta} = \frac{n^{\beta+1}}{\beta+1} + \frac{n^{\beta}}{2} + \frac{\beta\cdot n^{\beta-1}}{12} - \frac{\beta(\beta-1)(\beta-2)}{720}n^{\beta-3} + \ldots$$

we have

$$(n+1)\theta_n = 1 + \tfrac{1}{2}\{(\beta+1) - \frac{\beta(\beta+1)}{\beta}\}n^{-1} + 0(n^{-2})$$

$$= 1 + \frac{(3\beta+1)}{(\beta+1)}n^{-1} + 0(n^{-2})$$

$$>1 \text{ for large } n.$$

<u>LEMMA 2.4</u>: Let $\{a_n\}$ be a positive sequence which tends to zero. Let $\{n^{-\beta}a_n\}$ be monotonically decreasing for some non-negative integer β. Then the convergence of $\sum_{n=1}^{\infty} n^{p-\alpha p-2}A_n{}^p$ implies the convergence of the series

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2}a_n{}^p,$$

where

$$A_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

<u>Proof</u>: Since $\{n^{-\beta}a_n\}$ is a monotonically decreasing sequence, then it follows that

$$A_n = \frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{n} k^{-\beta}a_k k^{\beta}$$

$$\geq \frac{1}{n}n^{-\beta}a_n\sum_{k=1}^{n} k^{\beta} = Ka_n, \text{ for some } K$$

so that

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2} a_n{}^p \leq K \sum_{k=1}^{\infty} n^{p-\alpha p-2} A_n{}^p < \infty$$

and hence the result follows.

Proof of the Theorem 2.1: The necessary part follows from Theorem B as a particular case.

Sufficiency. Since $\{A_n\}$ is a positive null sequence and due to Lemma 2.3, $\{n^{-\beta} A_n\}$ is monotonically decreasing for some non-negative integer β, it follows from Lemma 2.2 that if $\sum_{n=1}^{\infty} A_n \cos nx$ is the Fourier series of a function $F(x) \in L(p,\alpha)$, then $\sum_{n=1}^{\infty} n^{p-\alpha p-2} A_n{}^p < \infty$.

Applying Lemma 2.4, we have $\sum_{n=1}^{\infty} n^{p-\alpha p-2} a_n{}^p < \infty$.

Hence by Lemma 2.1, $f(x) \in L(p,\alpha)$, and consequently $\sum_{n=1}^{\infty} a_n \cos nx$ is the Fourier series of $f(x)$.

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