Internat. J. Math. & Math. Sci.
VOL. 13 NO. 1 (1990) 61-66

COMMON FIXED POINTS OF COMPATIBLE MAPPINGS

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(Received October 19, 1988 and in revised form September 27, 1988)

ABSTRACT. In this paper, we present a common fixed point theorem for compatible mappings, which extends the results of Ding, Diviccaro-Sessa and the third author.

KEY WORDS AND PHRASES. Common fixed points, commuting mappings, weakly commuting mappings and compatible mappings.
1980 AMS MATHEMATICS SUBJECT CLASSIFICATION CODE. 54H25.

1. INTRODUCTION.

In [1], the concept of compatible mappings was introduced as a generalization of commuting mappings. The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park-Bae [2]. In [3], the third author extended a result of Singh-Singh [4] by employing compatible mappings in lieu of commuting mappings and by using four functions as opposed to three. On the other hand, Diviccaro-Sessa [5] proved a common fixed point theorem for four mappings, using a well known contractive condition of Meade-Singh [6] and the concept of weak commutativity of Sessa [7]. Their theorems generalize results of Chang [8], Imdad Khan [9], Meade-Singh [6], Sessa-Fisher [10] and Singh-Singh [4].

In this paper, we extend the results of Ding [11], Diviccaro-Sessa [5] and the third author [3].

The following Definition 1.1 is given in [1].

DEFINITION 1.1. Let A and B be mappings from a metric space (X,d) into itself. Then A and B are said to be compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z in X.

Thus, if $d(ABx_n, BAx_n) \to 0$ as $d(Ax_n, Bx_n) \to 0$, then A and B are compatible.

Mappings which commute are clearly compatible, but the converse is false. S. Sessa [7] generalized commuting mappings by calling mappings A and B from a metric space (X,d) into itself a weakly commuting pair if $d(ABx, BAx) \leq d(Ax, Bx)$ for all x in X. Any weakly commuting pair are obviously compatible, but the converse is false [3]. See [1] for other examples of the compatabile pairs which are not weakly commutative and hence not commuting pairs.

LEMMA 1.1 ([1]). Let A and B be compatible mappings from a metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z in X. Then $\lim_{n\to\infty} BAx_n = Az$ if A is continuous.

2. A FIXED POINT THEOREM.

Throughout this paper, suppose that the function $\Phi: [0,\infty)^5 \to [0,\infty)$ satisfies the following conditions:

(1) Φ is nondecreasing and upper semicontinuous in each coordinate variable,

(2) For each $t > 0$, $\psi(t) = \max \{\Phi(0,0,t,t,t), \ \Phi(t,t,t,2t,0), \Phi(t,t,t,0,2t)\} < t.$ (2.1)

LEMMA 2.1 ([12]). Suppose that $\Psi: [0,\infty) \to [0,\infty)$ is nondecreasing and upper semicontinuous from the right. If $\Psi(t) < t$ for every $t > 0$, then $\lim_{n\to\infty} \Psi^n(t) = 0$, where $\Psi^n(t)$ denotes the composition of $\Psi(t)$ with itself n-times.

Now, we are ready to state our main Theorem.

THEOREM 2.2. Let A,B,S, and T be mappings from a complete metric space (X,d) into itself. Suppose that one of A,B,S and T is continuous, the pairs A,S and B, T are compatible and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If the inequality

$$d(Ax,By) \leq \Phi(d(Ax,Sx), d(By,Ty), d(Sx,Ty), d(Ax,Ty), d(By,Sx)) \qquad (2.2)$$

holds for all x and y in X, where Φ satisfies (1) and (2), then A,B, S and T have a unique common fixed point in X.

PROOF. Let $x_o \in X$ be given. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we can choose x_1 in X such that $y_1 = Tx_1 = Ax_o$ and, for this point x_1, there exists a point x_2 in X such that $y_2 = Sx_2 = Bx_1$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1}. \qquad (2.3)$$

By (2.2) and (2.3), we have

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(Ax_{2n}, Bx_{2n+1}) \\ &\leq \Phi(d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}), \\ &\quad d(Ax_{2n}, Tx_{2n+1}), d(Bx_{2n+1}, Sx_{2n})) \\ &\leq \Phi(d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ &\quad 0, d(y_{2n+2}, y_{2n})) \\ &\leq \Phi(d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \\ &\quad 0, d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})). \end{aligned}$$

If $d(y_{2n+1}, y_{2n+2}) > d(y_{2n}, y_{2n+1})$ in the above inequality, then we have

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &\leqq \Phi(d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), \\ &\qquad d(y_{2n+1}), y_{2n+2}), 0, 2d(y_{2n+1}, y_{2n+2})) \\ &\leqq \Psi(d(y_{2n+1}, y_{2n+2})) < d(y_{2n+1}, y_{2n+2}), \end{aligned}$$

which is a contradiciton. Thus,

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &\leq \Phi(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1})\ d(y_{2n}, y_{2n+1}), \\ &\qquad 0, 2d(y_{2n}, y_{2n+1})) \\ &\leq \Psi(d(y_{2n}, y_{2n+1})). \end{aligned} \tag{2.4}$$

Similarly, we have

$$d(y_{2n+2}, y_{2n+3}) \leq \Psi(d(y_{2n+1}, y_{2n+2})). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$d_n = d(y_n, y_{n+1}) \leq \Psi(d(y_{n-1}, y_n)) \leq \ldots . \leq \Psi^{n-1}(d(y_1, y_2)). \tag{2.6}$$

By (2.6) and Lemma 2.1, we obtain

$$\lim_{n\to\infty} d_n = 0. \tag{2.7}$$

In order to show that $\{y_n\}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ such that, for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \varepsilon \text{ for } 2m(k) > 2n(k) \geq 2k. \tag{2.8}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2.8), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon \text{ and } d(y_{2n(k)}, y_{2m(k)}) > \varepsilon. \tag{2.9}$$

Then, for each even integer $2k$,

$$\varepsilon < d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

It follows from (2.7) and (2.9) that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon. \tag{2.10}$$

By the triangle inequality,

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1} \text{ and}$$

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1} + d_{2n(k)}.$$

From (2.7) and (2.10), as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon \text{ and } d(y_{2n(k)+1}, y_{2m(k)-1}) \to \varepsilon.$$

By (2.2) and (2.3), we have

$$d(y_{2n(k)}, y_{2m(k)}) \leq d_{2n(k)} + d(Ax_{2n(k)}, Bx_{2m(k)-1})$$
$$\leq d_{2n(k)} + \Phi(d_{2n(k)}, d_{2m(k)-1}, d(y_{2n(k)}, y_{2m(k)-1}),$$
$$d(y_{2n(k)+1}, y_{2m(k)-1}), d(y_{2m(k)}, y_{2n(k)})).$$

Since Φ is upper semicontinuous,

$$\varepsilon \leq \Phi(0, 0, \varepsilon, \varepsilon, \varepsilon) < \varepsilon \text{ as } k \to \infty,$$

which is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence and it converges to some point z in X. Consequently the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ converge to z. Suppose that S is continuous. Since A and S are compatible, Lemma 1.2 implies that

$$SSx_{2n} \text{ and } ASx_{2n} \to Sz.$$

By (2.2), we obtain

$$d(ASx_{2n}, Bx_{2n-1}) \leq \Phi(d(ASx_{2n}, SSx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}),$$
$$d(SSx_{2n}, Tx_{2n-1}), d(ASx_{2n}, Tx_{2n-1})\ \ d(Bx_{2n-1}, SSx_{2n})).$$

Letting $n \to \infty$, we have

$$d(Sz, z) \leq \Phi(0, 0, d(Sz, z), d(Sz, z), d(z, Sz)),$$

so that $z = Sz$. By (2.2), we also obtain

$$d(Az, Bx_{2n-1}) \leq \Phi(d(Az, Sz), d(Bx_{2n-1}, Tx_{2n-1}), d(Sz, Tx_{2n-1}),$$
$$d(Az, Tx_{2n-1}), d(Bx_{2n-1}, Sz)).$$

Letting $n \to \infty$, we have

$$d(Az, z) \leq \Phi(d(Az, Sz), 0, d(Sz, z), d(Az, z), d(z, Sz)),$$

so that $z = Az$. Since $A(X) \subset T(X)$, $z \in T(X)$ and hence there exists a point w in X such that $z = Az = Tw$.

$$d(z, Bw) = d(Az, Bw) \leq \Phi(0, d(Bw, Tw), d(Sz, Tw), d(Az, Tw), d(Bw,z)),$$

which implies that $z = Bw$. Since B and T are compatible and $Tw = Bw = z$, $d(TBw, BTw) = 0$ and hence $Tz = TBw = BTw = Bz$. Moreover, by (2.2),

$$d(z, Tz) = d(Az, Bz) \leq \Phi(0, d(Bz, Tz), d(z, Tz), d(z, Tz), d(Bz, z)),$$

so that $z = Tz$. Therefore, z is a common fixed point of A,B,S and T. Similarly, we can complete the proof in the case of the continuity of T. Now, suppose that A is continuous. Since A and S are compatible, Lemma 1.2 implies that

$$AAx_{2n} \text{ and } SAx_{2n} \to Az.$$

By (2.2), we have

$$d(AAx_{2n}, Bx_{2n-1}) \leq \Phi(d(AAx_{2n}, SAx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}),$$
$$d(SAx_{2n}, Tx_{2n-1}), d(AAx_{2n}, Tx_{2n-1}), d(Bx_{2n-1}, SAx_{2n})).$$

Letting $n \to \infty$, we obtain

$$d(Az, z) \leq \Phi(0, 0, d(Az, z), d(Az, z)\ d(z, Az)),$$

so that $z = Az$. Hence, there exists a point v in X such that $z = Az = Tv$.

$$d(AAz_{2n}, Bv) \le \Phi(d(AAx_{2n}, SAx_{2n}), d(Bv, Tv), d(SAx_{2n}, Tv),$$
$$d(AAx_{2n}, Tv)\ d(Bv, SAx_{2n})),$$

Letting $n \to \infty$, we have

$$d(z, Bv) \leqslant \Phi(0, d(Bv, Tv), d(z, Tv), d(Az, Tv), d(Bv,z)),$$

which implies that $z = Bv$. Since B and T are compatible and $Tv = Bv = z$, $d(TBv, BTv) = 0$ and hence $Tz = TBv = BTv = Bz$. Moreover, by (2.2), we have

$$d(Ax_{2n}, Bz) \leqslant \Phi(d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Sx_{2n}, Tz),$$
$$d(Ax_{2n}, Tz), d(Bz, Sx_{2n})).$$

Letting $n \to \infty$, $d(z, Bz) \leqslant \Phi(0, d(Bz, Tz), d(z, Tz), d(z, Tz), d(Bz, z))$, so that $z = Bz$. Since $B(X) \subset S(X)$, there exists a point w in X such that $z = Bz = Sw$.

$$d(Aw, z) = d(Aw, Bz) \leqslant \Phi(d(Aw, Sw), 0, d(Sw, z), d(Aw, z), d(z, Sw)),$$

so that $Aw = z$. Since A and S are compatible and $Aw = Sw = z$, $d(SBw, BSw) = 0$ and hence $Sz = SAw = ASw = Az$. Therefore z is a common fixed point of A,B,S and T. Similarly, we can complete the proof in the case of the continuity of B. It follows easily from (2.2) that z is a unique common fixed point of A,B,S and T.

COROLLARY 2.3. Let A,B,S and T be mappings from a complete metric space (X,d) into itself. Suppose that one of A,B,S and T is continuous, the pairs A,S and B,T are compatible and that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. If the inequality (2.2) holds for all x and y in X, where Φ satisfies (1) and (2.11);

$$\psi(t) = \max\{\Phi(t,t,t,t,t),\ \Phi(t,t,t,2t,0),\ \Phi(t,t,t,0,2t)\} < t \tag{2.11}$$

for each $t > 0$, then A,B,S and T have a unique common fixed point in X.

REMARK 2.4. From Theorem 2.2 and Corollary 2.3, we extend the results of Ding [11] and Diviccaro-Sessa [5] by employing compatibility in lieu of commuting and weakly commuting mappings, respectively. Further our theorem extends also a result of Ding [11] by using one continuous function as opposed to two.

REMARK 2.5. From Theorem 2.2 defining $\Phi: [0, \infty)^5 \to [0, \infty)$ by

$$\Phi(t_1,t_2,t_3,t_4,t_5) = h \max\{t_1,t_2,t_3, \tfrac{1}{2}(t_4+ t_5)\}$$

for all $t_1,t_2,t_3,t_4,t_5 \in [0,\infty)$ and $h \in [0, 1)$, we obtain a result of the third author [3] even if one function is continuous as opposed to two.

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