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THE DIOPHANTINE EQUATION $n^i + 1 = k(dn - 1)$

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ABSTRACT. The Diophantine equation of the title is solved for $i = 3,4$ and an infinite family of solutions were found for $i \geqq 5$.

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1. INTRODUCTION.

In this note we will find an infinite family of solutions of

$$n^i + 1 = k(dn - 1), \ k, \ d > 0, \tag{1.1}$$

and for $i = 3,4$, all solutions will be obtained.

This equation, for $i = 3$, was a problem in [1] and solved by Ligh [2]. The solutions were $n = 1,2,3$ and 5. For $i = 4$, the problem was proposed in [3] by K. Wilke and solved by the proposer in [4]. He showed that there are infinitely many values of n and all can be obtained from a single recurrence relation. It is the purpose of this note to show that there are infinite solutions for $i \geqq 5$ and, unlike $i = 4$, no single recurrence relation will yield all solutions.

2. SOLUTIONS FOR ARBITRARY i.

Reducing equation (1.1) to a congruence modulo n yields

$$k \equiv -1 \pmod{n}.$$

Hence there is a positive integer r such that $k = (rn - 1)$ and (1.1) can be written as

$$n^i + 1 = (dn - 1)(rn - 1). \tag{2.1}$$

We wish to find the set S of all triples (d,n,r) which satisfy (2.1). Clearly, if (d,n,r) is in S, then so is (r,n,d).

Finding an infinite family of solutions of (2.1) is facilitated by the following result:

THEOREM 1. If (d,n,r) satisfies (2.1) then there are positive integers x and y such that (x,d,n) and (n,r,y) also satisfy (2.1).

PROOF. Multiplying (2.1) by d^{i-1} and dividing by dn - 1 yields the following:

$$n^{i-1} d^{i-2} + n^{i-2} d^{i-3} + \ldots + n + \frac{n + d^{i-1}}{dn - 1} = d^{i-1} (rn - 1), \quad (2.2)$$

Since the right side of (2.2) is an integer it follows that dn - 1 must divide $n + d^{i-1}$. Let x be the positive integer such that

$$\frac{n + d^{i-1}}{dn - 1} = x. \quad (2.3)$$

Now multiplying (2.3) by d and rearranging gives the following equation:

$$d^i + 1 = (xd - 1)(nd - 1). \quad (2.4)$$

Hence (x,d,n) satisfies (2.1) and similarly there is a positive integer y such that (n,r,y) also satisfies (2.1).

Now equation (2.1) can be rewritten as follows.

$$n_1^i + 1 = (n_0 n_1 - 1)(n_2 n_1 - 1), \quad (2.5)$$

where (n_0, n_1, n_2) satisfies (2.1). Thus for each positive integer j, according to Theorem 1, there are integers n_{j-1}, n_j and n_{j+1} such that (n_{j-1}, n_j, n_{j+1}) satisfies (2.1) and

$$n_j^i + 1 = (n_{j-1} n_j - 1)(n_{j+1} n_j - 1). \quad (2.6)$$

THEOREM 2. If i > 3, then (2.1) has an infinite family of solutions.

PROOF. For n = 1, (2,1,3) and (3,1,2) are the only triples satisfying (2.1). Starting with either one, we obtain an infinite family of solutions if for each j, $n_{j+1} > n_j$. Clearly $n_1 = 1$ and $n_2 > n_1$. Suppose $n_j > n_{j-1}$, solving for n_{j+1} in (2.6), we have

$$n_{j+1} = \frac{n_j^{i-1} + n_{j-1}}{n_j n_{j-1} - 1} > \frac{n_j^{i-1}}{n_j^2} = n_j^{i-3}.$$

Hence, by induction, $n_{j+1} > n_j$ if i > 3 and (2.1) has an infinite family of solutions:

$(2,1,3) \to \ldots \to (n_{j-1}, n_j, n_{j+1}) \to \ldots$, or $(3,1,2) \to \ldots \to (n_{j-1}, n_j, n_{j+1}) \to \ldots$.

3. THE CASE $i = 3$ OR $i = 4$.

For $i = 3$, the only solutions are $n = 1, 2, 3$ and 5 and can be found in [2].

For $i = 4$, the solution were given in [4]. It was shown that if (d,n,r) with $d \leqq r$ is a nontrivial triple (i.e. $n > 1$) satisfying (2.1), then $d < n < r$. Thus, from Theorem 1, there is a positive integer x so that (x,d,n) satisfies (2.1). Moreover, if $d > 1$, then $x < d < n$. By continuing this process, starting from any nontrivial solution of (2.1), eventually one will reach a solution of the form $(1,n,r)$. But when $d = 1$, the solutions of (2.1) with $i = 4$ are (1,2,9) and (1,3,14). Thus the integers n satisfying (2.1) with $i = 4$ are precisely those integers in the sequence $n_1 < n_2 < n_3 <$. where any three consecutive terms satisfy (2.6) with $i = 4$. Furthermore, (2.6) is equivalent to

$$n_{j-1}n_{j+1} = n_j^{i-2} + \frac{n_{j-1} + n_{j+1}}{nj}, \tag{3.1}$$

and thus for any j there is a positive integer k_j such that

$$n_{j-1} + n_{j+1} = k_j n_j. \tag{3.2}$$

It was shown in [4] that if $i = 4$, then $k_j = 5$ for all j. Thus the values of n satisfying (2.1) with $i = 4$ are those in the sequences (with $n_1 = 1$, $n_2 = 2$ or $n_1 = 1$, $n_2 = 3$):

1, 2, 9, 43, 206, 987, ...

and

1, 3, 14, 67, 321, 1538, ...

4. THE CASE $i \geqq 5$.

Even though (2.1) has an infinite family of solutions for $i > 3$, unlike the case $i = 4$, no single recurrence relation will yield all solutions for $i \geqq 5$. We will accomplish this by showing that for each $i \geqq 5$, there is a triple (d,n,r) satisfying (2.1) and yet it cannot be obtained from the two trivial solutions (3,1,2) and (2,1,3). We will call such a triple primitive. Furthermore, we will also prove that, unlike the case $i = 4$, the integer k_j in (3.2) is not a constant. Hence, even though each primitive triple generates an infinite family of solutions, no single recurrence relation, as in the case $i = 4$, can describe the family.

THEOREM 3. For each $i \geqq 5$, there is a primitive triple (d,n,r), $n \geqq 2$, satisfying (2.1) besides the two trivial ones, (3,1,2) and (2,1,3).

PROOF. We will exhibit a triple (d,n,r) satisfying (2.1) that cannot be obtained from either (3,1,2) or (2,1,3) by applying Theorem 1. For i odd or $i = 2^x j$, $x \geqq 1$, j odd and $j \geqq 3$, $n^i + 1$ can be factored. Hence there are integers a and b such that $(2,2,a)$ and $(2^{2^x-1} + 1,2,b)$ satisfy (2.1) for i odd and $i = 2^x j$ respectively. They are primitive because the only triples with 2 in the middle obtained from either (3,1,2) or (2,1,3) are $(1,2,y)$ and $(z,2,1)$.

For $i = 2^x$, there are two cases: (i) if $2^{2^x} + 1$ is not a prime, then $2^{2^x} + 1 = ab$ where a and b are odd. Thus $(d,2,r)$ is primitive where $d = \frac{a+1}{2} > 1$ and $r = \frac{b+1}{2} > 1$. (ii) if $2^{2^x} + 1$ is a prime, then the only known values of x are $x = 0,1,2,3,4$. We need to consider $x = 3$ and $x = 4$ only. If $x = 3$, then $i = 8$ and

$$3^8 + 1 = 17 \cdot 386 = (6 \cdot 3 - 1)(129 \cdot 3 - 1).$$

Hence $(6,3,129)$ is primitive because the only triples with 3 in the middle from $(3,1,2)$ or $(2,1,3)$ are $(x,3,1)$ and $(1,3,y)$. If $x = 4$, then $i = 16$ and

$$6^{16} + 1 = 1697 \cdot 1662410081 = (283 \cdot 6 - 1)(277068347 \cdot 6 - 1).$$

Again, starting with $(3,1,2)$ or $(2,1,3)$, applying Theorem 1, one does not get $(283,6,277068347)$. Hence it is primitive.

Recall that when $i = 4$, there are only two primitive triples satisfying (2.1), namely $(3,1,2)$ and $(2,1,3)$. Starting from either one, using Theorem 1, one obtains an infinite family of solutions and furthermore, all of them are given by the recurrence relation (3.1) with $k_j = 5$ for all j.

In order to show $k_j = 5$ for all j when $i = 4$, it was shown in [4] that, according to (3.1) and (3.2),

$$\frac{n_{j-1} + n_{j+1}}{n_j} = \frac{n_j + n_{j+2}}{n_{j+1}} \quad \text{for all } j. \tag{4.1}$$

However, for $i \geq 5$, (4.1) no longer holds.

THEOREM 4. $\dfrac{n_{j-1} + n_{j+1}}{n_j} = \dfrac{n_j + n_{j+2}}{n_{j+1}}$

if and only if $i = 4$.

PROOF. From (2.6),

$$n_{j+2} = \frac{n_{j+1}^{i-1} + n_j}{n_j n_{j+1} - 1} \quad \text{and} \quad n_{j-1} = \frac{n_j^{i-1} + n_{j+1}}{n_{j+1} n_j - 1}.$$

Using the above and simplify, we have

$$\frac{n_j + n_{j+2}}{n_{j+1}} = \frac{n_j^2 + n_{j+1}^{i-2}}{n_{j+1} n_j - 1} \quad \text{and} \quad \frac{n_{j-1} + n_{j+1}}{n_j} = \frac{n_j^{i-2} + n_{j+1}^2}{n_{j+1} n_j - 1}.$$

Hence $n_j^2 + n_{j+1}^{i-2} = n_j^{i-2} + n_{j+1}^2$ implies

$$n_{j+1}^2 (n_{j+1}^{i-4} - 1) = n_j^2 (n_j^{i-4} - 1). \tag{4.2}$$

By Theorems 2, $n_{j+1} > n_j$ for all j and thus (4.2) holds if and only if $i = 4$.

We can conclude from Theorem 4, even though one gets an infinite family of solutions starting with any primitive triple, for $i \geq 5$ no single relation describes all solutions.

5. CONCLUDING REMARKS.

Equation (2.1) can be written as

$$n^{i-1} - drn + (d + r) = 0. \tag{5.1}$$

A similar equation,

$$n^2 + (d + r)n = kdr, \tag{5.2}$$

was investigated by W.R. Utz in [5] and he obtained all solutions. Because of the similarity of (5.1) and (5.2), at least in appearance, and the fact that there are other primitive triples for $i \geq 5$ in (2.1), we conclude this note with the following problems:

Problem 1. Find the solutions of

$$n^i + (d + r)n = kdr.$$

Problem 2. Find all primitive triples, if possible, of equation (2.1) for $i \geq 5$.

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