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*—INDUCTIVE LIMITS AND PARTITION OF UNITY

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ABSTRACT In this note we define and discuss some properties of partition of unity on *–inductive limits of topological vector spaces. We prove that if a partition of unity exists on a *–inductive limit space of a collection of topological vector spaces, then it is isomorphic and homeomorphic to a subspace of a *–direct sum of topological vector spaces.

KEY WORDS AND PHRASES. Partition of unity, *–inductive limits, *–direct sum.
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1. INTRODUCTION

M. De Wilde [1] introduced the concept of partition of unity in an inductive limit space of a family of locally convex spaces which extends the usual partition of unity in function spaces. Around the same time S.O. Iyahen [2] introduced *–inductive limits of topological vector spaces, not necessarily locally convex, as a generalisation of inductive limits. In this paper, we consider the notion of partition of unity in *–inductive limit spaces of topological vector spaces and obtain some useful results some of which are analogous to De Wilde's results in [1]. In section 2, we briefly discuss the well–known concept of F–semi–norms in topological vector spaces. The details may be found in [6]. In section 3, we define the concept of partition of unity in *–inductive limit and using this, obtain a family of F–semi–norms defining the *–inductive limit topology. Finally we conclude with a representation theorem of *–inductive limit space with a partition of unity.

We prove that if a partition of unity exists on a *–inductive limit space of a collection of topological vector spaces, then it is isomorphic and homeomorphic to a subspace of a *–direct sum of topological vector spaces.

2. F–SEMI–NORMS

Let E be a vector space over k where k is the field real or complex numbers.

DEFINITION 2.1

An F–semi–norm on E is a mapping $\nu : E \to \mathbb{R}$ such that

(i) $\nu(x) \geq 0$ for all $x \in E$;

(ii) $\nu(\lambda x) \leq \nu(x)$ for all $x \in E$ and for all $|\lambda| \leq 1$;

(iii) $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in E$;

(iv) for each $x \in E$, $\nu(\lambda x) \to 0$ as $\lambda \to 0$.

Suppose that $V = \{\nu_\alpha : \alpha \in \Lambda\}$ is a family of F–semi–norms on E. Then V determines a linear topology η on E. A base of η–neighbourhoods of the origin in E consists of sets of the form

$$U_{\nu_{\alpha_1}, \nu_{\alpha_2}, \nu_{\alpha_n}, \epsilon} = \{x \in E : \nu_{\alpha_j}(x) < \epsilon, j = 1,2,\ldots,k\}$$

where ϵ is an arbitrary positive number and $\nu_{\alpha_1}, \nu_{\alpha_2}, \ldots, \nu_{\alpha_n}$ is any finite subcollection of V. Also, it is clear that each $\nu_\alpha \in V$ is η–continuous and η is the topology on E determined by the family Q of all η–continuous F–semi–norms on E. In fact, an F–semi–norm $\mu \in Q$ if and only if, for each $\epsilon > 0$ there exists a $\delta > 0$ and a finite collection $\nu_{\alpha_1}, \nu_{\alpha_2}, \ldots, \nu_{\alpha_n}$ of V such that

$$U_{\nu_{\alpha_1}, \nu_{\alpha_2}, \ldots, \nu_{\alpha_n}, \delta} \subseteq \{x : \mu(x) < \epsilon\}.$$

Conversely, we have the following:

THEOREM 2.1

A vector space topology on E can always be determined by a family of F–semi–norms.

Proof: see [6], chapter 1, Proposition 2.

3. PARTITION OF UNITY:

Let (E, τ) be the *–inductive limit of a family of topological vector spaces (E_i, τ_i) $i \in I$, an index family, relative to linear maps $u_i : E_i \to E$. Suppose further that the index set I is directed and that for each pair indices $i, j \in I$ with $i < j$, there is a continuous linear map $v_{ij} : E_i \to E_j$ such that $u_i = u_j \circ v_{ij}$.

DEFINITION 3.1 A partition of unity on E is defined to be a family of linear maps (T_i) $(i \in I)$, $T_i : E \to E_i$, which satisfies the following conditions.

(i) $T_i \circ u_j$ is continuous for each pair (i–j).

(ii) For each $j \in I$, $T_i \circ u_j = 0$ except for a finite number of $i \in I$.

(iii) $\sum_{i \in I} u_i \circ T_i$ is the identity map on E.

Remark: We note that the condition (i) is equivalent to the following condition:

(i) each $T_i : E \to E_i$ is continuous.

Example 3.2 Suppose (E, τ) is the inductive limit of locally convex spaces (E_i, τ_i) $(i \in I)$ with $\{T_i\}$ $(i \in I)$ is a partition of unity of (E,τ). Then since τ is coarser than the

*–inductive limit topology τ^* on E, it follows that $\{T_i\}$ $(i \in I)$ is also a partition of unity of (E, τ^*).

Example 3.3 Let $\{E_n\}$ $(n = 1,2,...)$ be a sequence of topological vector spaces, E be the *–direct sum of the E_n's as defined in [2], and let $\{P_n\}$ $(n = 1,2,...)$ be the projection maps of E onto E_n. Then, E is the *–inductive limit of the sequence $\{\overset{N}{\underset{i=1}{\oplus}} E_i\}$ $(N = 1,2,...)$ and the maps $\{P_n\}$ constitute a partition of unity.

We now consider some properties of the *–inductive limit space (E, τ) with a partition of unity $\{T_i\}$ $(i \in I)$ but first some notations.

For each $i \in I$, let P_i be a family of F–semi–norms on E_i. Then P_i determines a linear topology τ_i on E_i and let $Q_i = \{v_i^a : a \in \Gamma_i\}$ be the family of all τ_i–continuous F–semi–norms on E_i. For each collection s of F–semi–norms $\{v_i^a : v_i^a \in Q_i\}$ $(i \in I)$ and each set σ of positive real numbers $\{c_i\}$ $i \in I$, we define a non–negative real–valued function π_σ^s on E by the equation

$$\pi_\sigma^s (x) = \sum_{i \in I} c_i \, \nu_i^a (T_i x) \quad \text{for } x \in E. \tag{3.1}$$

It is easy to verify that π_σ^s is a well–defined, F–semi–norm on E. By Π we denote the family of all such F–semi–norms π_σ^s for every collection of σ and s.

THEOREM 3.4 The *–inductive limit topology τ on E is given by the family Π of F–semi–norms π_σ^s defined by the equation 3.1.

PROOF Let τ_Π be the linear topology on E generated by the collection Π We have to prove that $\tau = \tau_\Pi$. We will do this in two steps. First, to prove that τ_Π is coarser than τ, it is sufficient to show that each $u_j : (E_j, \tau_j) \to (E, \tau_\Pi)$ is continuous. See [4]. Now each u_j is continuous, if and only if for any $\pi_\sigma^s \in \Pi$, $\pi_\sigma^s \circ u_j : E_j \to \mathbb{R}$ is continuous

In fact, for each $x \in E_j$,

$$\pi_\sigma^s (u_j x) = \sum_{i \in I} c_i \, \nu_i^a (T_i u_j x).$$

But $T_i \circ u_j$ is equal to 0 except for a finite number of indices $i \in I$.

Let $J = \{i \in I: T_i \circ u_j \neq 0\}$.

Now each $T_i \circ u_j$ is continuous from E_j into E_i, and so $\nu_i^a (T_i \circ u_j)$ is τ_j–continuous Thus we can write $\pi_\sigma^s \circ u_j = \sum_{i \in J} c_i \, (v_i^a (T_i \circ u_j))$ and so $\pi_\sigma^s \circ u_j$ is continuous. From that it follows that $\tau_\Pi \subseteq \tau$

For each $x \in E$,

$$\nu(x) = \nu\left[\sum_{i \in I} u_i \circ T_i x\right]$$
$$\leq \sum_{i \in I} \nu(u_i T_i x).$$

Now $\nu_0 u_i$ is a τ_i–continuous F–semi–norm on E_i and so belongs to Q_i. Hence

$$\nu(x) \leq \sum (\nu_0 u_i)(T_i x) = \pi_\sigma^s (x).$$

Here $s = \{\nu_0 u_i\}$ $(i \in I)$, and $c_i = 1$ for each $i \in I$. This implies that the identity map $(E, \tau_\Pi) \to (E, \tau)$ is continuous and so τ is coarser that τ_Π as required. This completes the proof.

COROLLARY 3.5 If each E_i $(i \in I)$ is separated, then (E, τ) is separated.

THEOREM 3.6 If B is a bounded set in E, then $T_i b = 0$ except for a finite number of indices $i \in I$. Hence B is bounded in E if and only if there exists a continuous linear mapping T from E onto some E_i such that $B = u_i TB$.

The proof is analogous to that of the corresponding result in ([1], p3) and so is omitted here.

COROLLARY 3.7 If each $\{E_i\}$ is sequentially complete, then E is sequentially complete.

PROOF: Let $\{x_n\}$ be a Cauchy sequence in E. Then $\{x_n\}$ is a bounded set in E, and so, by theorem 3.6, there exists a continuous linear mapping T from E into some E_i such that $\{x_n\} = u_i\ T\{x_n\}$.

Since a continuous linear mapping from one topological vector space into another takes Cauchy sequences to Cauchy sequences, $T\{x_n\}$ is a Cauchy sequence in E_i. Now E_i is sequentially complete, and so $T\{x_n\}$ converges to a point x say in E_i. Therefore $u_i\ T\{x_n\}$ converges to $u_i\ x$, since u_i is a continuous linear mapping. Therefore $\{x_n\}$ converges to a point in E. Hence the result.

At present it is not known whether the completeness of each (E_i, τ_i) implies the completeness of $\{E,\tau)$. Lastly we prove that the collection of numbers in σ of Π_σ^s can be chosen in an economical way. An useful application of this is given in [4].

PROPOSITION 3.8 Let $\sigma' = \{c_i : c_i \geq 1\}$. If Π' denotes all F–semi–norms of the form $\Pi_{\sigma'}^s$, for various collections of s and σ', then $\tau_\Pi = \tau_{\Pi'}$ where τ_Π and $\tau_{\Pi'}$ denote the topology generated by Π and Π' respectively.

PROOF It is obvious that $\Pi' \subseteq \Pi$ and so it is clear that $\tau_{\Pi'}$ is coarser than τ_Π. Conversely let U be a τ_Π–neighbourhood of the origin in E. Then V contains a set V of the form

$$V = \{x \in E: \pi_{\sigma_n}^{s_n}(x) < \epsilon;\ n = 1,2,...m;\ \epsilon > 0\} \text{ where } \pi_{\sigma_n}^{s_n}(x)$$

$$= \sum c_i^{(n)}\ \nu_i^{a(n)}\ (T_i x)$$

Now let for any real number r, [r] denote the greatest integer $\leq$ r then
$c_i^{(n)} < [c_i^{(n)}] + 1;$

and if we denote $\sigma'_n = \{[c_i^{(n)}] + 1\}$, then it is easy to see that $\pi_{\sigma'_n}^{s_n}$ (x) for all $x \in E$. So we have $U \supseteq V \supseteq V'$, where $V' = \{x \in E: \pi_{\sigma'_n}^{s\ n}(x) < \epsilon; n = 1,2,\ldots,m, \epsilon > 0\}$ is a $\tau_{\Pi'}$–neighbourhood of the origin. Thus we have τ_{Π} is coarser than $\tau_{\Pi'}$, and so $\tau_{\Pi} = \tau_{\Pi'}$.

4. DIRECT SUM

In this section we give an analogue of a representation theorem given by D. Keim in [3]. Let (E, τ) be the *–inductive limit of topological vector spaces (E_i, τ_i) $(i \in I)$ relative to linear maps $u_i = E_i \to E$. Suppose, further that, a partition of unity $\{T_i\}$ is defined on (E, τ). Then we have the following representation theorem.

THEOREM 4.1 (E, τ) is isomorphic and homeomorphic to a subspace of a *–direct sum of topological vector spaces.

PROOF: Define a linear map Φ from (E, τ) into the *–direct sum of E_i's as follows:

$$\Phi : E \to \sum_{i \in I} E_i \text{ given by } \Phi(x) = (T_i x) \text{ for } x \in E.$$

This mapping is well–defined and one–to–one since $\{T_i\}$ satisfies the conditions (ii) and (iii) of partition of unity respectively. It is easy to check that Φ is a linear map and so, is an isomorphism. Moreover that Φ is continuous is shown as follows.

By condition (ii) of partition of unity, $T_i o u_j = 0$ except for a finite number of $i \in I$ and for each fixed $j \in I$. Let $i_1, i_2, \ldots, i_n$ be the finite number of indices such that $T_{i_k} o u_j = 0$ for $k = 1,2,\ldots,n$. Then $\Phi o u_j = (\sum I_{i_k} o T_{i_k}) o\, u_j$ where I_{i_k} is the injection map of $E_{i_k} \to \sum_{i \in I} E_i$. Now for each i_k, $k = 1,2,\ldots,m$, $I_{i_k} o T_{i_k} o u_j$ is continuous by condition (i) of partition of unity and continuity of each I_{i_k}, $k = 1,2,\ldots,n$. Therefore $\Phi o u_j$ is continuous for each $j \in I$. Consequently Φ is continuous [5], as required.

Conversely, let Φ' be a linear map defined by $\Phi' : \sum_{i \in I} E_i \to E$

$$\Phi'(x_i) = \sum_{i \in I} u_i(x_i).$$

This is well–defined since $x_i = 0$ except for a finite number of $i \in I$. Moreover, Φ' is linear and $\Phi' \mid \Phi(E) = \Phi^{-1}$. Also, $\Phi' o I_i = u_i$ is continuous from $E_j \to E$ for each $j \in I$. Hence Φ' is continuous. Thus Φ is an isomorphism and a homeomorphism from E onto $\sum_{i \in I} E_i$.

REFERENCES

1. DE WILDE, M., Inductive limits and partition of unity, *Manuscripta Math.*,**5**, (1971), 45–58.

2. IYAHEN, S.O., On certain classes of linear topological spaces, *Proc. London Math. Soc.* (3), **18**, (1968), 285–307.

3. KEIM, D. Induktive and projective limiten Mit Zerlegung der Einheit, *Manuscripta Math.*, 10, (1973), 191–195.

4. MURALI, V. Partition of unity on algebraic inductive limits, *Math. Japonica* 22, (1977), 497–500.

5. MURALI, V. Suprabarrels in topological vector spaces, *Math. Japonica* 32, (1987) 797–800.

6. WEALBROECK, L. Topological Vector Spaces and Algebras, Springer–Verlag, Lecture notes (1971) 230.