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A NOTE ON EQUIVALENT INTERVAL COVERING SYSTEMS FOR HAUSDORFF DIMENSION ON R

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ABSTRACT. The Hausdorff dimension of a set in R is usually defined by considering countable coverings of the set by general intervals. In this note we establish sufficient conditions under which coverings whose members are restricted to a particular family g of intervals will produce the same value for dimension. A result of Billingsley is then employed to obtain a general technique for computing the dimensions of sets defined by certain types of generalized expansions. A specific example is included.

KEY WORDS AND PHRASES. Hausdorff dimension, Vitali covering, self-similar.

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1. INTRODUCTION.

Let $\alpha \geq 0$. The α-outer Hausdorff measure of a set $E \subseteq R$ is usually defined by

$$H^{\alpha}(E) = \lim_{\delta \to 0^+} \inf_{\substack{\bigcup_j I_j \supseteq E \\ d(I_j) \leq \delta}} \sum_{j=1}^{\infty} d(I_j)^{\alpha}$$

where I_j is an interval in R and $d(I_j)$ denotes the diameter of I_j. It is easy to see that the value of $H^{\alpha}(E)$ is unchanged when the coverings of E are restricted to being closed intervals. (It will be convenient for us to consider only closed intervals in Section 2.) It is well known that there exists a unique point α_0 such that $H^{\alpha}(E) = \infty$ for $\alpha < \alpha_0$ and $H^{\alpha}(E) = 0$ for $\alpha > \alpha_0$. This value α_0 is called the Hausdorff dimension of E (denoted by $\dim(E)$).

In actually computing the dimension of a set it is frequently useful to be able to consider only coverings from a restricted family of intervals. In [1] Besicovitch established that coverings by dyadic intervals (i.e. intervals of the form $[j 2^{-n}, (j+1)2^{-n}))$ produce the same dimension for sets. Billingsley [2] extended this result to r-adic intervals where $r \geq 2$ is some positive integer. However in [3] Billingsley remarked that he knew of no general conditions on interval families which would guarantee preservation of the correct dimension value for all sets. We address this problem in Section 2.

2. COVERING RESULTS.

Let $F \subseteq \mathbf{R}$ be a closed interval. A collection g of intervals is a Vitali covering of F if for each $\epsilon>0$ and each $x \in F$ there exists $I \in g$ such that $x \in I$ and $d(I)<\epsilon$. If an interval collection g is a Vitali covering of F then, for each $E \subseteq F$, we can define $H_g^\alpha(E)$ and $\dim_g(E)$ by

$$H_g^\alpha(E) = \lim_{\delta \to 0^+} \inf_{\substack{\bigcup_j I_j \supseteq E \\ I_j \in g \\ d(I_j) \leq \delta}} \sum_{j=1}^{\infty} d(I_j)^\alpha$$

and $\dim_g(E) = \sup\{\alpha \mid H_g^\alpha(E) = \infty\} = \inf\{\alpha \mid H_g^\alpha(E) = 0\}$ where usual Hausdorff dimension $\dim(E)$ results when g is taken to be the collection of all closed intervals. We automatically have $\dim(E) \leq \dim_g(E)$ for any Vitali covering g so to demonstrate equality we need only show $\dim_g(E) \leq \dim(E)$. A property of dimension which will be useful in this is the result $\dim_g(\bigcup_k E_k) = \sup_k \dim_g(E_k)$ for any countable collection $\{E_k\}_k$.

We will call g a bounded Vitali covering of F if, for each $x \in F$, there exists a sequence of intervals from g (which we will denote by $\{I_j(x)\}_j$) such that

(i) $x \in I_j(x)$ for each j

(ii) $d(I_j(x)) \downarrow 0$

(iii) $\inf_j \dfrac{d(I_{j+1}(x))}{d(I_j(x))} = b(x) > 0.$

We will say that a bounded Vitali covering of F is open if, for each $x \in F$, the sequence $\{I_j(x)\}_j$ can be chosen so that $x \in I_j^0(x)$ for each j (where $I_j^0(x)$ denotes the interior of $I_j(x)$).

Theorem 2.1 deals with open bounded Vitali coverings.

THEOREM 2.1. Let g be an open bounded Vitali covering of a closed interval $F \subseteq \mathbf{R}$. Then $\dim_g(E) = \dim(E)$ for all $E \subseteq F$.

PROOF. Let $E \subseteq F$. Then we can write

$$E = \bigcup_{i=2}^{\infty} \bigcup_{m=1}^{\infty} E_{i,m}$$

where $E_{i,m} = \{x \in E \mid \inf_j \dfrac{d(I_{j+1}(x))}{d(I_j(x))} \geq 1/i$ and $d(I_1(x)) \geq 1/m\}$. To prove the theorem it is sufficient to show $\dim_g(E_{i,m}) \leq \dim(E_{i,m})$. Let $0<\delta<1/m$ be arbitrary and suppose $\{F_k\}_k$ is a countable covering of $E_{i,m}$ by closed intervals with $d(F_k) \leq \delta$ for each k. Without loss of generality we can assume $F_k \subseteq F$ and $F_k \cap E_{i,m} \neq \varnothing$. For each k we will show that there exist four intervals from g which cover $F_k \cap E_{i,m}$ and whose diameters do not exceed $i\, d(F_k)$. For every $x \in F_k \cap E_{i,m}$ there exists $I_{j(k)}(x)$ such that $d(I_{j(k)}(x)) > d(F_k) \geq d(I_{j(k)+1}(x))$. Writing $F_k = [a_k, b_k]$ then we must have $[a_k, x] \subseteq I_{j(k)}(x)$ or $[x, b_k] \subseteq I_{j(k)}(x)$. Without loss of generality we will assume there exists at least one $x \in F_k \cap E_{i,m}$ for which $[a_k, x] \subseteq I_{j(k)}(x)$ since the argument is analogous when beginning with the assumption $[x, b_k] \subseteq I_{j(k)}(x)$. Let $x_0 = \sup\{x \in F_k \cap E_{i,m} \mid [a_k, x] \subseteq I_{j(k)}(x)\}$. Now $x_0 \in F_k \subseteq F$ and so there exists k' such that $d(I_{k'}(x_0)) \leq i\, d(F_k)$. Let $x_0^- \leq x_0$ be some point in $F_k \cap E_{i,m}$ such that $[a_k, x_0^-] \subseteq I_{j(k)}(x_0^-)$ and $I_{j(k)}(x_0^-) \cap I_{k'}(x_0) \neq \varnothing$. This is possible from the definition of x_0 and the assumption $x_0 \in I_{k'}^0(x_0)$. If $x > x_0$ implies $x \notin F_k \cap E_{i,m}$ then $I_{j(k)}(x_0^-) \cup I_{k'}(x_0)$ covers $F_k \cap E_{i,m}$. If however there exists $x > x_0$ such that

$x \epsilon F_k \cap E_{i,m}$ then it follows that $[x, b_k] \subseteq I_{j(k)}(x)$ and we define $x_1 = \inf\{x \epsilon F_k \cap E_{i,m} \mid [x, b_k] \subseteq I_{j(k)}(x)\}$. We can then find $x_1^+ \geq x_1$ with $x_1^+ \epsilon F_k \cap E_{i,m}$ and k'' such that $[x_1^+, b_k] \subseteq I_{j(k)}(x_1^+)$, $d(I_{k''}(x_1)) \leq i\ d(F_k)$ and $I_{j(k)}(x_1^+) \cap I_{k''}(x_1) \neq \emptyset$. Then $F_k \cap E_{i,m} \subseteq I_{j(k)}(x_0^-) \cup I_{k'}(x_0) \cup I_{k''}(x_1) \cup I_{j(k)}(x_1^+)$. Now $\dfrac{d(I_{j(k)}(x_0^-))}{d(F_k)} \leq \dfrac{d(I_{j(k)}(x_0^-))}{d(I_{j(k)+1}(x_0^-))} \leq i$ and similarly $\dfrac{d(I_{j(k)}(x_1^+))}{d(F_k)} \leq i$. Thus we obtain

$$\sum_{k=1}^{\infty} [d(I_{j(k)}(x_0^-))^\alpha + d(I_{k'}(x_0))^\alpha + d(I_{k''}(x_1))^\alpha + d(I_{j(k)}(x_1^+))^\alpha] \leq 4i^\alpha \sum_{k=1}^{\infty} d(F_k)^\alpha.$$

Hence we conclude

$$\inf_{\substack{\bigcup_j I_j \supseteq E_{i,m} \\ I_j \epsilon g \\ d(I_j) \leq i\delta}} \sum_{j=1}^{\infty} d(I_j)^\alpha \leq 4i^\alpha \inf_{\substack{\bigcup_j F_j \supseteq E_{i,m} \\ F_j \text{ closed interval} \\ d(F_j) \leq \delta}} \sum_{j=1}^{\infty} d(F_j)^\alpha.$$

Letting $\delta \to 0^+$ we obtain $H_g^\alpha(E_{i,m}) \leq 4i^\alpha H^\alpha(E_{i,m})$. This shows $\dim_g(E_{i,m}) \leq \dim(E_{i,m})$ as required and completes the proof of the theorem. □

Of more general use is the following theorem in which the open requirement is relaxed. We will call a bounded Vitali covering of F <u>complete</u> if for each $x \epsilon F$ the sequence $\{I_j(x)\}_j$ can be chosen so that whenever x is an endpoint of $I_j(x)$ (if at all) there exists $I \epsilon g$ with $d(I) \leq d(I_j(x))$ and $\epsilon > 0$ such that

(i) $(x - \epsilon, x) \subseteq I$ if x is the left hand endpoint of $I_j(x)$ (but not the left hand endpoint of F)

(ii) $(x, x + \epsilon) \subseteq I$ if x is the right hand endpoint of $I_j(x)$ (but not the right hand endpoint of F).

THEOREM 2.2. Let F be a closed interval of $\mathbb{R}$ and g a complete bounded Vitali covering of F. Then $\dim_g(E) = \dim(E)$ for all $E \subseteq F$.

PROOF. The proof follows that of Theorem 2.1 until discussion of x_0 is reached. If $x_0 \epsilon I_{k'}^0(x_0)$ or is the right hand endpoint of $I_{k'}(x_0)$ then no modification is required. However if x_0 is the left hand endpoint of $I_{k'}(x_0)$ we need to add in the interval I provided by (i) in the definition of a complete covering. We then select $x_0^- \leq x_0$ so that $I_{j(k)}(x_0^-) \cap I \neq \emptyset$. The analogous modification must be done in the case of x_1 if x_1 is the right hand endpoint of $I_{j(k)}(x_1)$. Consequently $F_k \cap E_{i,m}$ can be covered by six intervals from g, none of whose diameters exceed $i\ d(F_k)$. □

The hypotheses of Theorem 2.2 are satisfied by a wide class of coverings which includes the r-adic intervals but is much more extensive. In the next section we consider a technique for obtaining the dimension of certain sets.

3. COMPUTING DIMENSION.

In [2], [3], and [4] Billingsley developed a technique for computing the dimensions of sets defined in terms of r-adic expansions by considering limits of the form $\lim_{n \to \infty} \dfrac{\log \gamma(I_n(x))}{\log \lambda(I_n(x))}$ where γ is a (suitably chosen) diffuse probability distribution on the Borel sets of $[0,1]$, λ represents Lebesgue measure, and $I_n(x)$ is the r-adic interval of length r^{-n} containing x. This technique for "r-adic sets" arose out of results of Billingsley [2, 4] concerning the dimensions of certain sets obtained from discrete-time stochastic processes. We will show that these same results, along with Theorem 2.2, can be used to develop the analogous technique for computing dimensions of sets determined by more generalized expansions.

We begin by reviewing the necessary definitions and results from [2] and [4]. Let $I_1, I_2, \ldots$ be a stochastic process on a probability space $(X, \mathcal{F}, \mu)$ taking values in a countable, possibly finite, set S. A subset of X of the form $c(i_1, i_2, \ldots, i_n) = \{x \in X \mid I_1(x) = i_1, I_2(x) = i_2, \ldots, I_n(x) = i_n\}$, where $i_1, i_2, \ldots, i_n$ are members of S, is called an n-cylinder. Assume that each "∞-cylinder" has μ-measure zero i.e. $\mu(\{x \in X \mid I_1(x) = i_1, I_2(x) = i_2, \ldots\}) = 0$ for each sequence $i_1, i_2, \ldots$. Let E be a subset of X. For each $0 \le \alpha \le 1$ define

$$L_\mu^\alpha(E) = \lim_{\delta \to 0^+} \inf_{\substack{\bigcup_i c_i \supseteq E \\ \mu(c_i) < \delta}} \sum_i \mu(c_i)^\alpha$$

where each c_i is an n-cylinder (for some n). With methods similar to those used in the case of α-outer Hausdorff measure it can be shown that L_μ^α is an outer measure on the subsets of X and for each $E \subseteq X$ there exists a unique point α_0, $0 \le \alpha_0 \le 1$, such that $L_\mu^\alpha(E) = \infty$ for $\alpha < \alpha_0$ while $L_\mu^\alpha(E) = 0$ for $\alpha > \alpha_0$. The number α_0 is called the μ-dimension of E and denoted by $\dim_\mu(E)$. This value generally depends on μ, E, and the underlying stochastic process $I_1, I_2, \ldots$ (since the cylinders are determined by the process). It is not difficult to show that $\mu(E) > 0$ implies $\dim_\mu(E) = 1$ and this fact will prove very useful. We will also need the following result.

THEOREM 3.1. (Billingsley): Let $(X, \mathcal{F}, \mu)$ and $I_1, I_2, \ldots,$ be as defined above and, for each $x \in X$, let $c_n(x)$ denote the n-cylinder containing x. That is, $c_n(x) = \{x' \in X \mid I_1(x') = I_1(x), \ldots, I_n(x') = I_n(x)\}$. Let γ be another probability distribution on $(X, \mathcal{F})$ which assigns measure zero to each ∞-cylinder. If

$$E \subseteq \left\{ x \in X \mid \lim_{n \to \infty} \frac{\log \gamma(c_n(x))}{\log \mu(c_n(x))} = \theta \right\} \tag{3.1}$$

then $\dim_\mu(E) = \theta \dim_\gamma(E)$. □

Billingsley's idea was to compute the μ-dimension of E by constructing some measure γ for which $\gamma(E) > 0$ and (3.1) holds, thereby obtaining $\dim_\mu(E) = \theta$. He showed how this technique could be used to calculate the usual Hausdorff dimension of certain subsets of $[0,1]$ by identifying each point with its r-adic expansion (r chosen suitably), the sequence of r-adic digits comprising the stochastic process and an n-cylinder corresponding to an r-adic interval of length r^{-n}. Since coverings by r-adic intervals produce usual Hausdorff dimension, Theorem 3.1 can be applied with suitable γ and $\mu =$ Lebesgue measure. We now extend this technique to generalized expansions.

A generalized expansion of a number in $[0,1]$ will be defined as follows. For each $n = 1,2,\ldots$ let $k_n \ge 2$ be an integer and choose values $0 < a_{n,1} < \cdots < a_{n,k_n-1} < 1$, setting $a_{n,0} = 0$ and $a_{n,k_n} = 1$. The initial proportions $a_{1,1}, \ldots, a_{1,k_1-1}$ determine a division of $[0,1]$ into the disjoint intervals $[a_{1,i}, a_{1,i+1})$, $i = 0,1,\ldots,k_1-2$, and $[a_{1,k_1-1}, 1]$. We will indicate that a point x in $[0,1]$ falls into the i^{th} interval $(i = 0,1,\ldots,k_1-1)$ by the notation $I_1(x) = i$. $I_1(x)$ will be the first term in the expansion of x (with respect to the choices $a_{n,i}$). At the second stage each interval $\{x \mid I_1(x) = i\}$ is divided into k_2 disjoint subintervals determined by the given proportions $a_{2,1}, \ldots, a_{2,k_2-1}$. This splits $[0,1]$ into $k_1 k_2$ disjoint intervals which are most conveniently expressed in the form $\{x \mid I_1(x) = i,\ I_2(x) = j\}$ for some choice of $i = 0,1,\ldots,k_1-1$ and $j = 0,1,\ldots,k_2-1$. Letting $d_{n,i} = a_{n,i+1} - a_{n,i}$ for each n and i, we can alternately write $\{x \mid I_1(x) = i, I_2(x) = j\} = \{x \mid a_{1,i} + a_{2,j} d_{1,i} \le x < a_{1,i} + a_{2,j+1} d_{1,i}\}$ (but including the right hand endpoint if $i = k_1-1$ and $j = k_2-1$). $I_2(x)$ will be the second term in the expansion of x. Each interval $\{x \mid I_1(x) = i, I_2(x) = j\}$ is then divided according to the proportions $a_{3,1}, \ldots, a_{3,k_3-1}$. Continuing this subdivision process, the n^{th} stage produces a splitting of $[0,1]$ into $k_1 k_2 \cdots k_n$ disjoint intervals

$$\{x \mid I_1(x)=i_1, I_2(x)=i_2, \ldots, I_n(x)=i_n\}$$
$$= \{x \mid a_{1,i_1} + a_{2,i_2}d_{1,i_1} + a_{3,i_3}d_{2,i_2}d_{1,i_1} + \cdots + a_{n,i_n}d_{n-1,i_{n-1}} \cdots d_{1,i_1}$$
$$\leq x < a_{1,i_1} + a_{2,i_2}d_{1,i_1} + \cdots + a_{n,i_n+1}d_{n-1,i_{n-1}} \cdots d_{1,i_1}\}.$$

The sequence $I_1(x)$, $I_2(x)$,... is the generalized expansion of x, taking values in the countable set $S = \{0,1, \ldots, k_n-1 \mid n = 1,2,\ldots\}$. If $r \geq 2$ is a positive integer and $k_n = r$, $a_{n,i} = i/r$ for each n, then the result is the usual r-adic expansion of x. (If x has more than one r-adic expansion this method produces the terminating one.)

We will be using coverings composed of intervals belonging to the collection g of n-cylinders $c(i_1, \ldots, i_n) = \{x \mid I_1(x)=i_1, \ldots, I_n(x)=i_n\}$ generated by the generalized expansions. (Note that some n-cylinders may be empty; this occurs if some $i_j \in S \setminus \{0,1, \ldots, k_j-1\}$.) In order that g be a complete bounded Vitali covering of $[0,1]$ we need to make the restriction:

The diameters of the (nonempty) n-cylinders shrink at a controlled rate. If $c_n(x)$ is the n-cylinder containing the point x then

$$\inf_n \frac{d(c_{n+1}(x))}{d(c_n(x))} = \inf_n d_{n+1,I_{n+1}(x)} = b(x) > 0. \tag{3.2}$$

It is easy to see that (3.2) implies the diameter of each n-cylinder shrinks to zero as $n \to \infty$ and forces S to be a finite set.

We now proceed to the main result.

THEOREM 3.2. Let $I_1(x)$, $I_2(x)$,... represent the generalized expansion of $x \in [0,1]$ with respect to a choice of proportions $a_{n,i}$, $i = 1, \ldots, k_n-1$, $n = 1,2,\ldots$ and suppose the resulting interval collection g of n-cylinders satisfies (3.2). Let γ be defined over the n-cylinders by the relations:

$$\gamma(c(i_1,i_2, \ldots, i_n)) = p_n(i_1,i_2, \ldots, i_n) \tag{3.3}$$

where $0 \leq p_n(i_1, \ldots, i_n) \leq 1$, $p_n(i_1, \ldots, i_n) = 0$ if one or more $i_j \geq k_j$,
$\sum_{i=0}^{k_n-1} p_n(i_1, \ldots, i_{n-1},i) = p_{n-1}(i_1, \ldots, i_{n-1})$ (consistency condition), $\sum_{i=0}^{k_1-1} p_1(i) = 1$, and
$\lim_{n \to \infty} p_n(i_1, \ldots, i_n) = 0$.

Then γ extends uniquely to a diffuse probability distribution on the Borel sets B of $[0,1]$, and if

$$E \subseteq \left\{ x \in [0,1] \mid \lim_{n \to \infty} \frac{\log p_n(I_1(x), I_2(x), \ldots, I_n(x))}{\log d_{1,I_1(x)} d_{2,I_2(x)} \cdots d_{n,I_n(x)}} = \theta \right\} \tag{3.4}$$

then $\dim(E) = \theta \dim_\gamma(E)$. If $\gamma(E) > 0$ then $\dim(E) = \theta$.

PROOF. This will follow from Theorem 3.1 and Theorem 2.2. It is clear from the construction of the n-cylinders (and restriction (3.2)) that the n-cylinders ($n = 1,2,\ldots$) generate the Borel sets of $[0,1]$ and that (3.3) defines a diffuse probability measure γ that extends uniquely to the Borel sets. Regarding the generalized expansion $I_1, I_2, \ldots$ as a stochastic process on the probability space $([0,1], B, \lambda)$, (λ = Lebesgue measure) and noting that $\gamma(c_n(x)) = p_n(I_1(x), I_2(x), \ldots, I_n(x))$ while $\lambda(c_n(x)) = d_{1,I_1(x)} d_{2,I_2(x)} \cdots d_{n,I_n(x)}$, it follows from Theorem 3.1 that if E satisfies the hypotheses of Theorem 3.2 then $\dim_\lambda(E) = \theta \dim_\gamma(E)$. Now $\dim_\lambda(E)$ and $\dim_\gamma(E)$ are defined by coverings from the n-cylinders generated by the process $I_1, I_2, \ldots$, and restriction (3.2) ensures the n-cylinders form a complete bounded Vitali covering of $[0,1]$. From Theorem 2.2 we conclude $\dim_\lambda(E) = \dim(E)$ and the result follows. □

Theorem 3.2 can frequently be applied to Cantor sets built from generalized expansions. We say C is a generalized Cantor set if it can be expressed in the form $C = \{x \mid (I_1(x), I_2(x), \ldots) \in S^*\}$ where S^* is some subset of the countable product $\mathop{X}\limits_{n=1}^{\infty} \{0,1, \ldots, k_n - 1\}$. The simplest case occurs when $S^* = \mathop{X}\limits_{n=1}^{\infty} S_n$ where $S_n \subseteq \{0,1, \ldots, k_n - 1\}$ is the set of "allowable" digits at the n^{th} stage. The resulting Cantor set is called "independent" and can be written as $C = \{x \mid I_n(x) \in S_n$ for all $n\}$. The usual Cantor set (minus a countable collection of "endpoints" corresponding to some numbers with more than one triadic expansion) is an example, resulting when $k_n = 3$, $a_{n,i} = i/3$, and $S_n = \{0,2\}$ for all n. We have the following corollary.

COROLLARY 3.2. Let C be a generalized independent Cantor set built from generalized expansions whose n-cylinders satisfy (3.2). Let s_n denote the size of the set of allowable digits S_n at the n^{th} stage. Suppose there exists d_n such that $d_{n,i} = d_n$ for each $i \in S_n$. If the limit

$$\lim_{n \to \infty} \frac{\log (s_1 s_2 \cdots s_n)^{-1}}{\log d_1 d_2 \cdots d_n} \text{ exists and equals } \theta \tag{3.5}$$

then $\dim(C) = \theta$.

PROOF. We apply Theorem 3.2 with C in the role of E and γ defined by

$$\gamma(c_n(i_1, \ldots, i_n)) = \begin{cases} (s_1 s_2 \cdots s_n)^{-1} & \text{if } i_j \in S_j \text{ for all } j \\ 0 & \text{otherwise.} \end{cases}$$

γ corresponds to choosing uniformly and independently among the allowable digits at each stage. □

We apply Theorem 3.2 to compute the Hausdorff dimension of a certain generalized "Markov" Cantor set (i.e. a Cantor set in which the allowable digits at the n^{th} stage depend on the digit chosen at the $(n-1)^{\text{th}}$ stage). While techniques exist in the literature for calculating the dimension of self-similar sets (see [5], [6], [7]) by obtaining the so-called "similarity dimension", the following set is only self-similar in a limiting sense. (It can be partitioned as a countably infinite union of similitudes.) For each n take $k_n = 5$, and for n even set $d_{n,0} = d_{n,2} = d_{n,4} = \alpha$ and $d_{n,1} = d_{n,3} = a$ (where $\alpha > 0$, $a > 0$ satisfy $3\alpha + 2a = 1$) while for n odd set $d_{n,1} = d_{n,3} = \beta$ and $d_{n,0} = d_{n,2} = d_{n,4} = b$ (where $\beta > 0$, $b > 0$ satisfy $2\beta + 3b = 1$.) We set $S_1 = \{1,3\}$, allowing only "1" or "3" to be selected at the first stage. Letting $S_n(i)$ denote the allowable digits at the n^{th} stage given that "i" is selected at the $(n-1)^{\text{th}}$ stage, we use the rules $S_n(0) = \{1\}$, $S_n(1) = \{0,2\}$, $S_n(2) = \{1,3\}$, $S_n(3) = \{2,4\}$, and $S_n(4) = \{3\}$. (These rules correspond to the permissible moves in a random walk on $\{0,1,2,3,4\}$ with reflecting barriers at 0 and 4.) We will show the resulting Cantor set $C = \{x \mid I_1(x) \in S_1$ and $I_n(x) \in S_n(I_{n-1}(x))$ for each $n\}$ has dimension $\log 1/3 / \log \alpha\beta$. Construct a Markov probability rule on the n-cylinders according to the initial distribution $p_1(1) = p_1(3) = 1/2$ and transition probabilities $p(1 \mid 0) = 1$, $p(0 \mid 1) = 1/3$, $p(2 \mid 1) = 2/3$, $p(1 \mid 2) = p(3 \mid 2) = 1/2$, $p(2 \mid 3) = 2/3$, $p(4 \mid 3) = 1/3$, $p(3 \mid 4) = 1$, and $p(i \mid j) = 0$ otherwise. (This gives $p_2(i,j) = p_1(i)p(j \mid i)$ and, inductively, $p_n(i_1, \ldots, i_n) = p_{n-1}(i_1, \ldots, i_{n-1})p(i_n \mid i_{n-1})$.) The resulting distribution γ is clearly supported on the set C. Furthermore it is sufficient to compute the limit in (3.4) over odd integers and we obtain, for any $x \in C$,

$$\lim_{n \to \infty} \frac{\log p_{2n+1}(I_1(x), \ldots, I_{2n+1}(x))}{\log d_{1 I_1(x)} \cdots d_{2n+1, I_{2n+1}(x)}} = \lim_{n \to \infty} \frac{\log 2^{-1} 3^{-n}}{\log \alpha^n \beta^{n+1}}$$

and hence $\dim(C) = \log 1/3 / \log \alpha\beta$ as claimed.

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