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THE COMPUTATION OF THE INDEX OF A MORSE FUNCTION AT A CRITICAL POINT

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ABSTRACT. A theoretical approach in computing the index of a Morse function at a critical point on a real non-singular hypersurface V is given. As a consequence the Euler characteristic of V is computed. In the case where the hypersurface is polynomial and compact, a procedure is given that finds a linear function ℓ, whose restriction $\ell|_V$, is a Morse function on V.

KEY WORDS AND PHRASES. Morse function, critical point, index.
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1. INTRODUCTION.

Let $f(x_1, \ldots, x_n)$ be a real C^∞ function, and set

$$V = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | f(x_1, \ldots, x_n) = 0\}.$$

Suppose V is non-singular in $\mathbf{R}^n$. Furthermore, let $g(x_1, \ldots, x_n)$ be a real function whose restriction $g|_V$, on V, is a Morse function. Then we will first give a theoretical approach of how the Morse index of $g|_V$ at a critical point a can be computed. Using the above data, we can also compute the Euler characteristic, $\chi(V)$, of V.

Finally, in the case where f is a polynomial, we will say how we can obtain a polynomial function g, whose restriction $g|_V$, has no degenerate critical points on V.

2. The Basic Result.

We first recall some well known results from Morse Theory [3]. For A a $k \times k$ real non-singular symmetric matrix, we denote by the index of A, $i(A)$, the number of negative eigenvalues of A. Using the above definition, we may define the index of a Morse function at a critical point. Let $\mu : W \to \mathbf{R}$ be a real Morse function on a r-manifold W, and also let $w \in W$ be a critical point of μ. For $u_1, \dots, u_r$ local coordinates on W around w we can form the Hessian matrix of μ with respect to $u_1, \dots, u_r$, $H\mu(u)$, $H\mu(u) = \left(\frac{\partial^2 \mu}{\partial u_i \partial u_j}\right)$, $1 \leq i, j \leq r$. Although the Hessian matrix $H\mu(u)$ depends on the particular coordinates u, its index does not. We then define:

Definition 1. The index of μ at the critical point w, $i(w) = i(H\mu(u))$ for some coordinates u around w.

Let us now fix some notation. For $R(x_1, \dots, x_n)$ a real C^∞ function, we denote by $R_i = \frac{\partial R}{\partial x_i}$, $i = 1, \dots, n$, $R_{ij} = \frac{\partial^2 R}{\partial x_i \partial x_j}$, $i, j = 1, \dots, n$.

Let a be a critical point of $g|_V$. Without loss of generality, we may assume that $f_n(a) \neq 0$. Then using the Implicit Function Theorem we may "solve" the equation $f(x_1, \dots, x_n) = 0$ for x_n, i.e. near a, V can be thought as the graph of $x_n = x_n(x_1, \dots, x_{n-1})$, and, therefore, $x_1, \dots, x_{n-1}$ are local coordinates for V near a. If we differentiate the equation $f(x_1, \dots, x_n) = 0$ twice, and evaluate at a, we get:

(I) $0 = f_{ij} + f_{ni} \cdot x_{n_j} + f_{nj} \cdot x_{n_i} + f_{nn} \cdot x_{n_j} \cdot x_{n_i} + f_n \cdot x_{n_{ij}}$, $i, j = 1, \dots, n-1$.

At a again we have,

$$\text{(II)} \qquad g_i = \lambda f_i \,,\; i = 1, \dots, n,\; \lambda \in \mathbf{R}.$$

Now $g|_V$ with respect to the coordinates $x_1, \dots, x_{n-1}$ becomes $Q(x_1, \dots, x_{n-1}) = g(x_1, \dots, x_{n-1}, x_n(x_1, \dots, x_{n-1}))$. To compute therefore $i(a)$, it is enough to calculate the Hessian matrix $HQ(x_1, \dots, x_{n-1})$, at a. We have

$$Q_i = g_i + g_n x_{n_i} \,, \text{ and}$$

$$(III) \qquad Q_{ij} = g_{ij} + g_{ni} \cdot x_{n_j} + g_{nj} \cdot x_{n_i} + g_{nn} \cdot x_{n_i} \cdot x_{n_j} + g_n \cdot x_{n_{ij}} \,.$$

Substituting in III what $x_{n_{ij}}$ is in I and taking II into account, we get

$$\text{(IV)} \qquad Q_{ij} = \frac{1}{f_n^2}\left(h_{ij} \cdot f_n^2 - h_{ni} f_j f_n - h_{nj} f_i f_n + h_{nn} f_i f_j\right),$$

where $h = g - \lambda f$, λ is the constant in II, and $1 \leq i, j \leq n-1$.

We computed the Hessian matrix $HQ(x_1, \dots, x_{n-1}) = (Q_{ij})$ at a. But, unfortunately, this matrix depends on the particular coordinates used at the point a. Let us now give a coordinate free matrix whose index is related to $i(a)$ in a linear manner.

Let a, h, f be as before. Consider the following real $(n+1) \times (n+1)$ symmetric matrix N, evaluated at a.

$$N = \begin{pmatrix} 0 & \nabla f \\ \nabla^t f & H(h) \end{pmatrix}, \text{ where } H(h) = h_{ij}\,,\ i,j = 1,\dots,n$$

The following proposition is the main result in this paper.

PROPOSITION 1. For a, h, f, N as above, N is a non-singular matrix. Furthermore, $i(a) = i(N) - 1$.

The proof of Proposition 1 will be in stages. First we will state some generalities and then come back to the proof.

For A a $n \times n$ real symmetric matrix we associate the real bilinear form $q(x,y) = x^t A y$. We say that q is non-degenerate if $(q(x,y) = 0\ \forall y) \Rightarrow x = 0$. This is equivalent in saying that A is an invertible matrix. Since A is symmetric there exists an invertible matrix P such that $P^t A P$ is diagonal. Furthermore, $i(A) = i(P^t A P)$ [2].

Suppose $A = (a_{ij}), i,j = 1,\dots,n$ is a real symmetric matrix. Let $v = (v_1,\dots,v_n) \in \mathbf{R}^n$ so that $v_n \neq 0$. Consider the following real symmetric matrix B,

$$B = \begin{pmatrix} 0 & v \\ v^t & A \end{pmatrix}.$$

For $e_0, e_1, \dots, e_n$ the usual basis of $\mathbf{R}^{n+1}$, we have

$$< Be_0, e_0 > = 0$$
$$< Be_0, e_i > = v_i,\ i = 1,\dots,n$$
$$< Be_i, e_j > = a_{ij},\ i,j = 1,\dots,n,$$

where $<\ \ >$ denotes dot product.

Now introduce a new basis $e_0, \phi_1, \dots, \phi_{n-1}, e_n$ on $\mathbf{R}^{n+1}$ so that $\phi_i = v_n e_i - v_i e_n$, $i = 1,\dots,n-1$. With respect to those coordinates, the bilinear form $r(x,y) = x^t B y$ gets transformed to one whose matrix is $P^t B P$, where P is the following matrix

$$P = \begin{pmatrix} 1 & 0 & ,\dots, & 0 \\ 0 & v_n & ,\dots, & 0 \\ 0 & \vdots & \ddots & 0 \\ & & v_n & 0 \\ 0 & -v_1,\dots, & -v_{n-1} & 1 \end{pmatrix}, \text{ and } P^t B P \text{ becomes}$$

$$P^t B P = \begin{pmatrix} 0, & 0,\dots,0, & v_n \\ 0 & & \\ \vdots & \Gamma & \\ 0 & & \\ v_n & & \end{pmatrix}, \text{ where } \Gamma = (\gamma_{ij})\ i,j = 1,\dots,n$$

and $\gamma_{ij} = v_n^2 a_{ij} - v_n v_j a_{in} - v_i v_n a_{nj} + v_i v_j a_{nn}$, $i,j = 1,\dots,n-1$. We then have:

LEMMA 1. Suppose $\Gamma' = (\gamma_{ij})$, $i,j = 1,\dots,n-1$, γ_{ij} as above, is non-singular. Then B is also non-singular and furthermore

$$i(B) = i(\Gamma') + 1.$$

PROOF. We observe that $\det(P^tBP) \neq 0$, since $v_n \neq 0$, and therefore B is non-singular. Let R be a real non-singular $(n-1)\times(n-1)$ matrix, so that $R^t\Gamma' R$ is diagonal. Since Γ' is non-singular, all of the diagonal elements of $R^t\Gamma' R$ are non-zero. Let R' be the following non-singular matrix

$$R' = \begin{pmatrix} 1, & 0, & \dots, & 0 \\ 0 & R & & \\ & & & 0 \\ 0, & 0, & \dots, & 1 \end{pmatrix}.$$

Now consider $S = R''(P^tBP)R'$, S has the form

$$S = \begin{pmatrix} 0, & 0, & \dots, & v_n \\ \vdots & \gamma_1 & & b_1 \\ & \ddots & & \vdots \\ & & \gamma_{n-1} & \\ v_n & b_1, & \dots, & b_n \end{pmatrix}.$$

Let E_i be the following $(n+1)\times(n+1)$ elementary matrix.

$$E_i = \begin{pmatrix} I & 0, & 0 \\ 0, & \frac{-b_i}{\gamma_i}, & 1 \end{pmatrix},$$

where $-\frac{b_i}{\gamma_i}$ appears in the $(i+1)^{th}$ column for $i = 1, \dots, n-1$. Observe that each E_i is invertible. Furthermore, a computation shows that $\prod_{i=1}^{n-1} E_{n-i}^t \cdot S \cdot \prod_{i=1}^{n-1} E_i = S'$ has the following form

$$S' = \begin{pmatrix} 0, & 0, & \dots, & 0, & v_n \\ 0 & \gamma_1 & & & 0 \\ \vdots & & \ddots & & \\ 0 & & & \gamma_{n-1} & 0 \\ v_n & 0 & \dots & 0 & b \end{pmatrix}, \text{ where } b = b_n - \sum_{i=1}^{n-1} \frac{b_i^2}{\gamma_i}.$$

On the other hand, $i(S') = i(S)$. To complete the proof of the lemma it is enough to show that

(V) $$i(S') = \#\{i's \mid \gamma_i < 0\} + 1.$$

To achieve that we look at the $\det(S' - \lambda I) = (-\lambda)(b-\lambda)\prod_{i=1}^{n-1}(\gamma_i - \lambda) - v_n^2 \prod_{i=1}^{n-1}(\gamma_i - \lambda) = \prod_{i=1}^{n-1}(\gamma_i - \lambda)\cdot(\lambda^2 - b\lambda - v_n^2)$. But the real roots of $\lambda^2 - b\lambda - v_n^2$ are exactly two, one positive and one negative. ∎

PROOF OF PROPOSITION 1. With the same notation and the same change of coordinates, we take B to be N, then Γ' becomes A.

And now Lemma 1 says

$$i(N) = i(Q) + 1 = i(a) + 1. \quad ∎$$

To compute the index, $i(N)$, of N we first look at the negative zeros of $D(x) = \det(N - xI)$. To determine the number of negative zeros of $D(x)$ we can use the following argument: Let $d_0 = g.c.d(D, D')$, $d_1 = g.c.d(d_0, d_0'), \dots,$ and $d_i = g.c.d(d_{i-1}, d_{i-1}')$, $i = 1, \dots, k$, with d_k constant where $D' = \frac{dD}{dx}$, and

$d'_{i+1} = \frac{dd_i}{dx}$, $i = 0, \ldots, k-1$. Then we observe that

$$\frac{1}{d_k} \cdot D = \frac{D}{d_0} \cdot \frac{d_0}{d_1} \cdot \frac{d_1}{d_2} \cdots \frac{d_{k-1}}{d_k} = \delta_0 \cdot \delta_1 \cdots \delta_k \,, \delta_j = \frac{d_{j-1}}{d_j} \,, j = 0, \ldots, k, d_{-1} = D \,.$$

Furthermore, we note that each δ_j has simple roots and # (negative roots of D) $= \sum_{j=0}^{k}$ # (negative roots of δ_j). Finally we can use Sturm's Theorem to decide the number of negative zeros of each δ_j [1].

If N happens to be nice, in the sense that no more than two consecutive principal minors of N are singular, then $i(N)$ = variation of sign of the determinants of its principal minors [1].

The computation of the Euler characteristic, $\chi(V)$, of V does not require the computation of the index of N, but rather the sign of its determinant. We have

$$\chi(V) = \sum_{\substack{p \text{ is a critical} \\ \text{point of } g|_V}} (-1)^{i(p)} \,.$$

But $(-1)^{i(p)} =$ sign $\det(M)(p) = -$ sign $\det(N)(p)$.

Therefore, $\chi(V) = -\sum_p$ sign $\det(N)(p)$.

3. A THEORETICAL PROCEDURE.

From now on suppose that f is a polynomial of even degree and $V = \{f = 0\}$ is compact and non-singular in $\mathbf{R}^n$.

Let $\mathcal{L} = \{\ell : \mathbf{R}^n \to \mathbf{R}^n \,,\ \ell$ is linear, $\ell \neq 0\}$. Then $\mathcal{L}$ can be identified with $\mathbf{R}^n - \{0\}$. We have:

LEMMA 2. For almost all elements ℓ of $\mathcal{L}$, $\ell|_V$ is a Morse function on V.

PROOF. Let $\eta : V \to S^{n-1}$ be the Gauss Map. Then from Sard's Theorem we get that the set of critical values of η has measure zero in S^{n-1}. For $\ell \in \mathcal{L}$, $\ell|_V$ is not a Morse function on V if and only if $\frac{\nabla \ell}{\|\nabla \ell\|}$ is a critical value of η [4]. ∎

DEFINITION. For $f(x_1, \ldots, x_n)$ a real polynomial of degree $d, d \geq 1$, the bordered Hessian, $BH(f)$, of f is the following $(n+1) \times (n+1)$ real symmetric matrix

$$BH(f) = \begin{pmatrix} 0 & \nabla f \\ \nabla^t f & H(f) \end{pmatrix} \,, \text{ where } H(f) \text{ is the Hessian matrix of } f \,.$$

Let now $a = (a_1, \ldots, a_n) \in \mathbf{R}^n - \{0\}$, and consider the linear function $\ell(x) =< a, x >$. Let $L = \ell|_V$, and p a critical point of L. We may suppose $f_n(p) \neq 0$. Then if $h = \ell - \lambda f$, where $\ell_i = \lambda f_i$, $i = 1, \ldots, n$. p is a non-degenerate critical point of L if and only if the following matrix N is non-singular

$$N = \begin{pmatrix} 0 & \nabla f \\ \nabla^t f & H(h) \end{pmatrix} = \begin{pmatrix} 0 & \nabla f \\ \nabla^t f & -\lambda H(f) \end{pmatrix} [4].$$

Since $\nabla \ell = a$, and therefore $\lambda \neq 0$, Lemma 2 implies the following:

COROLLARY 2. For a, ℓ, L as above, L is a Morse function on V if and only if $\frac{a}{\|a\|}$ does not belong to the image of the set $\Delta = (\det BH(f) = 0) \cap V$ under the Gauss map η.

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