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ORDERED COMPACTIFICATION OF TOTALLY ORDERED SPACES

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ABSTRACT. A complete description of the T_2-ordered compactifications of a totally ordered space X is given in terms of the simple and essential singularities.

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0. Introduction.

A totally ordered space X is defined to be a totally ordered set equipped with a topology which is locally convex and T_2-ordered (i.e., the order is a closed subset of $X \times X$). A study of ordered compactifications was made previously by J. Blatter [1]. Our goal is to give a more intuitive treatment of this subject based on the notion of "singularity," which is the term that we use to designate a non-convergent, monotone, free, convex filter (or, equivalently, a non-convergent maximal c-filter).

The singularities of a totally ordered space X may be classified in several ways (e.g., bounded or unbounded, increasing or decreasing), but the most important distinction is between simple and essential singularities. For every T_2-ordered compactification of X, there is a unique compactification point corresponding to each simple singularity; it follows that a totally ordered space which has only simple singularities has a unique T_2-ordered compactification. Essential singularities always occur as ordered pairs, and to a given T_2-ordered compactification each pair of essential singularities contributes either one or two compactification points. There is (as Blatter showed earlier) a smallest T_2-ordered compactification obtained by assigning a single compactification point to each pair of essential singularities, and a largest T_2-ordered compactification obtained by assigning two compactification points to each essential pair. The latter is, of course, the Nachbin (or Stone-Čech ordered) compactification. Any other T_2-ordered compactification may be described by partitioning the set $\wp(X)$ of all essential pairs of singularities into two subsets, and assigning one compactification point to each pair in the first subset and two compactification points to each pair in the second. Thus there is a natural one-to-one correspondence between the subsets of $\wp(X)$ and the T_2-ordered compactifications of X.

Every T_2-ordered compactification of a totally ordered space is also totally ordered, and the compactification space always has the order topology.

1. *Totally Ordered Spaces.*

We shall assume throughout this paper that X is a totally ordered set. If a, b are distinct elements in X and $a \leq x \leq b$ implies $x = a$ or $x = b$, then b is said to *cover* a. A subset A of X is *increasing* (respectively, *decreasing*) if $a \in A$ and $a \leq x$ (respectively, $x \leq a$) implies $x \in A$. For $a \in X$, let $[a, \rightarrow)$ be the set of upper bounds of a, and let $(a, \rightarrow) = \{x \in X : a < x\}$ be the proper upper bounds of a; the sets $(\leftarrow, a]$ and $(\leftarrow, a)$ are defined dually. For $a, b \in X$, we define the "open" interval $(a, b) = (a, \rightarrow) \cap (\leftarrow, b)$ and the "closed" interval $[a, b] = [a, \rightarrow) \cap (\leftarrow, b]$.

If $A \subseteq X$, let $i(A) = \cup\{[a, \rightarrow) : a \in A\}$ denote the *increasing hull* of A, $d(A)$ the *decreasing hull* of A, and $A^\wedge = i(A) \cap d(A)$ the *convex hull* of A. If $A = A^\wedge$, then A is called a *convex set.* Let $A^\uparrow$ (respectively, $A^\downarrow$) designate the set of all upper (respectively, lower) bounds of A.

We shall always use the term "filter" to mean a proper set filter. A filter $\mathcal{F}$ is *free* if there is no point common to all the sets in $\mathcal{F}$. A filter which is not free is *fixed*; in particular, the symbol

$\dot{x}$ will denote the fixed ultrafilter generated by a point x.

For any filter $\mathcal{F}$, let $\mathcal{F}^\wedge$ be the filter generated by $\{F^\wedge : F \in \mathcal{F}\}$; if $\mathcal{F} = \mathcal{F}^\wedge$, then $\mathcal{F}$ is said to be *convex*. The set of upper bounds of a filter $\mathcal{F}$ is defined to be $\mathcal{F}^\uparrow = \cup\{F^\uparrow : F \in \mathcal{F}\}$; $\mathcal{F}^\downarrow$ is defined dually. For any free convex filter $\mathcal{F}$ on X, the sets $\mathcal{F}^\uparrow$ and $\mathcal{F}^\downarrow$ partition X. A filter $\mathcal{F}$ is said to be *increasing* (respectively, *decreasing*) if $\mathcal{F}^\downarrow \in \mathcal{F}$(*respectively*, $\mathcal{F}^\uparrow \in \mathcal{F}$); a filter which is either increasing or decreasing is said to be *monotone*.

The free, convex filters are of three different types, which may be described as follows.

Proposition 1.1 Each free, convex filter $\mathcal{F}$ on a totally ordered set X is of exactly one of the following types:

(a) Increasing, in which case $\mathcal{F}$ has a filter base consisting of sets of the form $(x, \rightarrow) \cap \mathcal{F}^\downarrow$, where $x \in \mathcal{F}^\downarrow$;

(b) Decreasing, in which case $\mathcal{F}$ has a filter base consisting of sets of the form $(\leftarrow, x) \cap \mathcal{F}^\uparrow$, where $x \in \mathcal{F}^\uparrow$;

(c) Non-monotone, in which case $\mathcal{F}$ has a filter base of sets of the form (a, b), where $a \in \mathcal{F}^\downarrow$ and $b \in \mathcal{F}^\uparrow$. In this case $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$, where $\mathcal{F}_1$ (generated by sets of the form $(x, \rightarrow) \cap \mathcal{F}^\downarrow, x \in \mathcal{F}^\downarrow$) is increasing and $\mathcal{F}_2$ (generated by sets of the form $(\leftarrow, x) \cap \mathcal{F}^\uparrow$, $x \in \mathcal{F}^\uparrow$) is decreasing.

If $\mathcal{G}$ is any free ultrafilter on X, one may easily verify that $\mathcal{G}^\wedge$ is a free, convex filter. Since the sets $\mathcal{G}^\uparrow$ and $\mathcal{G}^\downarrow$ partition X, at least one of these sets is in $\mathcal{G}$, and hence in $\mathcal{G}^\wedge$. Thus the convex hull of any free ultrafilter must be monotone. With the help of these observations and the preceding proposition, we obtain the next proposition.

Proposition 1.2 A free, convex filter $\mathcal{F}$ is monotone iff there is a free ultrafilter $\mathcal{G}$ on X such that $\mathcal{F} = \mathcal{G}^\wedge$. Furthermore if $\mathcal{F}$ is a monotone, free, convex filter and $\mathcal{H}$ is any ultrafilter finer than $\mathcal{F}$, then $\mathcal{F} = \mathcal{H}^\wedge$.

The *order topology* 0 on X has as an open subbase all sets of the form $(a, \rightarrow)$ and $(\leftarrow, b)$, for $a, b \in X$. This topology is *locally convex* (meaning that the neighborhood filter at each point has a filter base of convex sets) and T_2-ordered (meaning that the order relation is closed in $X \times X$).

We shall write "$\mathcal{F} \xrightarrow{0} x$" to indicate that a filter $\mathcal{F}$ converges to a point x in the order topology on X. It is well known that for any totally ordered set, order convergence coincides with convergence in the order topology; this can be stated as follows: $\mathcal{F} \xrightarrow{0} x$ iff $x = \sup \mathcal{F}^{\downarrow} = \inf \mathcal{F}^{\uparrow}$.

Proposition 1.3 Every free, convex filter $\mathcal{F}$ on X is of exactly one of the three following forms.

(a) $\mathcal{F}^{\uparrow} = \phi$ or $\mathcal{F}^{\downarrow} = \phi$, but not both;

(b) $\mathcal{F} \xrightarrow{0} x$ for some $x \in X$;

(c) $\mathcal{F}^{\uparrow} \neq \phi$ and $\mathcal{F}^{\downarrow} \neq \phi$, but $\sup \mathcal{F}^{\downarrow}$ and $\inf \mathcal{F}^{\uparrow}$ both fail to exist.

Proof. If a free convex filter $\mathcal{F}$ has the property $\mathcal{F}^{\uparrow} = \phi$, then $\mathcal{F}^{\downarrow} = X$. In particular, $\mathcal{F}^{\uparrow} = \phi$ and $\mathcal{F}^{\downarrow} = \phi$ cannot both hold. Next, suppose that (a) does not hold, so that $\mathcal{F}^{\uparrow}$ and $\mathcal{F}^{\downarrow}$ are both non-empty. If there is $x \in X$ such that $x = \inf \mathcal{F}^{\uparrow}$ (respectively, $x = \sup \mathcal{F}^{\downarrow}$), then one can show by a direct argument that $x = \sup \mathcal{F}^{\downarrow}$ (respectively, $x = \inf \mathcal{F}^{\uparrow}$). Thus the statement that either $x = \inf \mathcal{F}^{\uparrow}$ or $x = \sup \mathcal{F}^{\downarrow}$ is sufficient to guarantee that $\mathcal{F} \xrightarrow{0} x$. Therefore, if (a) and (b) both fail to hold, then $\mathcal{F}^{\uparrow}$ and $\mathcal{F}^{\downarrow}$ must both be non-empty and $\inf \mathcal{F}^{\uparrow}$ and $\sup \mathcal{F}^{\downarrow}$ must both fail to exist; consequently, (c) must hold. ∎

We define a *totally ordered space* (X, τ) to be a totally ordered set X equipped with a topology τ which is locally convex and T_2-ordered. For any totally ordered set X, $(X, 0)$ is a totally ordered space, and indeed 0 is the coarsest topology on X which is both T_2-ordered and locally convex. In particular, if τ is a compact, T_2-ordered, locally convex topology on X, then it follows from the preceding statement that $\tau = 0$. Thus every compact, totally ordered space has the order topology.

It should be noted that the term "totally ordered space" is defined in a more general way here than in [3], where this term is applied only to spaces with the order topology. We shall normally designate a totally ordered space (X, τ) simply by "X."

For any totally ordered space X, a *singularity* $\mathcal{F}$ on X is defined to be any non-convergent, monotone, free, convex filter, or, equivalently in view of Proposition 1.2, the convex hull of any non-convergent ultrafilter. We shall use Proposition 1.3 to define several different types of singularities:

(1) If $\mathcal{F}$ is a singularity such that $\mathcal{F}^{\uparrow} = \phi$ (respectively, $\mathcal{F}^{\downarrow} = \phi$), then $\mathcal{F}$ is an *increasing* (respectively, *decreasing*) *unbounded, simple, singularity.*

(2) If $\mathcal{F}$ is a singularity such that $\mathcal{F} \xrightarrow{0} x$ for some $x \in X$, then $\mathcal{F}$ is a *bounded, simple singularity.*

(3) If $\mathcal{F}$ is a singularity such that $\mathcal{F}^{\uparrow} \neq \phi$ and $\mathcal{F}^{\downarrow} \neq \phi$, then $\mathcal{F}$ is an *essential singularity.*

Let $\mathcal{S}(X)$ be the set of all singularities on a totally ordered space X. We totally order $\mathcal{S}(X)$ by imposing the relation: $\mathcal{F} \lesssim \mathcal{G}$ iff there is $F \in \mathcal{F}$ such that $F^{\uparrow} \in \mathcal{G}$. If X has no greatest (respectively, least) element, then there is an increasing (respectively, decreasing), unbounded, simple singularity which is the greatest (respectively, least) element in $\mathcal{S}(X)$. Thus there are at most two unbounded simple singularities. The next proposition shows that the essential singularities always occur as ordered pairs.

Proposition 1.4 A singularity $\mathcal{F}$ on a totally ordered space X is essential iff there is a singularity $\mathcal{G}$ such that $\mathcal{F} \cap \mathcal{G}$ is a non-monotone, convex filter. If $\mathcal{F}$ is an increasing (respectively, decreasing) essential singularity, then $\mathcal{G}$ is a decreasing (respectively, increasing) essential singularity and $\mathcal{F} \lesssim \mathcal{G}$ (respectively, $\mathcal{G} \lesssim \mathcal{F}$).

Proof. Let $\mathcal{F}$ be an increasing, essential singularity. Recall that $\mathcal{F}^{\downarrow}$ and $\mathcal{F}^{\uparrow}$ are decreasing and increasing sets, respectively, which partition X. Since $\mathcal{F}$ is essential, $\sup \mathcal{F}^{\downarrow}$ and $\inf \mathcal{F}^{\uparrow}$ both fail to exist; thus $\mathcal{F}^{\downarrow}$ contains no greatest element and $\mathcal{F}^{\uparrow}$ contains no least element. Thus the filter $\mathcal{G}$ generated by $\{(\leftarrow, a) \cap \mathcal{F}^{\uparrow} : a \in \mathcal{F}^{\uparrow}\}$ is well-defined, convex, and free; $\mathcal{G}$ is also decreasing (since $\mathcal{F}^{\uparrow} = \mathcal{G}^{\uparrow} \in \mathcal{G}$) and essential (since $\inf \mathcal{G}^{\uparrow}$ fails to exist). Furthermore $\mathcal{F}^{\downarrow} \in \mathcal{F}$ and $(\mathcal{F}^{\downarrow})^{\uparrow} = \mathcal{F}^{\uparrow} \in \mathcal{G}$, which implies $\mathcal{F} \lesssim \mathcal{G}$. Finally, one can verify that $\mathcal{F} \cap \mathcal{G}$ is a non-monotone, free, convex filter. If we assume, on the other hand, that $\mathcal{F}$ is a decreasing, essential singularity, one can similarly show that the filter $\mathcal{G}$ generated by $\{(a, \rightarrow) \cap \mathcal{F}^{\downarrow}$ and $a \in \mathcal{F}^{\downarrow}\}$ is an increasing, essential singularity such that $\mathcal{F} \cap \mathcal{G}$ is a non-monotone, free, convex filter and $\mathcal{G} \lesssim \mathcal{F}$.

Conversely, let $\mathcal{F}$ be a singularity and assume that a singularity $\mathcal{G}$ exists such that $\mathcal{F} \cap \mathcal{G}$ is a non-monotone, convex filter; $\mathcal{F} \cap \mathcal{G}$ must also be free since $\mathcal{F}$ and $\mathcal{G}$ are both free. Thus $\mathcal{F} \cap \mathcal{G}$

has the form described in Proposition 1.1 (c), and one can easily show that $\mathcal{F}^{\uparrow} = \mathcal{G}^{\uparrow} = (\mathcal{F} \cap \mathcal{G})^{\uparrow}$ and $\mathcal{F}^{\downarrow} = \mathcal{G}^{\downarrow} = (\mathcal{F} \cap \mathcal{G})^{\downarrow}$. If $\mathcal{F}$ is not essential, then $\mathcal{F} \xrightarrow{\circ} x$ and this implies that $x \in (a, b)$ for all $a \in (\mathcal{F} \cap \mathcal{G})^{\downarrow}$ and $b \in (\mathcal{F} \cap \mathcal{G})^{\uparrow}$. But these "open" intervals form a base for $\mathcal{F} \cap \mathcal{G}$, contrary to the fact that $\mathcal{F} \cap \mathcal{G}$ is a free filter. Thus $\mathcal{F}$ is an essential singularity, and the same reasoning applies to $\mathcal{G}$. ∎

If $\mathcal{F}$ and $\mathcal{G}$ are essential singularities on a totally ordered space X such that $\mathcal{F} \cap \mathcal{G}$ is a non-monotone, convex filter and if $\mathcal{F} \precsim \mathcal{G}$, then the ordered pair $\langle \mathcal{F}, \mathcal{G} \rangle$ will be called an *essential pair of singularities*. Let $\wp(X)$ be the set of all such essential pairs on X.

Proposition 1.5 Let $\mathcal{F}$ be an increasing and $\mathcal{G}$ a decreasing singularity on a totally ordered space X. Then $\langle \mathcal{F}, \mathcal{G} \rangle \in \wp(X)$ iff $\mathcal{F}^{\uparrow} = \mathcal{G}^{\uparrow}$ and $\mathcal{F}^{\downarrow} = \mathcal{G}^{\downarrow}$.

Proof. If $\langle \mathcal{F}, \mathcal{G} \rangle \in \wp(X)$, then $\mathcal{F}^{\downarrow} = \mathcal{G}^{\downarrow}$ and $\mathcal{F}^{\uparrow} = \mathcal{G}^{\uparrow}$ was established in the proof of the preceding proposition. Conversely, $\mathcal{F}^{\uparrow} = \mathcal{G}^{\uparrow}$ and $\mathcal{F}^{\downarrow} = \mathcal{G}^{\downarrow}$ imply that $\mathcal{F}^{\uparrow} = \mathcal{G}^{\uparrow} = (\mathcal{F} \cap \mathcal{G})^{\uparrow}$ and $\mathcal{F}^{\downarrow} = \mathcal{G}^{\downarrow} = (\mathcal{F} \cap \mathcal{G})^{\downarrow}$. Sets of the form $(a, \rightarrow) \cap \mathcal{F}^{\downarrow}$, $a \in \mathcal{F}^{\downarrow}$ form a filter base for $\mathcal{F}$, and sets of the form $(\leftarrow, b) \cap \mathcal{G}^{\uparrow}$, $b \in \mathcal{G}^{\uparrow}$ form a filter base for $\mathcal{G}$; unions of such sets are "open" intervals of the form (a, b), $a \in (\mathcal{F} \cap \mathcal{G})^{\downarrow}$ and $b \in (\mathcal{F} \cap \mathcal{G})^{\uparrow}$, and these constitute a filter base for $\mathcal{F} \cap \mathcal{G}$. Thus $\mathcal{F} \cap \mathcal{G}$ is non-monotone and convex, and $\langle \mathcal{F}, \mathcal{G} \rangle \in \wp(X)$ follows by Proposition 1.4. ∎

If X is the set of rational numbers with the usual order and topology, then $S(X)$ consists of two unbounded, simple singularities and an uncountable number of essential singularities; in this case there are no bounded simple singularities since the usual topology is the order topology. Furthermore, there is a natural one-to-one correspondence between $\wp(X)$ and the "irrational numbers." On the other hand, if X is the Sorgenfrey line (i.e., the real line with the usual order and "half-open interval" topology), then $S(x)$ contains two unbounded simple singularities, uncountably many bounded, simple singularities, and no essential singularities. For an arbitrary totally ordered space X, $S(X) = \phi$ iff X is compact.

2. Ordered Compactifications.

If Y is a poset with a topology, (Z, ψ) is a topological compactification of Y, and Z is also a poset, then (Z, ψ) is an *ordered compactification* of the ordered topological space Y if the embedding $\psi : Y \rightarrow Z$ is increasing in both directions (i.e., $x < y$ implies $\psi(x) < \psi(y)$ and vice-versa.)

We shall be interested only in *T_2-ordered compactifications,* which have the additional requirement that Z be T_2-ordered.

Nachbin, [4], has characterized those spaces (which he calls *completely regular ordered spaces)* which allow T_2-ordered compactifications. In particular, completely regular ordered spaces must be T_2-ordered and locally convex. A T_2-ordered space is said to be *T_4-ordered (normally ordered* in [4]) if, for each pair A, B of disjoint closed sets such that A is increasing and B is decreasing, there are disjoint open sets U and V, with U increasing and V decreasing, such that $A \subseteq U$ and $B \subseteq V$. Every T_4-ordered, locally convex space is completely regular ordered (see [2]). Furthermore, it is a simple matter to show that every totally ordered space is T_4-ordered. Thus we have established

Proposition 2.1 A totally ordered set X with a topology has a T_2-ordered compactification iff X is a totally ordered space.

In general, ordered compactifications of totally ordered spaces need not be totally ordered unless one imposes the restriction that the compactification be T_2-ordered. Recall the following characterization of T_2-ordered spaces: If $\mathcal{F} \to x, \mathcal{G} \to y$, and the product filter $\mathcal{F} \times \mathcal{G}$ has a trace on the order, then $x \leq y$.

Proposition 2.2 A T_2-ordered compactification of any totally ordered space is a totally ordered space with the order topology.

Proof. Let X be a totally ordered space, and let Y be a compact, T_2-ordered space (not necessarily totally ordered) which contains X as a dense subset. If $y_1, y_2 \in Y - X$, then y_1 and y_2 are limits of singularities on X, and since $\mathcal{S}(X)$ is totally ordered it follows that $y_1 \leq y_2$ or $y_2 \leq y_1$. If $x \in X$ and $y \in Y - X$, then y is the limit of a singularity $\mathcal{F} \in \mathcal{S}(X)$, and since $\mathcal{F}^\uparrow$ and $\mathcal{F}^\downarrow$ partition X , x is in $\mathcal{F}^\uparrow$ or $\mathcal{F}^\downarrow$. If $x \in \mathcal{F}^\uparrow$, then $\mathcal{F} \times \dot{x}$ has a trace on the order of X (and hence on the order of Y). Since Y is T_2-ordered, $y \leq x$. Similarly, if $x \in \mathcal{F}^\downarrow$, then $x \leq y$. Thus Y is totally ordered, and we recall that every compact, totally ordered space has the order topology. ∎

The familiar procedure for "ordering" the T_2 compactifications of a completely regular (non-ordered) space extends in a natural and obvious way to the T_2-ordered compactifications of a completely regular ordered space. If (Y_1, σ_1) and (Y_2, σ_2) are T_2-ordered compactifications of X, we say that $(Y_1, \sigma_1) \leq (Y_2, \sigma_2)$ if there is a continuous, increasing function $f : (Y_2, \sigma_2) \to (Y_1, \sigma_1)$ which makes the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma_2"] \arrow[dr, "\sigma_1"'] & Y_2 \arrow[d, "f"] \\ & Y_1 \end{tikzcd}$$

commute. Two T_2-ordered compactifications of X are *equivalent* if each is larger than the other in this sense.

Our goal is to describe all the T_2-ordered compactifications of a totally ordered space, and we begin with the largest, which is called the *Nachbin* (or *Stone-Čech ordered) compactification* ([2], [4]). It turns out that for totally ordered spaces, the Nachbin compactification is equivalent to the *Wallman ordered compactification* [3], and it will be useful to give a brief description of the latter compactification at this point.

Let Y be a T_2-ordered topological space (partially but not necessarily totally ordered) with a subbase of monotone open sets; it is shown in [4] that all completely regular ordered spaces (and, hence, all totally ordered spaces) satisfy these requirements. A subset A of Y is called a *c-set* if it is the intersection of a closed increasing set and a closed decreasing set. Note that every c-set is closed and convex, but the converse is generally false. A *c-filter* is one which has filter base of c-sets. The maximal c-filters on Y form the underlying set for w_0Y, the Wallman ordered compactification of Y.

For any subset A of Y, let $A^* = \{\mathcal{F} \in w_0Y : A \in \mathcal{F}\}$. The collection $\mathcal{U}^* = \{U^* :$ U a monotone, open subset of $X\}$ constitutes an open subbase for the compactification topology of w_0Y. The partial order relation on w_0Y is defined by: $\mathcal{F} \leq^* \mathcal{G}$ iff $I(\mathcal{F}) \subseteq \mathcal{G}$ and $D(\mathcal{G}) \subseteq \mathcal{F}$, where $I(\mathcal{F})$ is the filter generated by all closed, increasing sets in $\mathcal{F}$ and $D(\mathcal{G})$ is generated by all the closed, decreasing sets in $\mathcal{G}$. The embedding $\varphi_Y : Y \to w_0Y$ is the obvious one: $\varphi_Y(y) = \dot{y}$, for all $y \in Y$.

Proposition 2.3 Let X be a totally ordered space.

(a) The c-sets of X are precisely the closed, convex subsets.

(b) The non-convergent, maximal c-filters are precisely the singularities of X, and thus $w_0X = \{\dot{x}: x \in X\} \cup S(X)$.

(c) The Wallman ordered compactification is the largest T_2-ordered compactification of X.

Proof. The first assertion is obvious. The second follows by first observing that each singularity has a filter base of closed, convex sets and then applying Proposition 1.2. In order to prove (c), it is necessary and sufficient (by Corollaries 1.4 and 1.5 of [3]) to show that X is T_4-ordered and satisfies the additional condition: For each closed, convex subset A of $X, i(A)$ and $d(A)$ are both closed. But these conditions clearly hold for totally ordered spaces. ∎

Proposition 2.4 Let X be a totally ordered space. If $\mathcal{F}, \mathcal{G} \in w_0X$, then $\mathcal{F} \leq^* \mathcal{G}$ iff exactly one of the following is true:

(a) There are $x, y \in X$ such that $\mathcal{F} = \dot{x}$, $\mathcal{G} = \dot{y}$, and $x \leq y$ in X;

(b) $\mathcal{F} = \dot{x}$ for some $x \in X$, $\mathcal{G} \in S(X)$, and $x \in \mathcal{G}^{\downarrow}$;

(c) $\mathcal{G} = \dot{y}$ for some $y \in X, \mathcal{F} \in S(X)$, and $y \in \mathcal{F}^{\uparrow}$;

(d) $\mathcal{F}, \mathcal{G} \in S(X)$ and $\mathcal{F} \lesssim \mathcal{G}$.

Lemma 2.5 Let X be a totally ordered space. Then $\langle \mathcal{F}, \mathcal{G} \rangle \in \wp(X)$ iff $\mathcal{F}, \mathcal{G} \in S(X)$ and there is no $x \in X$ such that $\mathcal{F} \leq^* \dot{x} \leq^* \mathcal{G}$.

Proof. Using Proposition 2.4, we see that $\mathcal{F} \leq^* \dot{x} \leq^* \mathcal{G}$ is equivalent to $x \in \mathcal{F}^{\uparrow} \cap \mathcal{G}^{\downarrow}$. If $\langle \mathcal{F}, \mathcal{G} \rangle \in \wp(X)$, then by Proposition 1.5, $\mathcal{F}^{\uparrow} = \mathcal{G}^{\uparrow}$, which implies $\mathcal{F}^{\uparrow} \cap \mathcal{G}^{\downarrow} = \phi$. Conversely if $x \in \mathcal{F}^{\uparrow} \cap \mathcal{G}^{\downarrow}$, then $\mathcal{F}^{\uparrow} = \mathcal{G}^{\uparrow}$ is impossible, and so by Proposition 1.5 $\langle \mathcal{F}, \mathcal{G} \rangle \notin \wp(X)$. ∎

Proposition 2.6 Let X be a totally ordered space. If $\mathcal{F}, \mathcal{G} \in w_0X$, then $\mathcal{G}$ covers $\mathcal{F}$ iff exactly one of the following is true.

(a) There are $x, y \in X$ such that $\mathcal{F} = \dot{x}, \mathcal{G} = \dot{y}$, and y covers x.

(b) $\mathcal{F} = \dot{x}$ for some $x \in X$, $\mathcal{G}$ is a decreasing, simple, bounded singularity, and $\mathcal{G} \xrightarrow{0} x$.

(c) $\mathcal{G} = \dot{x}$ for some $x \in X$, $\mathcal{F}$ is an increasing, simple, bounded singularity, and $\mathcal{F} \xrightarrow{0} x$.

(d) $\langle \mathcal{F}, \mathcal{G} \rangle \in \wp(X)$.

Proof. We limit the proof to the case where $\mathcal{F}$ and $\mathcal{G}$ are both singularities. In this case, it follows from Lemma 2.5 that $\mathcal{G}$ covers $\mathcal{F}$ iff $\langle \mathcal{F}, \mathcal{G} \rangle \in \wp(X)$. ∎

Since w_0X has the order topology for any totally ordered space X, we may use Lemma 2.5 and Proposition 2.6 to describe the basic neighborhoods in w_0X for each compactification point. If $\mathcal{F}$ is an increasing singularity (either simple or essential, bounded or unbounded), then "closed" intervals in w_0X of the form $[\dot{x}, \mathcal{F}]$, for $x \in \mathcal{F}^{\downarrow}$, form a base for the neighborhood filter at $\mathcal{F}$. Likewise for a decreasing singularity $\mathcal{F}$, intervals in w_0X of the form $[\mathcal{F}, \dot{x}]$, for $x \in \mathcal{F}^{\uparrow}$, constitute a basic family of neighborhoods.

We next make use of our knowledge of w_0X to describe an arbitrary T_2-ordered compactification of a totally ordered space X. Since $(w_0X,\ \varphi_X)$ is the largest T_2-ordered compactification of X, any other T_2-ordered compactification (Y, ψ) of X is related to the former by: $(Y, \psi) \circledcirc\!\!\!\!\!<\ (w_0X,\ \varphi_X)$. Thus there is an increasing, continuous function $\sigma_Y : w_0X \to Y$ which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_X} & w_0X \\ & \underset{\psi}{\searrow} & \downarrow \sigma_Y \\ & & Y \end{array}$$

commute.

We can think of Y as the quotient space of w_0X obtained by identifying the compactification points (i.e., the singularities of X) in an appropriate way.

Proposition 2.7 Let (Y, ψ) be a T_2-ordered compactification of a totally ordered space X, and let $\sigma_Y : w_0X \to Y$ be the function described in the preceding paragraph. If $\mathcal{F}$ and $\mathcal{G}$ are distinct singularities of X such that $\mathcal{F} \lesssim \mathcal{G}$, then $\sigma_{\mathcal{F}} = \sigma_{\mathcal{G}}$ implies $\langle \mathcal{F}, \mathcal{G} \rangle \in \wp(X)$.

Proof. If $\langle \mathcal{F}, \mathcal{G} \rangle \notin \wp(X)$, then by Lemma 2.5 there is $x \in X$ such that $\mathcal{F} \leq^* \dot{x} \leq^* \mathcal{G}$. Since σ_Y is an increasing function and $\sigma_Y(\mathcal{F}) = \sigma_Y(\mathcal{G})$, we must conclude that $\sigma_Y(\mathcal{F}) = \sigma_Y(\dot{x})$. But $\sigma_Y(\mathcal{F}) \in Y - \psi(X)$, whereas $\sigma_Y(\dot{x}) \in \psi(X)$. ∎

Thus the only possible way that singularities can be identified in order to form a T_2-ordered compactification Y as a quotient of w_0X is to identify essential pairs of singularities to single elements in Y. To be more explicit, let $P \subseteq \wp(X)$ be a set of essential pairs on X, and let Y_P be the topological quotient space obtained from w_0X as follows: for each pair $\langle \mathcal{F}, \mathcal{G} \rangle \in P$, the distinct elements $\mathcal{F}$ and $\mathcal{G}$ in w_0X we identified to a single element, denoted by $\langle \mathcal{F}, \mathcal{G} \rangle$. Let $\sigma_P : w_0X \to Y_P$ be the canonical quotient map; σ_P is continuous by construction, and we impose on Y_P the unique total order relative to which σ_P is increasing. Let $\psi_p : X \to Y_P$ be the composition $\sigma_P \circ \varphi_X$.

In order to get a clearer picture of the T_2-ordered compactification (Y_P, ψ_P), we may identify the set Y_P with $P' \cup P$, where $P' = \{\mathcal{H} \in w_0X : \mathcal{H}$ does not belong to an essential pair in $P\}$. Y_P consists of equivalence classes of members of w_0X, each of which contains either one or two elements. Thus P' consists of the elements of w_0X which determine singleton classes in Y_P, whereas P can be identified with the two element classes in Y_P. If $\mathcal{H} \in P'$, the upper bounds of $\mathcal{H}$ in Y_P consist of those elements in P' which are upper bounds of $\mathcal{H}$ in w_0X, along with those elements $\langle \mathcal{F}, \mathcal{G} \rangle \in P$ such that $\mathcal{H} \leq^* \mathcal{F}$ (or equivalently, $\mathcal{H} \leq^* \mathcal{G}$) in w_0X. If $\langle \mathcal{F}, \mathcal{G} \rangle \in P$, the upper bounds of $\langle \mathcal{F}, \mathcal{G} \rangle$ in Y_P consist of those elements in P' which are above $\mathcal{F}$ in w_0X and also those elements $\langle \mathcal{F}', \mathcal{G}' \rangle \in P$ such that $\mathcal{F} \leq^* \mathcal{F}'$ in w_0X. For those elements of Y_P in P', the basic neighborhoods have the same form in Y_P as in w_0X except of course, that the intervals involved must be construed as lying in Y_P instead of w_0X. On the other hand, for $\langle \mathcal{F}, \mathcal{G} \rangle \in P$, the neighborhood filter in Y_P has a filter base of "closed" intervals in Y_P of the form $[\dot{x}, \dot{y}]$, where $\dot{x} \leq^* \mathcal{F} \leq^* \dot{y}$.

If P, Q are subsets of $\wp(X)$ and $P \subseteq Q$, then Y_Q can be regarded as the quotient space of Y_P obtained by identifying those essential pairs $\mathcal{F}, \mathcal{G}$ in P' such that $\langle \mathcal{F}, \mathcal{G} \rangle \in Q$. The canonical quotient map $\sigma_{PQ} : Y_P \to Y_Q$ is continuous and increasing, and the quotient map $\sigma_Q : w_0X \to Y_Q$ is given by $\sigma_Q = \sigma_{PQ} \circ \sigma_P$. Our results on T_2-ordered compactifications of a totally ordered space X may be summarized in the following theorems.

Theorem 2.8 Let X be a totally ordered space, and let P be an arbitrary subset of $\wp(X)$. Then (Y_P, ψ_P) is a T_2-ordered compactification of X. If P, Q are subsets of $\wp(X)$, then $P \subseteq Q$ iff $(Y_Q, \psi_Q) \circledcirc (Y_P, \psi_P)$. If $P = \emptyset$, then (Y_P, ψ_P) is equivalent to (w_0X, φ_X) and is the largest

T_2-ordered compactification of X. If $Q = \wp(X)$, then (Y_Q, ψ_Q) is the smallest T_2-ordered compactification of X.

Theorem 2.9 If (Y, ψ) is any T_2-ordered compactification of a totally ordered space X, then there is a subset P of $\wp(X)$ such that (Y, ψ) is equivalent to (Y_P, ψ_P).

Corollary 2.10 A totally ordered space X has a unique ordered compactification iff X has no essential singularities.

Examples of T_2-ordered spaces with unique T_2-ordered compactification include the real line (with usual order and topology), the Sorgenfrey line, and the discrete line (the real numbers with usual order and discrete topology). In the case of the Sorgenfrey line, one compactification point corresponding to each real number is added, along with least element $-\infty$ and greatest element ∞. The T_2 ordered compactification of the discrete line is similar, except that two compactification points are added for each real number (one on each side).

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