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ON THE DUAL SPACE OF A WEIGHTED BERGMAN SPACE ON THE UNIT BALL OF C^n

J.S. CHOA and H.O. KIM

Department of Applied Mathematics
KAIST
P.O. Box 150, Cheongryang
Seoul, KOREA

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ABSTRACT. The weighted Bergman space $A^p_\alpha(B_n)(0 < p < 1)$, of the holomorphic functions on the unit ball B_n of C^n forms an F-space. We find the dual space of $A^p_\alpha(B_n)$ by determining its Mackey topology.

KEY WORDS AND PHRASES. Hardy space, Bergman space, Mackey topology.
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1. INTRODUCTION.

Let B_n be the unit ball of C^n, ν be the normalized Lebesgue measure and σ be the rotation invariant positive Borel measure on S, the boundary of B_n, with $\sigma(S) = 1$. The weighted Bergman space $A^p_\alpha(B_n)(0 < p < \infty, \alpha \geq -1)$ consists of all functions holomorphic in B_n for which

$$\|f\|^p_{p,\alpha} = \begin{cases} \int_0^1 M^p_p(r;f)(1-r)^\alpha \, d\sigma(\zeta) 2nr^{2n-1} dr < \infty, & \text{if } \alpha > -1, \\ \sup_{0 \leq r < 1} M^p_p(r;f) < \infty, & \text{if } \alpha = -1, \end{cases}$$

where $M^p_p(r;f) = \int_S |f(r\zeta)|^p d\sigma(\zeta)$. Note that the weighted Bergman space $A^p_\alpha(B_n)$ is, in fact, the Hardy space $H^p(B_n)$ if $\alpha = -1$ (See [1]).

The purpose of this paper is to compute the dual space $(A^p_\alpha(B_n))^*$ for $0 < p \leq 1$ by determining the Mackey topology of $A^p_\alpha(B_n)$. The corresponding problems for the case $n = 1$ are settled by Duren, Romberg and Shields [2], Shapiro [3] and Ahern [4]. Our computations are very similar to those of them.

Throughout this work, $C_{\alpha,\beta,\ldots}$ denotes a positive constant depending only on $\alpha,\beta,\ldots$ which may vary in the various places, and the notation $a(z) \sim b(z)$ means that the ratio $a(z)/b(z)$ has a positive finite limit as $|z| \to 1$.

2. SOME PRELIMINARY RESULTS.

LEMMA 2.1. If $f \in A^p_\alpha(B_n)(0 < p < \infty, \alpha \geq -1)$, then

$$|f(z)| \leq C_{n,p,\alpha} \|f\|_{p,\alpha} (1 - |z|)^{-\frac{n+\alpha+1}{p}} .$$

PROOF. The case $\alpha = -1$ is proved in [1, Thm 7.2.5]. For the proof of the case $\alpha > -1$, it is enough to prove the result for $\frac{1}{2} \leqq |z| < 1$ since $|f(z)|$ is bounded for $|z| \leqq \frac{1}{2}$. For this range of ρ we have:

$$\begin{aligned}
\|f\|_{p,\alpha}^{p} &= \int_0^1 \int_S |f(r\zeta)|^p (1-r)^{\alpha} 2nr^{2n-1} dr d\sigma(\zeta) \\
&\geqq C_{n,p} \int_\rho^1 \int_S |f(r\zeta)|^p (1-r)^{\alpha} d\sigma(\zeta) dr \\
&\geqq C_{n,p}\, M_p^p(\rho;f) \int_\rho^1 (1-r)^{\alpha} dr \\
&= C_{n,p,\alpha}\, M_p^p(\rho;f)(1-\rho)^{\alpha+1}.
\end{aligned}$$

By the result of the case $\alpha = -1$ and the above result, we get

$$\begin{aligned}
|f(\rho z)| &\leqq C_{n,p} M_p(\rho;f)(1-|z|)^{-\frac{n}{p}} \\
&\leqq C_{n,p,\alpha} \|f\|_{p,\alpha} (1-\rho)^{-\frac{\alpha+1}{p}} (1-|z|)^{-\frac{n}{p}}.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
|f(z)| &= |f(\sqrt{r}\sqrt{r}\zeta)| \qquad (z = r\zeta) \\
&\leqq C_{n,p,\alpha} \|f\|_{p,\alpha} (1-\sqrt{r})^{-\frac{\alpha+1}{p}} (1-\sqrt{r})^{-\frac{n}{p}} \\
&\leqq C_{n,p,\alpha} \|f\|_{p,\alpha} (1-r)^{-\frac{n+\alpha+1}{p}}.
\end{aligned}$$

COROLLARY 2.2. (a) The convergence of $A_\alpha^p(B_n)$ with its invariant metric

$$d(f,g) = \begin{cases} \|f-g\|_{p,\alpha}^p, & (0 < p < 1), \\ \|f-g\|_{p,\alpha}, & (p \geqq 1) \end{cases}$$

implies the uniform convergence on any compact subset of B_n.

(b) $A_\alpha^p(B_n)$ is an F-space if $0 < p < 1$ and a Banach space if $p \geqq 1$.

PROOF. (a) follows immediately from Lemma 2.1. The proof of (b) is routine and is omitted.

COROLLARY 2.3. $A_\alpha^p(B_n) \subset A_\beta^q(B_n)$ if $0 < p < q$ and $\frac{n+\alpha+1}{p} = \frac{n+\beta+1}{q}$.
In particular,

$$A_\alpha^p(B_n) \subset A_\sigma^1(B_n), \quad \text{where } \sigma = \frac{n+\alpha+1}{p} - (n+1).$$

PROOF. First, we prove the case $\alpha > -1$. We use Lemma 2.1 in the first inequality of the following.

$$\|f\|_{q,\beta}^q = \int_0^1 \int_S |f(r\zeta)|^q (1-r)^\beta 2nr^{2n-1} dr d\sigma(\zeta)$$

$$\leq \int_0^1 \int_S |f(r\zeta)|^p C_{n,p,\alpha,q} \left[(1-r)^{-\frac{n+\alpha+1}{p}} \right]^{q-p} \|f\|_{p,\alpha}^{q-p}$$

$$\times (1-r)^\beta 2nr^{2n-1} dr d\sigma(\zeta)$$

$$\leq C_{n,p,\alpha,q} \|f\|_{p,\alpha}^{q-p} \int_0^1 \int_S |f(r\zeta)|^p (1-r)^\alpha 2nr^{2n-1} dr d\sigma(\zeta)$$

$$\leq C_{n,p,\alpha,q} \|f\|_{p,\alpha}^q .$$

This completes the proof of the case $\alpha > -1$. The remaining case is essentially a result of Hardy and Littlewood, but we give a proof using Ahern's technique in [5]. Let $f \in H^p(B_n)$. By Lemma 2.1, we have

$$|f(z)| \leq K_{n,p}(1-|z|)^{-\frac{n}{p}} \|f\|_p .$$

Set

$$Mf(\zeta) = \sup_{0 \leq r < 1} |f(r\zeta)| .$$

Then we have

$$\int_0^1 |f(r\zeta)| (1-r)^{(\frac{1}{p}-1)n-1} 2nr^{2n-1} dr \tag{2.1}$$

$$\leq K_{n,p} \|f\|_p \int_0^\lambda (1-r)^{-n-1} dr + C_{n,p} Mf(\zeta) \int_\lambda^1 (1-r)^{(\frac{1}{p}-1)n-1} dr$$

$$\leq K_{n,p} \|f\|_p \frac{(1-\lambda)^{-n}}{n} + C_{n,p} Mf(\zeta) \frac{(1-\lambda)^{(\frac{1}{p}-1)n}}{(\frac{1}{p}-1)n} . \tag{2.2}$$

If $Mf(\zeta) \leq K_{n,p}\|f\|_p$, by setting $\lambda = 0$ in (2.2), (2.1) is dominated by $C_{n,p}\|f\|_p$. If $Mf(\zeta) \geq K_{n,p}\|f\|_p$, by setting

$$\lambda = 1 - \left[\frac{K_{n,p}\|f\|_p}{Mf(\zeta)} \right]^{\frac{p}{n}}$$

in (2.2), (2.1) is dominated by

$$C_{n,p}\|f\|_p^{1-p} Mf(\zeta)^p .$$

Hence, for any $\zeta \in S$,

$$(2.1) \leq C_{n,p}\|f\|_p + C_{n,p}\|f\|_p^{1-p} Mf(\zeta)^p . \tag{2.3}$$

Integrating (2.3) with respect to $d\sigma(\zeta)$ over S and using the complex maximal theorem [1, Thm. 5.6.5], we obtain

$$\int_S \int_0^1 |f(r\zeta)|(1-r)^{(\frac{1}{p}-1)n-1} 2nr^{2n-1} dr d\sigma(\zeta)$$

$$\le C_{n,p} \|f\|_p + C_{n,p} \|f\|_p^{1-p} \|f\|_p^p$$

$$\le C_{n,p} \|f\|_p .$$

3. THE MACKEY TOPOLOGY OF $A^p_\alpha(B_n)$.

In this section, we will show that the Mackey topology of $A^p_\alpha(B_n)$ is the restriction of the topology of $A^1_\sigma(B_n)$, where $\sigma = \frac{n+\alpha+1}{p} - (n+1)$.

First we give necessary definitions.

DEFINITION 3.1. The Mackey topology of a non-locally convex topological vector space (X,τ) is the unique locally convex topology m on X satisfying the following conditions:

(1) m is weaker than τ,

(2) the τ-closure of the absolutely convex hull of each τ-neighborhood of the origin contains an m-neighborhood of the origin (See [6, Thm 1]).

DEFINITION 3.2. For $\beta > -n$ and $z, w \in B_n$, we define

$$K_\beta(z,w) = \binom{n+\beta}{n} \frac{(1-|w|^2)^\beta}{(1-\langle z,w\rangle)^{\beta+n+1}}$$

and

$$J_{\beta,\sigma}(w)(z) = (1-|w|^2)^{-\sigma} K_\beta(z,w).$$

The following proposition is useful in the sequel:

PROPOSITION 3.3. [1, p. 120] If $\beta > -n$, then $K_\beta(z,w)$ is a reproducing kernel for the holomorphic functions in $L^1\{(1-|w|^2)^\beta dv(w)\}$. In other words, if f is holomorphic on B_n and integrable with respect to the measure $(1-|w|^2)^\beta dv(w)$, then

$$f(z) = \int_{B_n} K_\beta(z,w) f(w) dv(w).$$

LEMMA 3.4. [1, Prop. 1.4.10] For $z \in B_n$ and c real, we define

$$I_c(z) = \int_S \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+c}} .$$

If $c > 0$, then $I_c(z) \sim (1-|z|^2)^{-c}$.

LEMMA 3.5. [7, Lemma 6] If $0 < r,\rho < 1$ and $\alpha - \beta + 1 < 0$, then

$$\int_0^1 (1-r)^\alpha (1-\rho r)^\beta dr \le C_{\alpha,\beta}(1-\rho)^{\alpha-\beta+1}$$

for some positive constant $C_{\alpha,\beta}$.

The next lemma is an easy application of the above two lemmas.

LEMMA 3.6. Let $0 < p < 1$ and fix $\beta > \frac{n+\alpha+1}{p} - (n+1) \equiv \sigma$. Then

$$\sup\{\|J_{\beta,\sigma}(w)\|^p_{p,\alpha} : w \in B_n\} < \infty.$$

PROOF. We only prove the case $\alpha > -1$. Let $w \in B_n$, and $0 < r < 1$.

Then we have, by Lemma 3.4 and 3.5,

$$\|J_{\beta,\sigma}(w)\|_{p,\alpha}^p$$

$$= \int_0^1 \int_S |J_{\beta,\sigma}(w)(r\zeta)|^p (1-r)^\alpha 2nr^{2n-1} dr d\sigma(\zeta)$$

$$\leq C_{n,p,\beta}(1-|w|^2)^{(\beta-\sigma)p} \int_0^1 (1-r)^\alpha \int_S \frac{d\sigma(\zeta)}{|1-<r\zeta,w>|^{(\beta+n+1)p}} dr$$

$$\leq C_{n,p,\beta}(1-|w|^2)^{(\beta-\sigma)p} \int_0^1 (1-r)^\alpha (1-r|w|^2)^{n-(\beta+n+1)p} dr$$

$$\leq C_{n,p,\alpha,\beta}(1-|w|^2)^{(\beta-\sigma)p}(1-|w|^2)^{\alpha+n+1-(\beta+n+1)p}$$

$$\leq C_{n,p,\alpha,\beta} .$$

Thus we have

$$\sup\{\|J_{\beta,\sigma}(w)\|_{p,\alpha}^p : w \in B_n\} < \infty.$$

The proofs of the following theorems are essentially the same as those of [3] (Prop. 4.4 and Prof. 4.5) and are omitted.

THEOREM 3.7. Let $0 < p < 1$ and $\beta > \frac{n+\alpha+1}{p} - (n+1) \equiv \sigma$. Then there exists $C_{n,p,\alpha,\beta} < \infty$ such that for each $f \in A_\sigma^1(B_n)$ there exist a sequence (w_j) of the points in B_n and a sequence (λ_j) of the complex numbers such that

$$\sum_j |\lambda_j| \leq C_{n,p,\alpha,\beta} \|f\|_{1,\sigma} \tag{3.1}$$

and

$$f = \sum_j \lambda_j J_{\beta,\sigma}(w_j), \tag{3.2}$$

where the last series converges in $A_\sigma^1(B_n)$.

THEOREM 3.8. The Mackey topology of $A_\alpha^p(B_n)$ is the restriction of the topology of $A_\sigma^1(B_n)$ where $\sigma = \frac{n+\alpha+1}{p} - (n+1)$.

4. THE DUAL SPACE OF $A_\alpha^p(B_n)$.

Finally, we will find the dual space of $A_\alpha^p(B_n)$. For the proof of this main result, the following definition is needed:

DEFINITION 4.1. (Radial fractional derivatives of holomorphic functions in B_n) Let $g(z) = \sum_{k=0}^\infty G_k(z)$ be the homogeneous expansion of g. For any real number q, the radial fractional derivative of g of order q is defined by

$$R^q g(z) = \sum_{k=0}^\infty (k+1)^q G_k(z).$$

Let

$$f(z) = \sum_{k=0}^\infty F_k(z) = \sum_{k=0}^\infty \sum_{|\gamma|=k} c(\gamma) z^\gamma,$$

and

$$g(z) = \sum_{\ell=0}^{\infty} G_\ell(z) = \sum_{\ell=0}^{\infty} \sum_{|\delta|=\ell} d(\delta) z^\delta$$

be the homogeneous expansions of f and g, respectively. We note that for $q > 0$, $0 \leq \rho < 1$, we have

$$\sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma)\overline{d(\gamma)} \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k$$

$$= \frac{2^q}{\Gamma(q)} \int_0^1 (\log \frac{1}{r})^{q-1} \int_S R^{-a} f(r\zeta)\overline{R^{q+a} g(r\rho\zeta)} 2nr^{2n-1} dr d\sigma(\zeta). \tag{4.1}$$

We can now prove the duality relation. We use the idea of Ahern [4] in the proof of the following.

THEOREM 4.2. Let $0 < p < 1$ and $\sigma = \frac{n+\alpha+1}{p} - (n+1)$. Then

$$(A^p_\alpha(B_n))^* = \{ f \in H(B_n) : \sup(1 - |z|)|R^{\sigma+2} f(z)| = \|f\|_{\Lambda_1} < \infty\}.$$

PROOF. By Theorem 3.8, $(A^p_\alpha)^* = (A^1_\sigma)^*$. It suffices to compute $(A^1_\sigma)^*$. For simplicity we assume $\sigma = 0$. Take g such that

$$\sup_{z \in B_n} (1 - |z|)|R^2 g(z)| < \infty$$

and let f be a polynomial. Then by (4.1)

$$\sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma)\overline{d(\gamma)} \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k$$

$$= 2 \int_0^1 \int_S R^{-1} f(r\zeta)\overline{R^2 g(r\rho\zeta)} 2nr^{2n-1} dr d\sigma(\zeta)$$

$$= 4 \int_0^1 \int_S (\log \frac{1}{r}) f(r\zeta)\overline{R^2 g(r\rho\zeta)} 2nr^{2n-1} dr d\sigma(\zeta).$$

Hence

$$\left| \lim_{\rho \to 1} \left(\sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma)\overline{d(\gamma)} \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k \right) \right|$$

$$\leq 4 \int_0^1 \int_S \frac{\log \frac{1}{r}}{1-r} |f(r\zeta)| \sup_{z \in B_n} (1-r) |R^2 g(r\zeta)| 2nr^{2n-1} dr d\sigma(\zeta). \tag{4.2}$$

Since $\log \frac{1}{r} \sim 1 - r$ as $r \to 1$, (4.2) is dominated by

$$C_{k,n} \|f\|_{A^1_0} \|g\|_{\Lambda_1}.$$

Since polynomials are dense in A^1_0, the mapping

$$\psi(f) = \lim_{\rho \to 1} \left(\sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma)\overline{d(\gamma)} \frac{(n-1)!k!}{(n-1+k)!} \frac{2n(k+1)^q}{(k+n)^q} \rho^k \right)$$

extends to be a bounded linear functional on A_0^1. Conversely, let $\psi \in (A_0^1)^*$. Since $A_0^1 \subset L^1(2nr^{2n-1}drd\sigma(\zeta))$, by the Hahn-Banach theorem ψ extends to be a bounded linear functional ψ on the space $L^1(2nr^{2n-1}drd\sigma(\zeta))$. But since $(L^1)^* = L^\infty$, there exists G in $L^\infty(2nr^{2n-1}drd\sigma(\zeta))$ such that

$$\psi(f) = \int_0^1 \int_S f(r\zeta)\overline{G(r\zeta)}2nr^{2n-1}drd\sigma(\zeta)$$

for each f in A_0^1. Let

$$H(z) = \int_0^1 \int_S \frac{G(w)}{(1-\langle z,w\rangle)^{n+1}} 2n\rho^{2n-1}d\rho d\sigma(\eta) \qquad (w = \rho\eta)$$

be the holomorphic projection of G. If f is a holomorphic polynomial, then

$$\psi(f) = \int_0^1 \int_S f(r\zeta)\overline{G(r\zeta)}2nr^{2n-1}drd\sigma(\zeta)$$

$$\int_0^1 \int_S f(r\zeta)\overline{H(r\zeta)}2nr^{2n-1}drd\sigma(\zeta)$$

$$\int_0^1 \int_S R^{-1} f(r\zeta)\overline{RH(r\zeta)}2nr^{2n-1}drd\sigma(\zeta)$$

$$\int_0^1 \int_S R^{-1} f(r\zeta)\overline{R^2 g(r\zeta)}2nr^{2n-1}drd\sigma(\zeta),$$

where g is defined to be $R^{-1}H$. The proof will be complete if we can show that

$$\sup_{z\in B_n} (1-|z|)|R^1H(z)| < \infty.$$

Since

$$\frac{\partial H(r\zeta)}{\partial r} = \int_0^1 \int_S \frac{(n+1)}{r} \frac{\langle r\zeta, \rho\eta\rangle G(\rho\eta)}{(1-\langle r\zeta, \rho\eta\rangle)^{n+2}} 2n\rho^{2n-1}d\rho d\sigma(\zeta), \qquad (z = r\zeta)$$

we have

$$r\left|\frac{\partial H(r\zeta)}{\partial r}\right| \le C_n\|G\|_\infty \int_0^1 \int_S \frac{d\sigma(\eta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+2}} d\rho$$

$$\le C_n\|G\|_\infty \int_0^1 \frac{1}{(1-\rho r)^2} d\rho$$

$$= C_n\|G\|_\infty \frac{1}{1-r}. \tag{4.3}$$

By (4.3) and $R^1H(r\zeta) = r\frac{\partial H(r\zeta)}{\partial r} + H(r\zeta)$, we have

$$\sup_{z\in B_n} (1-|z|)|R^1H(z)| < \infty .$$

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