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# SOME CLASSES OF ALPHA-QUASI-CONVEX FUNCTIONS

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ABSTRACT. Let $C[C,D]$, $-1\le D<C\le 1$ denote the class of functions $g, g(0)=0$ $g'(0)=1$, analytic in the unit disk E such that $\frac{(zg'(z))'}{g'(z)}$ is subordinate to $\frac{1+CZ}{1+DZ}$, $z\varepsilon E$. We investigate some classes of Alpha-Quasi-Convex Functions f, with $f(0)=f'(0)-1=0$ for which there exists a $g\varepsilon C[C,D]$ such that $(1-\alpha)\frac{f'(z)}{g'(z)}+\alpha\frac{(zf'(z))'}{g'(z)}$ is subordinate to $\frac{1+AZ}{1+BZ}$, $-1\le B<A\le 1$. Integral representation, coefficient bounds are obtained. It is shown that some of these classes are preserved under certain integral operators.

KEY WORDS AND PHRASES. Convex, starlike, quasi-convex, close-to-convex function, Integral representation, Alpha-quasi-convex functions.
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## 1. INTRODUCTION

Let $S, K, S^*$ and C denote the classes of analytic functions f: $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are respectively univalent, close-to-convex, starlike (with respect to the origin) and convex in the unit disc E. In [1], a new subclass $C^*$ of univalent functions was introduced and studied. A function f belongs to $C^*$ if there exists a convex function g such that, for $z\varepsilon E$,

$$\text{Re } \frac{(zf'(z))'}{g'(z)} > 0.$$

The functions in $C^*$ are called quasi-convex functions and $C\subset C^*\subset K\subset S$. It is also sknown that $f\varepsilon C^*$, if, and only if, $zf'\varepsilon K$. For complete study of $C^*$, see Noor [2].

A new class $Q_\alpha$ of $\alpha$-quasi-convex functions has been defined and discussed in some details in [3]. A function f belongs to the class $Q_\alpha$, $\alpha$ real, if and only if there exists a convex function g such that, for $z\varepsilon E$

$$\text{Re}\left[ (1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)} \right] > 0 \tag{1.1}$$

We note that $Q_0 = K$ and $Q_1 = C^*$.

In [4] , Janowski introduced the calss $P[A,B]$. For A and B, $1\le B<A\le 1$, a function p, analytic in E with $p(0)=1$ belongs to the class $P[A,B]$, if $p(z)$ is subordinate to $\frac{1+AZ}{1+BZ}$ . Also, given C and D, $-1\le D<C\le 1$, $C[C,D]$ and $S^*[C,D]$

denote the classes of functions f analytic in E with $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ such that $\frac{(zf'(z))'}{f'(z)} \varepsilon P[C,D]$ and $\frac{zf'(z)}{f(z)} \varepsilon P[C,D]$ respectively. For C=1 and D=-1 we note that $C[1,-1]=C$ and $S^*[1,-1]=S^*$. Silvia [5] defines the classes K[A,B;C,D] as follows:

Definition 1.1. A function $f: f(z) = z+\sum_{n=2}^{\infty} a_n z^n$, analytic in E, is said to be in the class K[A,B; C,D], $-1\le B<A\le 1$; $-1\le D<C\le 1$, if there exists a $g\varepsilon C[C,D]$ such that $\frac{f'(z)}{g'(z)} \varepsilon P[A,B]$.

It is clear that K[1,-1;1,-1] = K and

$$K[A,B;C,D] \subset K \subset S.$$

We now define the following:

Definition 1.2. Let $\alpha \geq 0$ be real and $f: f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E. Then $f\varepsilon Q_\alpha[A,B; C,D]$, $-1\le B<A\le 1$; $-1\le D<C\le 1$ if and only if there exists a function $g\varepsilon C[C,D]$ such that, for $z\varepsilon E$,

$$(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)} \varepsilon P[A,B].$$

It is clear that $Q_\alpha[1,-1;\ 1,-1] = Q_\alpha$.

## 2. MAIN RESULTS

We shall now study some of the basic properties of the class $Q_\alpha[A,B;C,D]$. From the definition 1.2, we immediately have:

THEOREM 2.1. Let $F(z) = (1-\alpha)f(z)+\alpha zf'(z)$, where $0<\alpha<1$ is real and $z\varepsilon E$. Then $f\varepsilon Q_\alpha[A,B;C,D]$, $-1\le B<A\le 1$; $-1\le D<C\le 1$ if and only if $F\varepsilon K[A,B;C,D]$.

We now give the integral representation for the functions in the class $Q_\alpha[A,B;C,D]$.

THEOREM 2.2. A function $f\varepsilon Q_\alpha[A,B;C,D]$, for $\alpha>0$, $-1\le B<A\le 1$; $-1\le D<C\le 1$, if and only if there exists a function $F\varepsilon K[A,B;C,D]$ such that, for $z\varepsilon E$,

$$f(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \zeta^{\frac{1}{\alpha}-2} F(\zeta)\, d\zeta \tag{2.1}$$

PROOF. From (2.1), it follows that

$$\left(\frac{1}{\alpha}-1\right) z^{\frac{1}{\alpha}-2} f(z) + \alpha z^{\frac{1}{\alpha}-1} f'(z) = z^{\frac{1}{\alpha}-2} F(z),$$

so

$$(1-\alpha)f(z)+\alpha zf'(z) = F(z)$$

and the result follows immediately from theorem 2.1.

THEOREM 2.3. Let $f\varepsilon Q_\alpha[A,B;C,D]$, $0<\alpha<1$ and $-1\le B<A\le 1$; $-1\le D<C\le 1$. Then $f\varepsilon K[A,B;C,D]$ and hence is univalent.

PROOF. Silvia [5] has proved that if $f_1 \varepsilon K[A,B;C,D]$, then so is

$$F_1(z) = \frac{1+\gamma_1}{\gamma_1 z} \int_0^z t^{\gamma_1 - 1} f_1(t)\, dt, \quad \text{Re } \gamma_1 > 0. \tag{2.2}$$

Using this result and the integral representation (2.2) with $\gamma_1 = \frac{1}{\alpha} - 1$ for $f\varepsilon Q_\alpha[A,B;C,D]$, we obtain the required result.

For our next theorem, we need the following result due to Silvia [5].

LEMMA 2.1. Let $F\epsilon K[A,B;C,D]$ and $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then

$$|b_2| \leq \frac{(C-D)+(A-B)}{2},$$

and

$$|b_3| \leq \begin{cases} \frac{C-D}{6} + \frac{(A-B)(C-D+1)}{3}, & |C-2D| \leq 1 \\ \frac{(C-D)(C-2D)}{6} + \frac{(A-B)(C-D+1)}{3}, & |C-2D| > 1. \end{cases}$$

THEOREM 2.4. Let $F\epsilon Q_\alpha[A,B;C,D]$, $0<\alpha<1$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Then

$$|a_2| \leq \frac{1}{1+\alpha}\left[\frac{(C-D) + (A-B)}{2}\right],$$

and

$$|a_3| \leq \frac{1}{(1+2\alpha)} \begin{cases} \frac{(C-D)}{6} + \frac{(A-B)(C-D+1)}{3}, & |C-2D| \leq 1 \\ \frac{(C-D)(C-2D)}{2} + \frac{(A-B)(C-D+1)}{3}, & |(C-2D)| > 1 \end{cases}$$

PROOF. Since $f\epsilon Q_\alpha[A,B; C,D]$, by theorem 2.1, the function

$$F(z) = (1-\alpha)f(z) + \alpha z f'(z)$$

belongs to $K[A,B;C,D]$. Let $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Thus

$$(1-\alpha)[z + \sum_{n=2}^{\infty} a_n z^n] + \alpha z[1 + \sum_{n=2}^{\infty} n a_n z^n] = z + \sum_{n=2}^{\infty} b_n z^n$$

or

$$(1-\alpha)\sum_{n=2}^{\infty} a_n z^n + \alpha \sum_{n=2}^{\infty} n a_n z^n = \sum_{n=2}^{\infty} b_n z^n.$$

Equating coefficients of $z^n$ on both sides, we have

$$[(1-\alpha) + \alpha n]a_n = b_n \tag{2.3}$$

Now, using Lemma 2.1 and the relation (2.3), we obtain the required result.

REMARK 2.1. Let $F\epsilon K[A,B;1,-1]$ and be given by $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Then

$$|b_2| \leq \frac{1}{2}(A-B+2).$$

This result is sharp for the function $F_0 \epsilon K[A,B,1,-1]$ and defined by

$$F_0(z) = \int_0^z \frac{(1+Aw)}{(1-w)^2(1+Bw)}\, dw.$$

3. THE CLASS $Q_\alpha[1-2\beta,-1;1-2\gamma,-1]$

In definition 1.2, if we put $A=1-2\beta$, $B=-1$; $C=1-2\gamma$, $D=-1$, then we have the following:

Definition 3.1. A function f, analytic in E, is said to be alpha-quasi-convex of order $\beta$ type $\gamma$, if, and only if, there exists a function

$g \in C[1-2\gamma,-1]$ such that

$$H(\alpha,f) = (1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)} \in P[1-2\beta,-1]$$

REMARK 3.1. Let g be analytic in E. Then $g\in C[1-2\gamma,-1]$ if and only if

$$\text{Re}\frac{(zg'(z))'}{g'(z)} > \gamma, \quad z\in E.$$

Thus $H(\alpha,f)\in P[1-2\beta,-1]$ implies that

$$\text{Re}\left[(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\right] > \beta, \quad z\in E.$$

REMARK 3.2. It follows , from the definition 3.1, that $f\in Q_\alpha[1-2\beta,-1;1-2\gamma,-1]$ if, and only if $\{(1-\alpha)f+\alpha zf'\} \in K[1-2\beta,-1;\ 1-2\gamma,-1]$.

We now have the following:

THEOREM 3.1. Let $f\in Q_\alpha[1-2\beta,\ -1;1-2\gamma,-1]$ and be given by $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$. Then we have, for $n\geq 2$

$$|a_n| \leq \frac{2(3-2\gamma)(4-2\gamma)\ldots\ldots(n-2\gamma)[n(1-\beta)+\beta-\gamma]}{n![1+\alpha(n-1)]}.$$

This result is sharp and the equality holds for the function $f_0$ defined as

$$f_0(z)=\begin{cases}\frac{1}{\alpha} z^{1-\frac{1}{\alpha}}\int_0^z \zeta^{(\frac{1}{\alpha}-2)}(\zeta(1-\gamma)(1-2\beta)+(\beta-\gamma)\ [1-(1-\zeta)^{2-2\gamma}])d\zeta, & \gamma\neq 1, \gamma\neq\frac{1}{2}\\ \frac{1}{\alpha} z^{1-\frac{1}{\alpha}}\int_0^z \zeta^{\frac{1}{\alpha}-2}\ [(1-2\beta)\log(1-\zeta)+\frac{2(1-\beta)\zeta}{1-\zeta}]\ d\zeta, & \gamma=\frac{1}{2}\\ \frac{1}{\alpha} z^{1-\frac{1}{\alpha}}\int_0^z \zeta^{\frac{1}{\alpha}-2}\ [2(\beta-1)\log(1-\zeta)+(2\beta-1)\zeta]d\zeta, & \gamma=1\end{cases}$$

PROOF. Since $f\in Q_\alpha[1-2\beta,-1;1-2\gamma,-1]$ , the function

$$F(z)=(1-\alpha)f(z)+\alpha zf'(z)$$

belong to $K[1-2\beta,-1;1-2\gamma,-1]$. Let $F(z)=z+\sum_{n=2}^{\infty} b_n z^n$.

Libera [6] has proved that , for $n\geq 2$,

$$|b_n| \leq \frac{2(3-2\gamma)(4-2\gamma)\ldots\ldots(n-2\gamma)[n(1-\beta)+\beta-\gamma]}{n!}, \tag{3.1}$$

Now, from relation (2.3), we have

$$a_n = \frac{b_n}{1+\alpha(n-1)}$$

Using this and (3.1), we obtain the required result

THEOREM 3.2. Let $0<\lambda\leq 1$ and $0\leq\beta<1$. Let f be defined as

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}}\int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta)d\zeta, \qquad \frac{1}{\lambda}\geq 1.$$

and $F\in Q_\alpha[1-2\beta,-1;1-2\gamma,-1]$ where $0\leq\lambda\leq 1$, $\alpha\geq 0$. Then $f\in Q_\alpha[1-2\beta,-1;1-2\gamma,-1]$

PROOF. Let

$$F_1(z) = (1-\alpha)F(z)+\alpha zF'(z), \tag{3.2}$$

and let

$$f_1(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}}\int_0^z \zeta^{\frac{1}{\lambda}-2} F_1(\zeta)d\zeta. \tag{3.3}$$

Since $F\in Q_\alpha[1-2\beta,-1,1-2\gamma,-1]$ , it follows from remark 3.2 that $F_1\in K[1-2\beta,-1;\ 1-2\gamma,-1]$ . We want to show that $f\in Q_\alpha[1-2\beta,-1;\ 1-2\gamma,-1]$ , where $f_1(z) = (1-\alpha)f(z)+\alpha zf'(z)$. Now (3.2) can be written as

$$F_1(z) = (1-\alpha)F(z)+\alpha zF'(z)$$

$$= \alpha z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}F(z))',$$

and using this, we obtain from (3.3)

$$f_1(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{2-\frac{1}{\alpha}} \zeta^{\frac{1}{\lambda}-2} (\zeta^{\frac{1}{\alpha}-1} F(\zeta))' d\zeta$$

$$= \frac{\alpha}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-\frac{1}{\alpha}} (\zeta^{\frac{1}{\alpha}-1} F(\zeta))' d\zeta$$

So, integrating by parts,

$$f_1(z) = \frac{\alpha}{\lambda} z^{1-\frac{1}{\lambda}} [z^{\frac{1}{\lambda}-\frac{1}{\alpha}} (z^{\frac{1}{\alpha}-1} F(z)) - \int_0^z (\frac{1}{\lambda}-\frac{1}{\alpha}) \zeta^{\frac{1}{\lambda}-\frac{1}{\alpha}-1} F(\zeta))\, d\zeta]$$

$$= \frac{\alpha}{\lambda} F(z) + \frac{\alpha}{\lambda} (\frac{1}{\alpha}-\frac{1}{\lambda}) z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta)\, d\zeta$$

$$=\alpha[\frac{1}{\lambda}F(z)]+\alpha\, [\frac{1}{\lambda}(1-\frac{1}{\lambda})+\frac{1}{\lambda}(\frac{1}{\alpha}-1)]\, z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta) d\zeta$$

$$=\alpha z[\frac{1}{\lambda} z^{-1}F(z)+\frac{1}{\lambda}(1-\frac{1}{\lambda}) z^{-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta)d\zeta]$$

$$+(1-\alpha)\, [\frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \zeta^{\frac{1}{\lambda}-2} F(\zeta)d\zeta].$$

$$= \alpha zf'(z)+(1-\alpha)f(z). \qquad (3.4)$$

Now in (3.3) $F_1 \varepsilon K[1-2\beta,-1;1-2\gamma,-1]$ and so $f_1 \varepsilon K[1-2\beta,-1;1-2\gamma,-1]$, where we have used (2.2) with $\gamma_1 = \frac{1}{\lambda}-1, A=1-2\beta, B=-1, C=1-2\gamma$ and $D=-1$. Thus it follows from remark 3.2 and the relation (3.4) that $f\varepsilon Q_\alpha[1-2\beta,-1;1-2\gamma,-1]$, and this completes the proof.

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