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ON A NONLINEAR WAVE EQUATION IN UNBOUNDED DOMAINS

CARLOS FREDERICO VASCONCELLOS

Instituto de Matemática
Universidade Federal do Rio de Janeiro
P.O. Box 68530 CEP 21944, Rio de Janeiro, RJ
Brazil

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ABSTRACT. We study existence and uniqueness of the nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} + M\left(x, \int |\nabla u(x,t)|^2 dx + \int |u(x,t)|^2 dx\right)(-\Delta u + u) = 0$$

in unbounded domains. The above model describes nonlinear wave phenomenon in non-homogeneous media. Our techniques involve fixed point arguments combined with the energy method.

KEY WORDS AND PHRASES. Nonlinear equation, unbounded domain, energy method, fixed point theorems.

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1. INTRODUCTION

In this paper we prove the existence and uniqueness of a local solution for the following problem:

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} + M(x, \|u(t)\|^2)Au = 0 \\ \\ u(x,0) = u_o \qquad u_t(x,0) = u_1(x) \end{array}\right. \tag{1.1}$$

where $M: \mathbb{R}^n x \mathbb{R} \to \mathbb{R}$,

$$\|u(t)\|^2 = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_j}(x,t)\right|^2 dx + \int_{\mathbb{R}^n} |u(x,t)|^2 dx, \qquad \forall\ t \geq 0$$

and

$$Au = -\Delta u + u = -\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + u.$$

Since the above problem is considered in unbounded domain we can not use the same method of existence of solutions used, for example, by P.H. Rivera ([1]), in which he studied the problem (1.1) when x runs in a bounded open subset of $\mathbb{R}^n$. He found a weak local solution for the problem using Galerkin method and the discrete spectrum of the Laplacian operator in bounded domains.

In the other hand, since that the mapping M depends explicity on x we can not use Fourier transform as was done, for example, by. G.P. Menzala ([2]) in which he studied the problem (1.1) when $M(x,\lambda) = M_o(\lambda)$, that is when the mapping M is independent of x.

Our assumptions about the mapping M are described below.

There exist functions $\varphi,\psi \in W^{1,\infty}(\mathbb{R}^n)$ and $m_o > 0$ such that $\varphi(x) \geq m_o > 0$ a.e. in $\mathbb{R}^n$, $\psi(x) \geq 0$ a.e. in $\mathbb{R}^n$ and $M(x,\lambda) = \varphi(x) + f(\lambda)\psi(x)$, $(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}$ where $f: \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $f(\lambda) \geq 0$ for $\lambda \geq 0$.

Here $W^{1,\infty}(\mathbb{R}^n) = \{\varphi \in L^\infty(\mathbb{R}^n): \frac{\partial\varphi}{\partial x_j} \in L^\infty(\mathbb{R}^n),\ j=1,\dots,n\}$. We also consider the usual Sobolev space $H^1(\mathbb{R}^n)$ with the norm

$$\|u\|^2 = \sum_{j=1}^{n} \int_{\mathbb{R}^n} |\frac{\partial u}{\partial x_j}(x)|^2 dx + \int_{\mathbb{R}^n} |u(x)|^2 dx.$$

Our main result in this paper will be:

There exists a unique local solution for problem (1.1) with the following properties:

There exists $T_2 > 0$ and a function $u: \mathbb{R}^n \times [0,T_2] \to \mathbb{R}$ which belongs to $C^2_\omega([0,T_2];L^2(\mathbb{R}^n)) \cap C^1([0,T_2];L^2(\mathbb{R}^n)) \cap C([0,T_2];H^1(\mathbb{R}^n))$. (1.2)

For each $t < T_2$ $u(\cdot,t) \in H^2(\mathbb{R}^n)$ and $\frac{\partial u}{\partial t}(\cdot,t) \in H^1(\mathbb{R}^n)$. (1.3)

Here $C^2_\omega([0,T_2];L^2(\mathbb{R}^n)) = \{u: [0,T_2] \to L^2(\mathbb{R}^n) : t \mapsto (u(t)|v)$ is twice continuously differentiable in $[0,T_2]$, $\forall\, v \in L^2(\mathbb{R}^n)\}$ where $(\cdot|\cdot)$ denotes the usual inner product in $L^2(\mathbb{R}^n)$. We also denote by $H^2(\mathbb{R}^n)$ the usual Sobolev space of order two.

The basic idea in order to obtain our result will be to use fixed point arguments together with the energy method in appropriate Banach spaces.

It is important to observe that our main result holds also in any open subset of $\mathbb{R}^n$.

Before concluding this introduction we would like to make a few comments on the literature. J.L. Lions ([3]) considered the problem:

$$\left| \begin{array}{ll} \frac{\partial^2 u}{\partial t^2} - M(\int_\Omega |\nabla u|^2 dx)\Delta u = 0 & \Omega \times (0,T) \\ u(x,0) = u_o(x), \quad u_t(x,0) = u_1(x) & \end{array} \right. \tag{1.4}$$

where $M(\lambda) \geq m_o > 0$ and Ω denotes a bounded open subset of $\mathbb{R}^n$. Arosio-Spagnolo ([4]) solved the problem (1.4) when $M(\lambda) > 0$, $\forall\, \lambda > 0$ in the analytic case. Recently, Ebihara-Miranda-Medeiros ([5]) studied problem (1.4) when $M(\lambda) \geq 0$, $\forall\, \lambda \geq 0$ for more general cases. Others authors like Andrade ([6]), Ball ([7]), Bernstein ([8]), Dickey ([9]), Greenberg-Hu ([10]), Medeiros ([11]), Menzala ([12]), Nishida ([13]), Nishihara ([14]), Pohozaev ([15]), Rivera ([16]), Ribeiro ([17]) and Yamada ([18,19]) also studied related problems.

2. A PRELIMINARY RESULT

In this section we prove the existence and uniqueness of a solution of the following "linearized" problem: Let $T > 0$. Let $v \in C^1([0,T];H^1(\mathbb{R}^n))$

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} + M(x,\|v(t)\|^2)Au = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T] \\ u(x,0) = u_o(x) \qquad u_t(x,0) = u_1(x) \end{array}\right. \tag{2.0}$$

From now on we shall denote by H the usual space $L^2(\mathbb{R}^n)$ in which we consider the norm $|u|^2 = \int_{\mathbb{R}^n} |u(x)|^2 dx$ and inner product $(u|v)$. Let us consider the linear operator $A: D(A) \subset H \to H$ defined by $Au = -\Delta u + u$, with $D(A) = H^2(\mathbb{R}^n)$. Clearly A is self-adjoint and satisfies:

$$(Au|u) \geq |u|^2, \qquad u \in D(A). \tag{2.1}$$

All functions we consider in this paper will be real valued. The square root of A, denoted by $A^{1/2}$ has domain $V = D(A^{1/2}) = H^1(\mathbb{R}^n)$. The inner product in V is defined by:

$$[u|v] = (A^{1/2}u|A^{1/2}v) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}\, dx + \int_{\mathbb{R}^n} u(x)v(x)dx$$

with norm $\|\cdot\|$ defined in §1.

For each $\lambda \in \mathbb{R}$ we define $B(\lambda): H \to H$ by $B(\lambda)u = M(\cdot,\lambda)u$. Because of our assumptions on $M(x,\lambda)$ the operator $B(\lambda)$ has the following properties:

For each $\lambda \in \mathbb{R}$, $B(\lambda)$ is a linear bounded symmetric operator on H. (2.2)

For each $\lambda \geq 0$ $(B(\lambda)u|u) \geq m_o|u|^2$, $u \in H$ (2.3)

For each $\lambda \geq 0$ $B(\lambda): V \to V$ is a linear continuous bijective operator (2.4)

$\forall\, T > 0$ $\exists\, \alpha_T > 0$ such that $\|B(\lambda_1)-B(\lambda_2)\|_{\mathcal{L}(V)} \leq \alpha_T|\lambda_1-\lambda_2|$ if $|\lambda_1|,|\lambda_2| \leq T$. Here $\mathcal{L}(V)$ is the space of linear continuous operators on V (2.5)

$\forall\, T > 0$ $\exists\, \beta_T > 0$ such that if $(u,v) \in D(A)\times V$ and $|\lambda| \leq T$ $|(B(\lambda)Au|v) - (B(\lambda)A^{1/2}u|A^{1/2}v)| \leq \beta_T\|u\|\,|v|$ (2.6)

$B: [0,+\infty) \to \mathcal{L}(H)$ is continuously differentiable. (2.7)

Here $\mathcal{L}(H)$ is the space of linear continuous operators on H.

LEMMA 1. Let v belonging to $C^1(\mathbb{R};V)$, then

$\forall\, t \in \mathbb{R}$ $N(t) = A^{1/2}\, B(\|v(t)\|^2)A^{1/2}$ is a self adjoint operator in H with domain $D(N(t)) = D(A)$, $\forall\, t \in \mathbb{R}$. (2.8)

$(N(t)u|u) \geq m_o|u|^2$ $\quad \forall\, t \in \mathbb{R}$ and $\forall\, u \in D(A)$ (2.9)

$\forall\, T > 0$ there is m_T such that (2.10)
$\|N(t)N^{-1}(0) - N(s)N^{-1}(0)\|_{\mathcal{L}(H)} \leq m_T\, |t-s|$
whenever $|t|,|s| \leq T$.

PROOF: By (2.4) we can show that $D(N(t)) = D(A)$, $\forall\ t \in \mathbb{R}$ and that the image of $N(t)$ is H. Hence since $A^{1/2}$ and $B(\|v(t)\|^2)$ are symmetric we obtain (2.8).

In the other hand if $u \in D(A)$ we obtain by (2.1) and (2.3) that $(N(t)u|u) = (B(\|v(t)\|^2)A^{1/2}u|A^{1/2}u) \geq m_o|A^{1/2}u|^2 \geq m_o|u|^2$ therefore (2.9) follows.

To prove (2.10) we observe by (2.8) and the closed graph theorem, that $N(t)[N(0)]^{-1} \in \mathcal{L}(H)$.

We consider $u \in H$ and $T > 0$, then by (2.5) we obtain,
$|N(t)[N(0)]^{-1}u-N(s)[N(0)]^{-1}u| \leq \alpha_T\|[B(\|v(0)\|^2]^{-1}\|_{\mathcal{L}(V)}|u||t-s|$
$|t|,|s| \leq T$ which proves (2.10).

PROPOSITION 1. Let $u_o \in H^3(\mathbb{R}^n) = D(A^{3/2})$, $u_1 \in H^2(\mathbb{R}^n)$ and $v \in C^1(\mathbb{R};V)$. Then there is a unique function $u\colon \mathbb{R} \to H^3(\mathbb{R}^n)$ such that:

$$u \in C^2(\mathbb{R};V) \cap C^1(\mathbb{R};D(A)) \tag{2.11}$$

$$\left|\begin{array}{l} u'' + B(\|v(t)\|^2)Au(t) = 0 \quad \text{in} \quad V\times\mathbb{R} \\ u(0) = u_o \qquad u'(0) = u_1 \end{array}\right. \tag{2.12}$$

PROOF: By Lemma 1 and a result due to J. Goldstein (see Theorem 2.2. in [20]) there is a unique function $w\colon \mathbb{R} \to H^2(\mathbb{R}^n)$ such that

$$w \in C^2(\mathbb{R};H) \cap C^1(\mathbb{R};V) \tag{2.13}$$

$$\left|\begin{array}{l} w'' + N(t)w(t) = 0 \quad \text{in} \quad H\times\mathbb{R} \\ w(0) = A^{1/2}u_o \qquad w'(0) = A^{1/2}u_1 \end{array}\right. \tag{2.14}$$

Let us consider $u(t) = A^{-1/2}w(t)$ for $t \in \mathbb{R}$ then $u\colon \mathbb{R} \to H^3(\mathbb{R}^n)$ satisfies (2.11) and (2.12).

Therefore it follows that u is the unique solution of (2.12) which satisfies (2.11).

PROPOSITION 2. Let T be a positive real number. Then given $v \in C^1(0,T;V)$, $u_o \in H^3(\mathbb{R}^n) = D(A^{3/2})$ and $u_1 \in H^2(\mathbb{R}^n)$. There is a unique function $u = u(v)\colon [0,T] \to H^3(\mathbb{R}^n)$ such that:

$$u \in C^2(0,T;V) \cap C^1(0,T;D(A)) \tag{2.15}$$

$$\left|\begin{array}{l} u'' + B(\|v(t)\|^2)Au(t) = 0 \\ u(0) = u_o \qquad v'(0) = u_1 \end{array}\right. \tag{2.16}$$

PROOF: We define

$$w(t) = \begin{cases} v(t) & \text{if} \quad 0 \leq t \leq T \\ v'(T)(t-T)+v(t) & \text{if} \qquad t > T \\ v'(0)t + v(0) & \text{if} \qquad t < 0 \end{cases}$$

$w \in C^1(\mathbb{R};V)$ and hence there is $u = u(w)\colon \mathbb{R} \to H$ which satisfies the Proposition 1, in particular u satisfies (2.15) and (2.16), with $u\colon [0,T] \to H^3(\mathbb{R}^n)$.

Remains to prove the uniqueness. Suppose that we have another solution z of (2.16) which satisfies (2.15).

Then $\sigma(t) = u(t)-z(t)$ satisfies

$$\begin{cases} \sigma''(t) + B(\|v(t)\|^2)A\sigma = 0 & \text{in} \quad V \times [0,T] \\ \sigma(0) = 0 \qquad \sigma'(0) = 0 \end{cases} \tag{2.17}$$

We consider $\tau(t) = \frac{1}{2}\{|\sigma'(t)|^2+(B(\|v(t)\|^2)A^{1/2}\sigma|A^{1/2}\sigma)\}$ then by (2.17) we obtain that

$$\tau'(t) = -(B(\|v(t)\|^2)A\sigma|\sigma') + (B(\|v(t)\|^2)A^{1/2}\sigma|A^{1/2}\sigma') + + [v(t)|v'(t)](B'(\|v(t)\|^2)A^{1/2}\sigma|A^{1/2}\sigma).$$

Hence by (2.6) and (2.7)

$$\tau'(t) \le \frac{\beta_T}{2}(\|\sigma(t)\|^2 + |\sigma'(t)|^2) + C_T\|\sigma(t)\|^2$$

where

$$C_T = \sup_{0\le t\le T} (\|B'(\|v(t)\|^2)\|_{\mathfrak{L}(H)} \|v(t)\|\|v'(t)\|).$$

Now, using (2.3) we obtain that there exists $\eta_T > 0$ such that:

$$\tau'(t) \le \eta_T\, \tau(t), \qquad t \in (0,T].$$

Hence, since $\tau(0) = 0$, it follows that $\tau(t) = 0$, $\forall\, t \in [0,T]$ which proves the Proposition 2.

COROLLARY 1. Let $v \in V^1(0,T;V(\mathbb{R}^n))$, $u_o \in H^3(\mathbb{R}^n)$ and $u_1 \in H^2(\mathbb{R}^n)$. Then there is a unique $u: [0,T] \to H^3(\mathbb{R}^n)$ such that: $u \in C^2([0,T];H^1(\mathbb{R}^n)) \cap C^1([0,T];H^2(\mathbb{R}^n))$ and satisfies (2.0).

3. SOLUTION OF PROBLEM (1.1)

We consider $T > 0$ and we denote by $X_T = \{u: [0,T] \to H :$ $u \in C^1(0,T;V) \cap C(0,T;D(A)\}$. Clearly X_T is a Banach space with the norm $\|u\|_{X_T} = \sup\limits_{0\le t\le T} \{\|u'(t)\| + |Au(t)|\}$. Now, we consider $u_o \in H^3(\mathbb{R}^n) = D(A^{3/2})$ and $u_1 \in D(A) = H^2(\mathbb{R}^n)$. We observe that given $v \in X_T$ there is a unique $u = S(v) \in X_T$ which satisfies (2.15) and (2.16). Let us call $\|u_1\|^2 + |Au_o|^2 = C$ and consider $E_{T,C} = \{v \in X_T : \|v(0)\|^2 < C\}$.

LEMMA 2. There are $r = r(C) > 0$ and $T_o = T_o(C) > 0$ such that if $v \in E_{T_o,C}$ and $\|v\|_{X_{T_o}} \le r$, then $\|S(v)\|_{X_{T_o}} \le r$.

PROOF: We consider $T > 0$ and $u = S(v)$ where $v \in X_T$ and we define $z(t) = \frac{1}{2}\{\|u'(t)\|^2 + (B(\|v(t)\|^2)Au|Au)\}$. Thus, since u satisfies (2.16) we obtain that $z'(t) = [v'(t)|v(t)](B'(\|v(t)\|^2)Au|Au)$. Therefore, by (2.3)

$$z'(t) \le \frac{2}{m_o} \|v\|^2_{X_T} \|B'(\|v(t)\|^2)\|_{\mathfrak{L}(H)} z(t). \tag{3.1}$$

Let

$$r = \sqrt{\frac{4C(1+\|B(\|v(0)\|^2)\|_{\mathfrak{L}(H)}}{\min(1,m_o)}}.$$

Therefore if $\|v\|_{X_T} \le r$ then $\|B'(\|v(t)\|)\| \le \gamma_c$, $0 \le t \le T$. Thus, by (3.1) we obtain that:

$$z(t) \le z(0)\, \exp(\frac{2}{m_o} r^2 \gamma_c t), \qquad 0 \le t \le T. \tag{3.2}$$

We choose $T_o = \min\{T,\mu\}$, where $\mu = \frac{m_o}{2r^2\gamma_c}\log 2$ then for each $0 \le t \le T_o$, we conclude that:

$$z(t) \le 2\, z(0) \le C(1 + \|B(\|v(0)\|^2)\|_{\mathcal{L}(H)})$$

and so, by (2.3)

$$(\|u'(t)\| + |Au(t)|)^2 \le r^2, \quad \text{for } 0 \le t \le T_o.$$

This completes the proof of Lemma 2.

Now, we define the space $Y_T = \{u: [0,T] \to H: u \in C(0,T;V) \cap \cap C^1(0,T;H)\}$ with the norm:

$$\|u\|_{Y_T} = \sup_{0\le t\le T} \{\|u(t)\| + |u'(t)|\}.$$

Clearly Y_T is a Banach space.

LEMMA 3. We consider r and $T_o > 0$ as in Lemma 2. Then there are $0 < T_1 \le T_o$ and $0 < \theta < 1$ such that

$$\|S(u)-S(v)\|_{Y_{T_1}} \le \theta\|u-v\|_{Y_{T_1}} \quad \text{for every } u \text{ and } v \text{ in } E_{T_1,C} \tag{3.3}$$

with $\|u\|_{X_{T_1}} \le r$ and $\|v\|_{X_{T_1}} \le r$.

PROOF: Let us consider u and v in $E_{T_o,C}$. Then $\sigma(t) = = S(u)(t)-S(v)(t)$ satisfies:

$$\left|\begin{array}{l} \sigma''(t) + B(\|u(t)\|^2)A\sigma + [B(\|u(t)\|^2)-B(\|v(t)\|^2)]AS(v) = 0 \\ \sigma(0) = 0 = \sigma'(0) \end{array}\right. \tag{3.4}$$

If we define

$$y(t) = \frac{1}{2}\{|\sigma'(t)|^2 + (B(\|u(t)\|^2)A^{1/2}\sigma|A^{1/2}\sigma)\}$$

by (2.6) and (3.4), then we obtain that:

$$y'(t) \le \beta_{T_o}\|\sigma(t)\|\,|\sigma'(t)| + $$
$$+ \|B(\|u(t)\|^2)-B(\|v(t)\|^2)\|_{\mathcal{L}(H)}|AS(v)|\,|\sigma'(t)| +$$
$$+ \|u'(t)\|\,|Au(t)|\,\|B'(\|u(t)\|^2)\|_{\mathcal{L}(H)}\|\sigma(t)\|^2.$$

If $\|u\|_{X_{T_o}} \le r$ and $\|v\|_{X_{T_o}} \le r$, then by (2.3), (2.5) and Lemma 2 above we obtain that

$$y'(t) \le \left(\frac{2}{\sqrt{m_o}}\beta_{T_o} + \frac{2}{m_o}\gamma_c r^2\right)y(t) + 2\alpha_{T_o} r^2\|u-v\|_{Y_{T_o}}\|\sigma\|_{Y_{T_o}}. \tag{3.5}$$

Let us consider $\Gamma = \frac{1}{\sqrt{m_o}}\beta_{T_o} + \frac{1}{m_o}\gamma_c r^2$. Then, since that $y(0) = 0$, we obtain by (3.5) that

$$y(t) \le \frac{\alpha_{T_o} r^2}{\Gamma}(e^{2\Gamma t} - 1)\,\|u-v\|_{Y_{T_o}}\|\sigma\|_{Y_{T_o}}, \qquad 0 \le t \le T_o. \tag{3.6}$$

Now, we choose $T_1 < \min\{T_o, \frac{1}{2\Gamma}\log\left(\frac{\min(1,m_o)\Gamma}{4\,\alpha_{T_o}^2 r^2} + 1\right)\}$.

If we repeat the proof for $0 \le t \le T_1 \le T_o$ follows, by (3.6), that:

$$(|\sigma'(t)| + \|\sigma(t)\|)^2 \le \theta\|u-v\|_{Y_{T_1}}\|\sigma\|_{Y_{T_1}}, \qquad 0 \le t \le T_1 \tag{3.7}$$

where $\theta = \frac{\alpha_{T_o} r^2}{\Gamma}(e^{2\Gamma T_1} - 1)\frac{4}{\min(1,m_o)}$.

THEOREM 1. Given $u_o \in D(A^{3/2})$ and $u_1 \in D(A)$. Then there exists $T_1 > 0$ and a unique function $u: [0,T_1] \to D(A)$ such that:

$$u \in C^2_\omega([0,T_1];H) \cap C^1([0,T_1];H) \cap C([0,T_1];V) \tag{3.8}$$

$$u'(t) \in V, \qquad 0 \le t \le T_1 \tag{3.9}$$

$$\frac{d}{dt}(u'(t)|v) + (B(\|u(t)\|^2)Au(t)|v) = 0, \qquad \forall\, v \in H \tag{3.10}$$

$$u(0) = u_o \qquad u'(0) = u_1 \tag{3.11}$$

Moreover, there is $r = r(c) > 0$ such that $\|u\|_{X_{T_1}} \le r$.

PROOF: Let T_1 be defined in Lemma 3.

We define $u \equiv 0$, $u \in E_{T_1,C}$ and consider $u_{\nu+1} = S(u_\nu)$, $\nu = 0,1,2,\ldots$ where $u_o = u = 0$.

We note that $u_\nu \in X_{T_1} \subset Y_{T_1}$, $\forall\, \nu$. Furthermore, by Lemma 1 (§3), we have that $\|u_\nu\|_{X_{T_1}} \le r$ $\forall\, \nu$. Thus by Lemma 3 we obtain that $\|u_{\nu+1}-u_\nu\|_{Y_{T_1}} \le \theta^\nu \|u_1\|_{X_{T_1}}$. Therefore for $\nu \ge \mu$

$$\|u_\nu - u_\mu\| \le \frac{\theta^\mu}{1-\theta}\|u_1\|_{Y_{T_1}}.$$

Which implies that there is $u \in Y_{T_1}$ such that:

$$\lim_{\nu\to+\infty} u_\nu = u \quad \text{in} \quad Y_{T_1} \tag{3.12}$$

By Lemma 2, (3.12) and (2.5), we obtain that:

$$\lim_{\nu\to+\infty} B(\|u_\nu(t)\|^2) = B(\|u(t)\|^2) \quad \text{in} \quad \mathcal{L}(V) \tag{3.13}$$

uniformly for $0 \le t \le T_1$.

By (3.12) and Lemma 2 we conclude that:

$$u(t) \in D(A) \quad \forall\, t \in [0,T_2] \quad \text{and} \quad |Au(t)| \le r, \quad t \in [0,T_1].$$

Moreover, for each v in V, $\lim_{\nu\to+\infty}(Au_{\nu+1}(t)|v) = (Au(t)|v)$ uniformly in $[0,T_1]$ hence

$$\lim_{\nu\to+\infty} (Au_{\nu+1}(t)|v) = (Au(t)|v) \text{ uniformly in } [0,T_1] \quad \forall\, v \in H, \tag{3.14}$$

because V is dense in H.

Now, we have that

$$(u''_{\nu+1}(t)|v) = -(B(\|u_\nu(t)\|^2)Au_{\nu+1}(t)|v), \qquad \forall\, v \in H$$

consequently, by using (3.13) and (3.14) we obtain, that

$$\lim_{\nu\to+\infty} (u''_{\nu+1}(t)|v) = -(B(\|u(t)\|^2)Au(t)|v), \quad \forall\, v \in H \text{ uniformly in } [0,T_1] \tag{3.15}$$

Therefore, by (3.12), then $u \in C^2_\omega([0,T_1];H)$ and

$$\frac{d}{dt}(u'(t)|v) = -(B(\|u(t)\|^2)Au(t)|v), \quad t \in [0,T_1] \quad \forall\, v \in H.$$

By Lemma 2 and (3.12) we obtain that $u'(t) \in V$ $t \in [0,T_2]$ and $\|u'(t)\| \le r$, $t \in [0,T_2]$.

It remains to prove uniqueness. We consider u and v satisfying the Theorem 1. We note that $|Au(t)| + \|u(t)\| \le r$ $\forall\, t \in [0,T_1]$

and $|Av(t)| + \|v(t)\| \le r \quad \forall\, t \in [0,T_1]$ then if we consider $\sigma(t) = u(t)-v(t)$ and using a similar proof of Lemma 3 we obtain that $\|\sigma\|_{Y_{T_1}} \le \theta\|\sigma\|_{Y_{T_1}} < \|\sigma\|_{Y_{T_1}}$ and therefore $\sigma(t) = 0 \quad \forall\, t, \quad t \in [0,T_1]$.

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