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# SOME LIMIT THEOREMS FOR RATIOS OF TRANSITION PROBABILITIES

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**ABSTRACT.** In this paper, infinite dimensional forward convergent stochastic chains have been considered in the framework of [1]. The main result of this paper deals with the observation that the total flow of probability from the C-states to the T-states is very small compared to that from the T-states to the C-states, if the chain is observed for a sufficiently long time. Some examples have been given to justify the assumptions involved.

KEY WORDS AND PHRASES. Infinite stochastic matrices, convergence, basis, C-states, T-States.

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## 1. INTRODUCTION.

Let $(P_n)$ be a sequence of finite or countably infinite stochastic matrices. Then $(P_n)$ is called a convergent chain iff for each $k \geq 1$, $\lim_{n\to\infty} P_{k,n} = Q_k$ exists and is a stochastic matrix. Here $P_{k,n} = P_{k+1} P_{k+2} \cdots P_n$ for $k < n$. It has been proved in [1] that for such a chain there exists a unique partition $\{T, C_1, C_2, \ldots\}$ of the state space $S$ such that for any limit point $Q$ of the $Q_k$'s the following are true:

$$\begin{aligned} Q_{ij} &= 0, \text{ for all } i \text{ if } j \varepsilon t \\ &= 0, \text{ if } i \text{ and } j \text{ belong to different C-classes} \\ &= Q_{kj}, \text{ if } i,j,k \text{ belong to the same C-classes.} \end{aligned}$$

One of the problems, here, is to identify the T-states and the C-states for infinite convergent chains. Also, the following theorem is known (see [2]) for finite chains.

**THEOREM 1.1.** Let $(P_n)$ be a finite convergent chain with basis $\{T, C_1, C_2, \ldots, C_p\}$. Let L T. Then for each positive integer $k \geq 1$,

$$\lim_{m \to \infty} \frac{\sum_{n=k}^{m} (P_n)_{TC_L}}{\sum_{n=k}^{m} (P_n)LL^c} = 0$$

where $(p_n)_{AB} = \sum_{\substack{i \varepsilon A \\ j \varepsilon B}} (p_n)_{ij}$.

The above theorem and some other results have been extended to the infinite-dimensional case with some modifications. Some examples have been discussed to justify the assumptions involved.

## 2. PRELIMINARIES.

**CONDITION (U).** A convergent, forward stochastic chain $(P_n)$ is said to satisy condition (U) if for each j in each C-class in the basis,

$$\lim_{k\to\infty} \operatorname{Sup}_{i\in T} \left|(Q_k)_{ij} - Q_{ij}\right| = 0$$

This condition yielded a number of finite space like results for infinite chains in [1]. However, the next few examples demonstrate that Theorem 1.1 fails to go over directly to the infinite state space, even for infinite chains which satisfy condition (U).

**EXAMPLES.** (a) For a convergent, infinite chain with basis $\{T, C_1, C_2, \ldots\}$ and any L T the ratio

$$\frac{\sum_{q=k}^{m} (P_q)_{T^c L}}{\sum_{q=k}^{m} (P_q)_{LL^c}}$$

for $k \geq 1$, may not go to zero as $m \to \infty$. For, if

$$P_n = \begin{bmatrix} 0 & 1/2 & 1/2^2 \ldots\ldots \\ 0 & 1/2 & 1/2^2 \ldots\ldots \\ \ldots & \ldots & \ldots \\ 0 & 1/2 & 1/2^2 \ldots\ldots \\ 1 & 0 & 0 \quad \ldots\ldots \\ 1 & 0 & 0 \quad \ldots\ldots \end{bmatrix} \left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\} \text{first } n \text{ rows}$$

then it has been shown in [1], that for each k,

$$\lim_{n\to\infty} P_{k,n} = \begin{bmatrix} 0. & 1/2 & 1/2^2 \ldots\ldots \\ 0 & 1/2 & 1/2^2 \ldots\ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

The basis of this chain is of course $\{T = \{1\}, C = \{2,3,\ldots\}\}$. But for all $k < m$,

$$\frac{\sum_{n=k}^{m} (P_n)_{T^c,\{1\}}}{\sum_{n=k}^{m} (P_n)_{\{1\},T^c}} = \infty$$

It can be seen that this chain satifies a condition stronger than (U) because $||Q_k - Q|| = 0$ for all k.

(b) This example is given to show that the previous limit may be finite but not zero. If $a_k = 2^{4(k-1)}$, $k \geq 1$ with $a_k \leq n < a_{k+1}$, let

$$(P_{3n-2})_{ij} = \begin{cases} 1/2^j & \text{if } i = 0,\ j \geq 1 \\ 1-1/2^k & \text{if } 1 \leq i = j \leq k \\ 1 & \text{if } i = j \geq k+1 \\ 1/2^k & \text{if } j = n \text{ and } 1 \leq i \leq k \end{cases}$$

$$(P_{3n-1})_{ij} = \begin{cases} 1 & \text{if } 0 \leq i = j \leq k \\ 1 & \text{if } j = n,\ i > k \end{cases}$$

$$(P_{3n})_{ij} = \begin{cases} 1 & \text{if } i = n,\ j = 0 \\ 1 & \text{if } i = j \neq n \end{cases}$$

Here, the state space is $\{0,1,2,\ldots\}$. It has been shown in [1] that

$$\lim_{n \to \infty} P_{k,n} = \begin{bmatrix} 0 & 1/2 & 1/2^2 \ldots\ldots \\ 0 & 1/2 & 1/2^2 \ldots\ldots \\ \ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots & & \end{bmatrix}$$

for all k and $||Q_k - Q|| = 0$. Here, $T = \{0\}$, $C = \{1,2,\ldots\}$. But

$$\frac{\sum_{n=3k+1}^{3m} (P_n)_{T^c,\{0\}}}{\sum_{n=3k+1}^{3m} (P_n)_{\{0\},T^c}} = \frac{m-k}{m-k} = 1, \text{ for } k < m$$

so that

$$\lim_{m \to \infty} \frac{\sum_{n=k}^{m} (P_n)_{T^c,\{0\}}}{\sum_{n=k}^{m} (P_n)_{\{0\},T^c}} = 1, \text{ for all } k$$

so that the limiting value of the above ratio is not zero.

## 3. MAIN RESULTS.

**LEMMA 3.1.** Let $(P_n)$ be any chain (finite or infinite). Also, let

$$T = \{j \mid \lim_{n\to\infty} (p_{m,n})_{ij} = 0, \text{ for all } m \text{ and for all } i\}$$

and assume that $Q_k = \lim_{l\to\infty} P_{k,l}$ are stochastic matrices. Then we have

$$\sum_n (P_n)_{LL^c} = \infty, \text{ where } (P_n)_{LL^c} = \sum_{\substack{i\in L\\ j\in L^c}} (P_n)_{ij} \text{ and } L \quad T.$$

**PROOF.** Suppose that the lemma is false. Then there exists $L \quad T$ such that $\sum_n (P_n)_{LL^c} < \infty$. Then for given $\varepsilon > 0$, there exists $k(\varepsilon)$ such that for all $k \geq k(\varepsilon)$,

$$\sum_{n \geq k} (P_n)_{LL^c} < \varepsilon.$$

For any $t\in L$, $(P_{k,n})_{tL^c} = (P_{k+1} P_{k+1,n})_{tL^c}$

$$= \sum_{i\in S} (P_{k+1})_{ti} (P_{k+1,n})_{iL^c}$$

$$= \sum_{i\in L} (P_{k+1})_{ti}(P_{k+1,n})_{iL^c} + \sum_{i\in L^c} (P_{k+1})_{ti}(P_{k+1,n})_{iL^c}$$

$$\leq \max_{i\in L} (P_{k+1,n})_{iL^c} + (P_{k+1})_{LL^c}.$$

Therefore, $\max_{t\in L} (P_{k,n})_{tL^c} \leq \max_{t\in L} (P_{k+1,n})_{tL^c} + (P_{k+1})_{LL^c}$.

Repeating the procedure,

$$\max_{t\in L} (P_{k,n})_{tL^c} \leq \sum_{j=1}^{n} (P_{k+j})_{LL^c}$$

Therefore, for $k \geq k(\varepsilon)$ and $t\in L$,

$$\sum_{j\in L^c} (Q_k)_{tj} < \varepsilon$$

which is a contradiction. Hence the supposition is false and the result follows.

**REMARK 3.2.** If $\pi_{k_0}$ is an infinite-dimensional initial probability row-vector and $(P_n)$ a sequence of infinite stochastic matrices, then let $\pi_m = \pi_{k_0} P_{k_0,m}$. If $\pi_{k_0}$ has all positive entries, then for each $j\in T$, choosing $k_0$ sufficiently large it follows that for a convergent chain, $\lim_{n\to\infty} \pi_n(j) > 0$.
Also, for $m < n$ and $i\in S$,

$$\pi_{n+1}(i) - \pi_n(i) = \sum_{j\neq i} \pi_n(j) (P_{n+1})_{ji} - \pi_n(i) [1 - (P_{n+1})_{ii}] \quad (3.1)$$

$$\pi_n(i) - \pi_m(i) = \sum_{j\neq i} \sum_{k=m}^{n-1} \pi_k(j) (P_{k+1})_{ji} - \sum_{k=m}^{n-1} \pi_k(i) [1 - (P_{k+1})_{ii}] \quad (3.2)$$

The above identities are use to get the following results:,

**THEOREM 3.3.** Let $(P_n)$ be a convergent chain with basis $\{T, C_1, C_2 \ldots\}$.

(A) Suppose that $t \varepsilon T$. Then the following hold:

(i) $$\sum_{n=1}^{\infty} [1 - (P_n)_{tt}] = \infty$$

(ii) For $j \varepsilon T$, $$\lim_{n\to\infty} \frac{(P_n)_{jt}}{1-(P_n)_{tt}} = 0$$

(B) Suppose that there is a state $t \varepsilon S$ such that

(i) $$\sum_{n=1}^{\infty} [1 - (P_n)_{tt}] = \infty$$

(ii) $$\lim_{n\to\infty} \frac{(P_n)_{jt}}{1 - (P_n)_{tt}} = 0,$$ uniformly for $j \neq t$.

Then $t \varepsilon T$.

(C) Suppose that there is only one T-state such that

$$\lim_{n\to\infty} \frac{(P_n)_{jt}}{1 - (P_n)_{tt}} = A_j \quad (0 \leq A_j < \infty) \quad \text{for each } j \neq t.$$

Then $t \varepsilon T$ iff $\sum_{n=1}^{\infty} [1 - P_n)_{tt}] = \infty$ and $A_j = 0$ for each $j \neq t$.

**PROOF.** (A) Part (i) follows since $(P_{k,n})_{tt} \geq (P_{k+1})_{tt} (P_{k+2})_{tt} \cdots\cdots\cdots (P_n)_{tt}$

So that $0 = (Q_k)_{tt} \geq \prod_{m=k+1}^{\infty} (P_{k+m})_{tt}$

which means that $\sum_{n=1}^{\infty} [1 - (P_n)_{tt}] = \infty$.

For part (ii), suppose that it is false for some $j_0 \varepsilon T$. Then there exists $\beta > 0$ and a positive integer $N$ such that for all $n \geq N$,

$$(P_n)_{j_0 t} \geq \beta [1 - (P_n)_{tt}].$$

Then from equation (3.2) for $n > m > N$, we have

$$\pi_n(t) - \pi_m(t) \geq \sum_{k=m}^{n=1} [\beta\, \pi_k(j_0) - \pi_k(t)][(1 - P_{k+1})_{tt}]$$

which is impossible because of part (i) and because the left side goes to zero as $m,n \to \infty$ but $\lim_{k\to\infty} \pi_k(j_0) > 0$ and $\lim_{k\to\infty} \pi_k(t) = 0$. Therefore the supposition is false and part (ii) is proved

(B) Suppose $t \in T$. Then there exists $n_0$ such that $n \geq n_0$ implies

$$\pi_n(t) > 2\beta > 0.$$

By (1),

$$2\beta\,[1 - (P_{n+1})_{tt}]$$

$$\leq \sum_{j\neq t} \pi_n(j)(P_{n+1})_{jt} + \pi_n(t) - \pi_{n+1}(t)$$

$$\leq \max_{j\neq t} (P_{n+1})_{jt} + \pi_n(t) - \pi_{n+1}(t).$$

By condition (ii), there exists $N > n_0$ such that $n \geq N$ implies

$$\max_{j\neq t} (P_{n+1})_{jt} < \beta\,[1 - (P_n)_{tt}]$$

then we have from above, for $n > N$,

$$\beta\,[1 - (P_{n+1})_{tt}] \leq \pi_n(t) - \pi_{n+1}(t)$$

which means that $\beta \sum_{n=N+1}^{m} [1 - (P_{n+1})_{tt}] \leq \pi_{N+1}(t) - \pi_{m+1}(t) < 2$ which contradicts (i).

(C) This part follows immediately from parts (A) and (B).

**THEOREM 3.4.** Let $(P_n)$ be a convergent infinite chain with $\{T, C_1, C_2, \ldots\}$ as its basis. Then for all $k \geq 1$,

$$\lim_{m\to\infty} \frac{\sum_{n=k}^{m} (P_n)_{BA}}{\sum_{n=k}^{m} (P_n)_{AA^c}} = 0$$

where A (finite set) T, B (finite set) $T^c$. Moreover, if

$$\underline{\lim}_{n\to\infty} [\inf_{t\in T} \sum_{i\notin C_s} (Q_n)_{ti}] > 0$$

then the above result holds for any $A \subset T$ and finite $B \subset C_s$.

**PROOF.** If $\pi_n(A) = \sum_{t\in A} \pi_n(t)$ and $m < n$, then

$$\pi_n(A) - \pi_m(A) = \sum_{t\in A} \sum_{j\neq t} \sum_{q=m}^{n-1} \pi_q(j))P_{q+1})_{jt}$$

$$- \sum_{t\in A} \sum_{q=m}^{n-1} \pi_q(j) \sum_{j\neq t} (P_{q+1})_{tj}$$

$$\geq \sum_{q=m}^{n-1} \sum_{j\in B} \pi_q(j)(P_{q+1})_{jA} - \sum_{q=m}^{n-1} \sum_{t\in A} \pi_q(t)(P_{q+1})_{tA^c}$$

Now, for all $t\in A$ (finite) and $j\in B$ (finite) and $q$ sufficiently large,

$$\pi_q(t) < \varepsilon\varepsilon_1 \quad \text{and} \quad \pi_q(j) > \varepsilon_1.$$

Therefore, for $m$ sufficiently large and $m < n$,

$$\pi_n(A) - \pi_m(A) > \varepsilon_1 \sum_{q=m}^{n-1} (P_{q+1})_{BA} - \varepsilon\varepsilon_1 \sum_{q=m}^{n-1} (P_{q+1})_{AA^c} \tag{3.3}$$

Let $(m_r)$ by any given sequence of positive integers. Then there exists $(p_r)$ $(m_r)$ such that for each $r \geq k$, there exists $\beta > 0$ such that

$$\sum_{q=k}^{r} (P_q)_{BA} \leq \varepsilon/2 \sum_{q=k}^{p_r} (P_q)_{AA^c} \tag{3.4}$$

since
$$\sum_q (P_q)_{AA^c} = \infty \quad \text{and} \quad \sum_{q=r}^{p_r} (P_q)_{AA^c} > \beta > 0. \tag{3.5}$$

But $\lim_{r\to\infty} \dfrac{\sum_{q=r}^{p_r} (P_q)_{BA}}{\sum_{q=r}^{p_r} (P_q)_{AA^c}} = 0$ from equation (3.3). Hence from (3.4) and (3.5),

for sufficiently large $r$, we have

$$\sum_{q=k}^{p_r} (P_q)_{BA} < \varepsilon \sum_{q=k}^{p_r} (P_q)_{AA^c}$$

and the first part of the theorem follows.

We refer the reader to a result obtained in [2] for the proof of the second part. It has been proved in [2] that for all $i \in C_s$,

$$\sum_{n=1}^{\infty} \sum_{t\in A} (P_n)_{it} < \infty.$$

Using the above result, the proof of the second part easily follows.

**EXAMPLE 3.5.** We now give an example to show that in Theorem 3.4, the condition that the flow in the denominator is from $A$ to $A^c$, is necessary.

Let $P_n = \begin{bmatrix} 0 & 1/n & 1-1/n & 0 & \cdots\cdots \\ 1/n & 1-1/n & 0 & 0 & \cdots\cdots \\ 0 & 1 & 0 & 0 & \cdots\cdots \\ 0 & 1 & 0 & 0 & \cdots\cdots \end{bmatrix}$

By Bernstein's theorem, the products are weakly ergodic. In fact,

$$(P_{k,n+1})_{i1} = \sum_s (P_{k,n})_{is}(P_{n+1})_{s1}$$

$$= \frac{1}{n+1}(P_{k,n})_{i2} \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for al\_ } i$$

$$(P_{k,n+1})_{i2} = \sum_s (P_{k,n})_{is}(P_{n+1})_{s2}$$

$$= 1 - (P_{k,n})_{i1} + \frac{1}{n+1}(P_{k,n})_{i1} - \frac{1}{n+1}(P_{k.n})_{i2}$$

$$\to 1 \quad \text{as} \quad n \to \infty.$$

So, the chain is strongly ergodic with $T^c = \{2\}$. But if $A = \{1\}$, $B = \{2\}$

then $\dfrac{\sum_{n=k}^{m} (P_n)_{21}}{\sum_{n=k}^{m} (P_n)_{12}} = 1$, for all $k \leq m$.

Hence, the assertion is true.

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