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*Research Article*

# Finite 1-Regular Cayley Graphs of Valency 5

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Let $\Gamma = \mathrm{Cay}(G, S)$ and $G \leq X \leq \mathrm{Aut}\Gamma$. We say $\Gamma$ is $(X, 1)$-regular Cayley graph if $X$ acts regularly on its arcs. $\Gamma$ is said to be core-free if $G$ is core-free in some $X \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$. In this paper, we prove that if an $(X, 1)$-regular Cayley graph of valency 5 is not normal or binormal, then it is the normal cover of one of two core-free ones up to isomorphism. In particular, there are no core-free 1-regular Cayley graphs of valency 5.

## 1. Introduction

We assume that all graphs in this paper are finite, simple, and undirected.

Let $\Gamma$ be a graph. Denote the vertex set, arc set, and full automorphism group of $\Gamma$ by $V\Gamma$, $A\Gamma$, and $\mathrm{Aut}\Gamma$, respectively. A graph $\Gamma$ is called *$X$-vertex-transitive* or *$X$-arc-transitive* if $X$ acts transitively on $V\Gamma$ or $A\Gamma$, where $X \leq \mathrm{Aut}\Gamma$. $\Gamma$ is simply called *vertex-transitive*, *arc-transitive* for the case where $X = \mathrm{Aut}\Gamma$. In particular, $\Gamma$ is called *$(X, 1)$-regular* if $X \leq \mathrm{Aut}\Gamma$ acts regularly on its arcs and then *1-regular* when $X = \mathrm{Aut}\Gamma$.

Let $G$ be a finite group with identity element 1. For a subset $S$ of $G$ with $1 \notin S = S^{-1} := \{x^{-1} \mid x \in S\}$, the Cayley graph $\mathrm{Cay}(G, S)$ of $G$ (with respect to $S$) is defined as the graph with vertex set $G$ such that $x, y \in G$ are adjacent if and only if $yx^{-1} \in S$. It is easy to see that a Cayley graph $\mathrm{Cay}(G, S)$ has valency $|S|$, and it is connected if and only if $\langle S \rangle = G$.

Li proved in [1] that there are only finite number of core-free $s$-transitive Cayley graphs of valency $k$ for $s \in \{2, 3, 4, 5, 7\}$ and $k \geq 3$ and that, with the exceptions $s = 2$ and $(s, k) = (3, 7)$, every $s$-transitive Cayley graph is a normal cover of a core-free one. It was proved in [2] that there are 15 core-free $s$-transitive cubic Cayley graphs up to isomorphism, and there are no core-free 1-regular cubic Cayley graphs. A natural problem arises. Characterize 1-transitive Cayley graphs, in particular, which graphs are 1-regular? Until now, the result about 1-regular graphs mainly focused constructing examples. For example, Frucht gave the first example of cubic 1-regular graph in [3]. After then, Conder and Praeger constructed two infinite families of cubic 1-regular graphs in [4]. Marušič [5] and Malnič et al. [6] constructed two infinite families of tetravalent 1-regular graphs. Classifying such graphs has aroused great interest. Motivated by above results and problem, we consider 1-regular Cayley graphs of valency 5 in this paper.

A graph $\Gamma$ can be viewed as a Cayley graph of a group $G$ if and only if $\mathrm{Aut}\Gamma$ contains a subgroup that is isomorphic to $G$ and acts regularly on the vertex set. For convenience, we denote this regular subgroup still by $G$. If $X \leq \mathrm{Aut}\Gamma$ contains a normal subgroup that is regular and isomorphic to $G$, then $\Gamma$ is called an *$X$-normal Cayley graph* of $G$; if $G$ is not normal in $X$ but has a subgroup which is normal in $X$ and semiregular on $V\Gamma$ with exactly two orbits, then $\Gamma$ is called an *$X$-bi-normal Cayley graph*; furthermore if $X = \mathrm{Aut}\Gamma$, $\Gamma$ is called *normal* or *bi-normal*. Some characterization of normal and bi-normal Cayley graphs has given in [1, 2].

For a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, $\Gamma$ is said to be *core-free* (with respect to $G$) if $G$ is core-free in some $X \leq \mathrm{Aut}\Gamma$; that is, $\mathrm{Core}_X(G) = \cap_{x \in X} G^x = 1$.

The main result of this paper is the following assertion.

**Theorem 1.** *Let $\Gamma = \mathrm{Cay}(G, S)$ be an $(X, 1)$-regular Cayley graph of valency 5, where $G \leq X \leq \mathrm{Aut}\Gamma$. Let $n(G)$ be the number of nonisomorphic core-free $(X, 1)$-regular Cayley*

Table 1

| Number | $X$ | $G$ | $n(G)$ | 1-regular | Remark |
|---|---|---|---|---|---|
| 1 | $A_5$ | $A_4$ | 1 | No | Icosahedron |
| 2 | $S_5$ | $S_4$ | 1 | No | |

*graph of valency 5 with the regular subgroup equal to G. Then either*

(i) $\Gamma$ *is an X-normal or X-bi-normal Cayley graph or*

(ii) $\Gamma$ *is a nontrivial normal cover of one line of Table 1.*

*In particular, there are no core-free 1-regular Cayley graphs of valency 5.*

By Theorem 1, we can get the following remark immediately.

*Remark 2.* Let $\Gamma = \mathrm{Cay}(G, S)$ be an 1-regular Cayley graph of valency 5. Then $\Gamma$ is normal or bi-normal.

## 2. Examples

In this section we give some examples of graphs appearing in Theorem 1.

*Example 3.* Let $M = \langle a\rangle \cong \mathbb{Z}_{11}$ be a cyclic group. Assume that $\tau \in \mathrm{Aut}(M)$ is of order 10 and $X = M : \langle\tau\rangle \cong \mathbb{Z}_{11} : \mathbb{Z}_{10}$. Let

$$G = M : \left\langle \tau^5 \right\rangle \cong D_{22}. \quad (1)$$

Suppose that

$$S = g^{\langle\tau^2\rangle} = \left\{g, g^{\tau^2}, g^{\tau^4}, g^{\tau^6}, g^{\tau^8}\right\}, \quad (2)$$

where $g \in G$ is an involution such that $g \neq \tau^5$. Let $\Gamma = \mathrm{Cay}(G, S)$ be the Cayley graph of the dihedral group $G$ with respect to $S$. Then $\Gamma$ is a connected $(X, 1)$-regular Cayley graph of valency 5. In particular, $\Gamma$ is $X$-normal.

*Proof.* Let

$$G = \langle a\rangle : \langle b\rangle = \left\{1, a, a^2, \ldots, a^{10}, b, ab, a^2b, \ldots, a^{10}b\right\} \cong D_{22}, \quad (3)$$

where $b = \tau^5$.

Noting $o(a) = 11$, we may assume that $a^\tau = a^9$. Since the involution $g \in G$ is not equal to $b$, we may let $g = a^ib$ for some $1 \le i < 11$ such that $(9, i) = 1$. Then $g^{\tau^2} = (a^ib)^{\tau^2} = a^{81i}b = a^{4i}b$, and so $g^{\tau^2}g^{-1} = a^{3i} \in \langle g^{\langle\tau^2\rangle}\rangle = \langle S\rangle$. Thus the element $g^{\tau^2}g^{-1}$ is of order 11 as $(3i, 11) = 1$. So $\langle S\rangle = \langle g^{\langle\tau^2\rangle}\rangle = G$; that is, $\Gamma = \mathrm{Cay}(G, S)$ is connected.

Obviously, $G \lhd X \le \mathrm{Aut}\Gamma$ and $X_1 = \langle\tau^2\rangle$. However, $|X| = 55 = |A\Gamma|$; then $\Gamma$ is an $(X, 1)$-regular normal Cayley graph of $G$ of valency 5. □

*Example 4.* Let $G = \langle a, b \mid a^5 = b^2 = 1, a^b = a^{-1}\rangle \cong D_{10}$. Set $S = \{b, ab, a^2b, a^3b, a^4b\}$ and $\Gamma = \mathrm{Cay}(G, S)$. Then $\Gamma \cong K_{5,5}$ and $\mathrm{Aut}\Gamma = S_5 \wr S_2$. Let $X = (\mathbb{Z}_5 \times \mathbb{Z}_5) : \mathbb{Z}_2 \ncong D_{10} \times \mathbb{Z}_5$ such that $G \le X \le \mathrm{Aut}\Gamma$. It follows that $\mathrm{Core}_X(G) \cong \mathbb{Z}_5$. Then $X_\alpha \cong \mathbb{Z}_5$ for $\alpha \in V\Gamma$, and furthermore $\Gamma$ is $(X, 1)$-regular. Obviously $G$ is not normal in $X$. However, $\mathrm{Core}_X(G) \trianglelefteq X$ is semiregular and has exactly two orbits on $V\Gamma$; then $\Gamma$ is an $(X, 1)$-regular Cayley graph of valency 5. In particular, $\Gamma$ is $X$-bi-normal.

## 3. The Proof of Main Results

In this section, we will prove our main results. We first present some properties about normal Cayley graphs.

For a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, we have a subgroup of $\mathrm{Aut}(G)$:

$$\mathrm{Aut}(G, S) = \left\{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\right\}. \quad (4)$$

Clearly it is a subgroup of the stabilizer in $\mathrm{Aut}\Gamma$ of the vertex corresponding to the identity 1 of $G$. Since $\Gamma$ is connected, $\mathrm{Aut}(G, S)$ acts faithfully on $S$. By Godsil [7, Lemma 2.1], the normalizer $N_{\mathrm{Aut}\Gamma}(G) = G : \mathrm{Aut}(G, S)$. So $\Gamma = \mathrm{Cay}(G, S)$ is a normal Cayley graph if and only if $\mathrm{Aut}(G, S) = (\mathrm{Aut}\Gamma)_1$.

Let $\Gamma = \mathrm{Cay}(G, S)$ be an $(X, 1)$-regular Cayley graph of valency 5 such that $G \le X \le \mathrm{Aut}\Gamma$. Then $S$ contains at least one involution. Let $K = \mathrm{Core}_X(G)$, which is the core of $G$ in $X$.

**Lemma 5.** *Assume that* $K = 1$. *Then* $(X, G) = (A_5, A_4)$ *or* $(S_5, S_4)$.

*Proof.* Let $H$ be the stabilizer in $X$ of the vertex corresponding to the identity of $G$. Then $H \cong \mathbb{Z}_5$, $H \cap G = 1$, and $X = GH$. Let $[X : G]$ be the set of right cosets of $G$ in $X$. Consider the action of $X$ on $[X : G]$ by the right multiplication. Then we get that $X$ is a primitive permutation group of degree 5 and $G$ is a stabilizer of $X$. Since $\Gamma$ has valency 5, $|G| = |V\Gamma| \ge 6$, and so $|X| = |G||H| \ge 30$. Then we can show $X \cong A_5$ or $S_5$, and then $G = A_4$ or $S_4$, respectively. □

**Lemma 6.** *Suppose that* $G = A_4$ *and* $X = A_5$. *Then* $\Gamma$ *is the icosahedron graph. Moreover,* $\mathrm{Aut}\Gamma = A_5 \times \mathbb{Z}_2$ *and* $\Gamma$ *is not 1-regular.*

*Proof.* Note that $X = GH$, where $X \cong A_5$, $G \cong A_4$, and $H \cong \mathbb{Z}_5$. Since $X$ has no nontrivial normal subgroup, $\Gamma$ is not bipartite. So $\Gamma$ is the icosahedron graph. Further by Magma [8], $\mathrm{Aut}\Gamma = A_5 \times \mathbb{Z}_2$, so $\Gamma$ is not 1-regular. □

**Lemma 7.** *Suppose that* $G = S_4$ *and* $X = S_5$. *Then the graph* $\Gamma$ *is not 1-regular and there is only one isomorphism class of these graphs.*

*Proof.* Note that $G = S_4$, $X = S_5$, and $X = GH$. Let $H = \langle\sigma\rangle$, where $\sigma = (1\ 2\ 3\ 4\ 5)$. By considering the right multiplication action of $X$ on the right cosets of $G$ in $X$, $G$ can be viewed as a stabilizer of $X$ acting on $\{1, 2, 3, 4, 5\}$. Without lost generality, we may assume that 1 is fixed by $G$. Take an involution $\tau \in S$. Then, by [2], $\tau \in S_5 \setminus N_{S_5}(H)$ and we can identify $S$ with $H\tau H \cap G$. Note that $\tau \in G \le S_4$

and $N_{S_5}(H) = H : \mathrm{Aut}(H) = \langle(1\ 2\ 3\ 4\ 5)\rangle : \langle(2\ 3\ 5\ 4)\rangle \cong \mathbb{Z}_5 : \mathbb{Z}_4$; then $\tau$ is one of the following: $(2\ 5)$, $(3\ 5)$, $(2\ 3)$, $(3\ 4)$, $(4\ 5)$, $(2\ 4)$, $(2\ 3)(4\ 5)$, and $(2\ 4)(3\ 5)$. Note $H = \langle(1\ 2\ 3\ 4\ 5)\rangle$. Assume that $\tau = (2\ 5)$; by calculation, we have $(2\ 5) := h_1$, $\tau \cdot (1\ 2\ 3\ 4\ 5) = (1\ 2)(3\ 4\ 5) := h_2$, $\tau \cdot (1\ 3\ 5\ 2\ 4) = (1\ 3\ 5\ 4) := h_3$, $\tau \cdot (1\ 4\ 2\ 5\ 3) = (1\ 4\ 2\ 3) := h_4$, and $\tau \cdot (1\ 5\ 4\ 3\ 2) = (1\ 5)(2\ 4\ 3) := h_5$. Then $H(2\ 5)H = \{Hh_1, Hh_2, Hh_3, Hh_4, Hh_5\} = \{(2\ 5), (1\ 5)(2\ 3\ 4), (1\ 4\ 5\ 3), (1\ 2)(3\ 5\ 4), (1\ 3\ 2\ 4), (1\ 5)(2\ 4\ 3), (1\ 4\ 2\ 3), (1\ 3\ 5\ 4), (1\ 2)(3\ 4\ 5), (2\ 5\ 3\ 4), (1\ 5\ 2\ 4), (1\ 4\ 5)(2\ 3), (1\ 3), (1\ 3\ 5\ 2), (1\ 2\ 5)(3\ 4), (2\ 4), (1\ 5\ 4)(2\ 3), (1\ 2\ 3)(4\ 5), (3\ 5), (1\ 5\ 2)(3\ 4), (1\ 4\ 2\ 5), (1\ 4), (1\ 3\ 2)(4\ 5), (1\ 2\ 5\ 3), (2\ 4\ 3\ 5)\}$. Thus the corresponding $S$ is $\{(2\ 5), (2\ 5\ 3\ 4), (2\ 4), (3\ 5), (2\ 4\ 3\ 5)\}$ since $1^s = 1$ for each $s \in H(2\ 5)H$. By similar argument, for every $\tau$, we can work out $S$ explicitly, which is one of the following four cases: $S_1 = \{(2\ 5), (2\ 5\ 3\ 4), (2\ 4), (3\ 5), (2\ 4\ 3\ 5)\}$, $S_2 = \{(2\ 3), (3\ 4), (4\ 5), (2\ 3\ 4\ 5), (2\ 5\ 4\ 3)\}$, $S_3 = \{(2\ 3)(4\ 5), (2\ 3\ 5), (2\ 5\ 3), (2\ 4\ 5), (2\ 5\ 4)\}$, and $S_4 = \{(2\ 4)(3\ 5), (2\ 4\ 3), (3\ 5\ 4), (2\ 3\ 4), (3\ 4\ 5)\}$.

Now let $A = \mathrm{Aut}\Gamma$. We declare that $X \neq A$. Assume that $X = A$. Note that $G = N_A(G) = G\mathrm{Aut}(G,S)$; then $\mathrm{Aut}(G,S) = 1$. Let $\sigma = (2\ 5)(3\ 4)$. Since $\sigma = (2\ 5\ 3\ 4)^{(2\ 5)} \cdot (3\ 5) = (3\ 4)^{(2\ 3)} \cdot (2\ 3\ 4\ 5) = (2\ 3)(4\ 5) \cdot (2\ 5\ 3) \cdot (2\ 5\ 4) = (2\ 4)(3\ 5) \cdot (2\ 4\ 3) \cdot (3\ 5\ 4)$, $\sigma \in G$ and $S^\sigma = S$ for any possible $S$. Therefore $\sigma \in \mathrm{Aut}(G,S)$, which leads to a contradiction. So the assertion is right; that is, $\Gamma$ is not 1-regular.

Let $G_i = \langle S_i\rangle$ and $\Gamma_i = \mathrm{Cay}(G_i, S_i)$ for $i \in \{1,2,3,4\}$. Set $\gamma = (2\ 4\ 5\ 3)$, then $S_1^\gamma = S_2$ and $S_3^\gamma = S_4$. It follows that $G_1^\gamma = G_2$ and $G_3^\gamma = G_4$, namely, $\Gamma_1 \cong \Gamma_2$ and $\Gamma_3 \cong \Gamma_4$. Now we consider $G_1 = \langle S_1\rangle$. Note that $(2\ 4) = (2\ 5)^{(2\ 4\ 3\ 5)}$ and $(3\ 5) = (2\ 5\ 3\ 4)^{(2\ 5)} \cdot (2\ 5\ 3\ 4)^2$, then $G_1 = \langle(2\ 5), (2\ 5\ 3\ 4)\rangle$. Since $(2\ 5\ 4) = (2\ 3)(4\ 5) \cdot (2\ 3\ 5)$, $G_3 = \langle S_3\rangle = \langle(2\ 3)(4\ 5), (2\ 3\ 5)\rangle$. On the other hand, $(2\ 5\ 3\ 4)^4 = (2\ 5)^2 = ((2\ 5) \cdot (2\ 5\ 3\ 4))^3 = 1$ and $(2\ 3\ 5)^3 = ((2\ 3)(4\ 5))^2 = ((2\ 3)(4\ 5) \cdot (2\ 3\ 5))^3 = 1$, then $G_1 \cong S_4$ and $G_3 \cong A_4$. By the assumption, $\Gamma_3$ is not the graph satisfying conditions. So far we get the result that there is only one isomorphism class of graphs when $G = S_4$. □

To finish our proof, we need to introduce some definitions and properties. Assume that $\Gamma$ is an $X$-vertex transitive graph with $X$ being a subgroup of $\mathrm{Aut}\Gamma$. Let $N$ be a normal subgroup of $X$. Denote the set of $N$-orbits in $V\Gamma$ by $V_N$. The normal quotient $\Gamma_N$ of $\Gamma$ induced by $N$ is defined as the graph with vertex set $V_N$, and two vertices $B, C \in V_N$ are adjacent if there exist $u \in B$ and $v \in C$ such that they are adjacent in $\Gamma$. It is easy to show that $X/N$ acts transitively on the vertex set of $\Gamma_N$. Assume further that $\Gamma$ is $X$-edge-transitive. Then $X/N$ acts transitively on the edge set of $\Gamma_N$, and the valency $\mathrm{val}(\Gamma) = m\mathrm{val}(\Gamma_N)$ for some positive integer $m$. If $m = 1$, then $\Gamma$ is called a normal cover of $\Gamma_N$.

*Proof of Theorem 1.* Let $\Gamma = \mathrm{Cay}(G,S)$ be an $(X,1)$-regular Cayley graph of valency 5, where $G \leq X \leq \mathrm{Aut}\Gamma$. Then it is trivial to see that $\Gamma$ is connected. Let $N = \mathrm{Core}_X(G)$ be the core of $G$ in $X$. Assume that $N$ is not trivial. Then either $G = N$ or $|G : N| \geq 2$. The former implies $G \trianglelefteq X$; that is, $\Gamma$ is an $X$-normal Cayley graph with respect to $G$. For the case where $|G : N| = 2$, it is easy to verify $\Gamma$ is an $X$-bi-normal Cayley graph. Suppose that $|G : N| > 2$; namely, $N$ has at least three orbits on $V\Gamma$. Since $\mathrm{val}(\Gamma) = 5$ is a prime and $\Gamma$ is $(X,1)$-regular, $\Gamma$ is a cover of $\Gamma_N$ and $G/N \leq X/N \leq \mathrm{Aut}\Gamma_N$. We have that $\Gamma_N$ is a Cayley graph of $G/N$ and $\Gamma_N$ is core-free with respect to $G/N$. Now suppose that $N$ is trivial, then $\Gamma$ is a core-free one. According to Lemmas 5, 6, and 7, there are two core-free $(X,1)$-regular Cayley graphs of valency 5 (up to isomorphism) as in Table 1. As far, Theorem 1 holds.


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