Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 561573, 12 pages
doi:10.1155/2011/561573

Research Article

Systems of Generalized Quasivariational Inclusion Problems with Applications in $L\Gamma$-Spaces

Ming-ge Yang,[1,2] Jiu-ping Xu,[3] and Nan-jing Huang[1,3]

[1] *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China*
[2] *Department of Mathematics, Luoyang Normal University, Luoyang, Henan 471022, China*
[3] *College of Business and Management, Sichuan University, Chengdu, Sichuan 610064, China*

Correspondence should be addressed to Nan-jing Huang, nanjinghuang@hotmail.com

Received 27 September 2010; Accepted 22 October 2010

Academic Editor: Yeol J. E. Cho

We first prove that the product of a family of $L\Gamma$-spaces is also an $L\Gamma$-space. Then, by using a Himmelberg type fixed point theorem in $L\Gamma$-spaces, we establish existence theorems of solutions for systems of generalized quasivariational inclusion problems, systems of variational equations, and systems of generalized quasiequilibrium problems in $L\Gamma$-spaces. Applications of the existence theorem of solutions for systems of generalized quasiequilibrium problems to optimization problems are given in $L\Gamma$-spaces.

1. Introduction

In 1979, Robinson [1] studied the following parametric variational inclusion problem: given $x \in \mathbb{R}^n$, find $y \in \mathbb{R}^m$ such that

$$0 \in g(x, y) + Q(x, y), \tag{1.1}$$

where $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a single-valued function and $Q : \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^p$ is a multivalued map. It is known that (1.1) covers variational inequality problems and a vast of variational system important in applications. Since then, various types of variational inclusion problems have been extended and generalized by many authors (see, e.g., [2–7] and the references therein).

On the other hand, Tarafdar [8] generalized the classical Himmelberg fixed point theorem [9] to locally H-convex uniform spaces (or LC-spaces). Park [10] generalized the result of Tarafdar [8] to locally G-convex spaces (or LG-spaces). Recently, Park [11] introduced the concept of abstract convex spaces which include H-spaces and G-convex spaces as special cases. With this new concept, he can study the KKM theory and its applications in abstract convex spaces. More recently, Park [12] introduced the concept of $L\Gamma$-spaces which include LC-spaces and LG-spaces as special cases. He also established the Himmelberg type fixed point theorem in $L\Gamma$-spaces. To see some related works, we refer to [13–21] and the references therein. However, to the best of our knowledge, there is no paper dealing with systems of generalized quasivariational inclusion problems in $L\Gamma$-spaces.

Motivated and inspired by the works mentioned above, in this paper, we first prove that the product of a family of $L\Gamma$-spaces is also an $L\Gamma$-space. Then, by using the Himmelberg type fixed point theorem due to Park [12], we establish existence theorems of solutions for systems of generalized quasivariational inclusion problems, systems of variational equations, and systems of generalized quasiequilibrium problems in $L\Gamma$-spaces. Applications of the existence theorem of solutions for systems of generalized quasiequilibrium problems to optimization problems are given in $L\Gamma$-spaces.

2. Preliminaries

For a set X, $\langle X\rangle$ will denote the family of all nonempty finite subsets of X. If A is a subset of a topological space, we denote by $\operatorname{int} A$ and $\overline{A}$ the interior and closure of A, respectively.

A multimap (or simply a map) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the values $T(x) \subset Y$ for all $x \in X$. Given a map $T : X \multimap Y$, the map $T^- : Y \multimap X$ defined by $T^-(y) = \{x \in X : y \in T(x)\}$ for all $y \in Y$, is called the (lower) inverse of T. For any $A \subset X, T(A) := \bigcup_{x\in A} T(x)$. For any $B \subset Y, T^-(B) := \{x \in X : T(x)\cap B \neq \emptyset\}$. As usual, the set $\{(x, y) \in X \times Y : y \in T(x)\} \subset X \times Y$ is called the graph of T.

For topological spaces X and Y, a map $T : X \multimap Y$ is called

(i) closed if its graph Graph(T) is a closed subset of $X \times Y$,

(ii) upper semicontinuous (in short, u.s.c.) if for any $x \in X$ and any open set V in Y with $T(x) \subset V$, there exists a neighborhood U of x such that $T(x') \subset V$ for all $x' \in U$,

(iii) lower semicontinuous (in short, l.s.c.) if for any $x \in X$ and any open set V in Y with $T(x)\cap V \neq \emptyset$, there exists a neighborhood U of x such that $T(x')\cap V \neq \emptyset$ for all $x' \in U$,

(iv) continuous if T is both u.s.c. and l.s.c.,

(v) compact if $T(X)$ is contained in a compact subset of Y.

Lemma 2.1 (see [22]). *Let X and Y be topological spaces, $T : X \multimap Y$ be a map. Then, T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there exists a net $\{y_\alpha\}$ in Y such that $y_\alpha \in T(x_\alpha)$ for each α and y_α converges to y.*

Lemma 2.2 (see [23]). *Let X and Y be Hausdorff topological spaces and $T : X \multimap Y$ be a map.*

(i) *If T is an u.s.c. map with closed values, then T is closed.*

(ii) *If Y is a compact space and T is closed, then T is u.s.c.*

(iii) *If X is compact and T is an u.s.c. map with compact values, then $T(X)$ is compact.*

In what follows, we introduce the concept of abstract convex spaces and map classes $\mathfrak{R}$, $\mathfrak{RC}$ and $\mathfrak{RO}$ having certain KKM properties. For more details and discussions, we refer the reader to [11, 12, 24].

Definition 2.3 (see [11]). An abstract convex space $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap E$ with nonempty values. We denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

In the case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$. It is obvious that any vector space E is an abstract convex space with $\Gamma = \text{co}$, where co denotes the convex hull in vector spaces. In particular, $(\mathbb{R}; \text{co})$ is an abstract convex space.

Let $(E, D; \Gamma)$ be an abstract convex space. For any $D' \subset D$, the Γ-convex hull of D' is denoted and defined by

$$\text{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E, \tag{2.1}$$

(co is reserved for the convex hull in vector spaces). A subset X of E is called a Γ-convex subset of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$; that is, $\text{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D' \rangle})$ itself is an abstract convex space called a subspace of $(E, D; \Gamma)$. When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-convex if $\text{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' = X \cap D$. When $(E; \Gamma) = (\mathbb{R}; \text{co})$, Γ-convex subsets reduce to ordinary convex subsets.

Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A), \quad \forall A \in \langle D \rangle, \tag{2.2}$$

then G is called a KKM map with respect to F. A KKM map $G : D \multimap E$ is a KKM map with respect to the identity map 1_E. A map $F : E \multimap Z$ is said to have the KKM property and called a $\mathfrak{K}$-map if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}. \tag{2.3}$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map is defined for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z). \tag{2.4}$$

Note that if Z is discrete, then three classes $\mathfrak{K}$, $\mathfrak{KC}$ and $\mathfrak{KO}$ are identical. Some authors use the notation $\text{KKM}(E, Z)$ instead of $\mathfrak{KC}(E, Z)$.

Definition 2.4 (see [24]). For an abstract convex space $(E, D; \Gamma)$, the KKM principle is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$.

A KKM space is an abstract convex space satisfying the KKM principle.

Definition 2.5. Let $(Y;\Gamma)$ be an abstract convex space, Z be a real t.v.s., and $F : Y \multimap Z$ a map. Then,

(*i*) F is $\{0\}$-quasiconvex-like if for any $\{y_1, y_2, \ldots, y_n\} \in \langle Y \rangle$ and any $\overline{y} \in \Gamma(\{y_1, y_2, \ldots, y_n\})$ there exists $j \in \{1, 2, \ldots, n\}$ such that $F(\overline{y}) \subset F(y_j)$,

(*ii*) F is $\{0\}$-quasiconvex if for any $\{y_1, y_2, \ldots, y_n\} \in \langle Y \rangle$ and any $\overline{y} \in \Gamma(\{y_1, y_2, \ldots, y_n\})$ there exists $j \in \{1, 2, \ldots, n\}$ such that $F(y_j) \subset F(\overline{y})$.

Remark 2.6. If Y is a nonempty convex subset of a t.v.s. with $\Gamma = \mathrm{co}$, then Definition 2.5 (i) and (ii) reduce to Definition 2.4 (iii) and (vi) in Lin [5], respectively.

Definition 2.7 (see [25]). A uniformity for a set X is a nonempty family $\mathcal{U}$ of subsets of $X \times X$ satisfying the following conditions:

(*i*) each member of $\mathcal{U}$ contains the diagonal Δ,

(*ii*) for each $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$,

(*iii*) for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$,

(*iv*) if $U \in \mathcal{U}$, $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$,

(*v*) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a uniform space. Every member in $\mathcal{U}$ is called an entourage. For any $x \in X$ and any $U \in \mathcal{U}$, we define $U[x] := \{y \in X : (x, y) \in U\}$. The uniformity $\mathcal{U}$ is called separating if $\bigcap\{U \subset X \times X : U \in \mathcal{U}\} = \Delta$. The uniform space $(X, \mathcal{U})$ is Hausdorff if and only if $\mathcal{U}$ is separating. For more details about uniform spaces, we refer the reader to Kelley [25].

Definition 2.8 (see [12]). An abstract convex uniform space $(E, D; \Gamma; \mathcal{B})$ is an abstract convex space with a basis $\mathcal{B}$ of a uniformity of E.

Definition 2.9 (see [12]). An abstract convex uniform space $(E \supset D; \Gamma; \mathcal{B})$ is called an $L\Gamma$-space if

(*i*) D is dense in E, and

(*ii*) for each $U \in \mathcal{B}$ and each Γ-convex subset $A \subset E$, the set $\{x \in E : A \cap U[x] \neq \emptyset\}$ is Γ-convex.

Lemma 2.10 (see [12, Corollary 4.5]). *Let $(E \supset D; \Gamma; \mathcal{B})$ be a Hausdorff KKM $L\Gamma$-space and $T : E \multimap E$ a compact u.s.c. map with nonempty closed Γ-convex values. Then, T has a fixed point.*

Lemma 2.11 (see [24, Lemma 8.1]). *Let $\{(E_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of abstract convex spaces. Let $E := \prod_{i \in I} E_i$ and $D := \prod_{i \in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then, $(E, D; \Gamma)$ is an abstract convex space.*

Lemma 2.12. *Let I be any index set. For each $i \in I$, let $(X_i; \Gamma_i; \mathcal{B}_i)$ be an $L\Gamma$-space. If one defines $X := \prod_{i \in I} X_i$, $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$ for each $A \in \langle X \rangle$ and $\mathcal{B} := \{\bigcap_{j=1}^{n} U^j : U^j \in \mathcal{S}, j = 1, 2, \ldots, n$ and $n \in \mathbb{N}\}$, where $\mathcal{S} := \{\{(x, y) \in X \times X : (x_i, y_i) \in U_i\} : i \in I, U_i \in \mathcal{B}_i\}$. Then, $(X; \Gamma; \mathcal{B})$ is also an $L\Gamma$-space.*

Proof. By Lemma 2.11, $(X;\Gamma)$ is an abstract convex space. It is easy to check that $\mathcal{S}$ is a subbase of the product uniformity of X. Since $\mathcal{B}$ is the basis generated by $\mathcal{S}$, we obtain that $\mathcal{B}$ is a basis of the product uniformity, and the associated uniform topology on X.

Now, we prove that for each $U\in\mathcal{B}$ and each Γ-convex subset $A\subset X$, the set $\{x\in X : A\cap U[x]\neq\emptyset\}$ is Γ-convex. Firstly, we show that for each $i\in I$, $\pi_i(A)$ is a Γ_i-convex subset of X_i. For any $N_i\in\langle\pi_i(A)\rangle$, we can find some $N\in\langle A\rangle$ with $\pi_i(N)=N_i$. Since A is a Γ-convex subset of X, we have $\Gamma(N)\subset A$. It follows that $\Gamma_i(\pi_i(N))=\Gamma_i(N_i)\subset\pi_i(A)$. Thus, we have shown that $\pi_i(A)$ is a Γ_i-convex subset of X_i. Secondly, we show that the set $\{x\in X : A\cap U[x]\neq\emptyset\}$ is Γ-convex. Since each $U^j\in\mathcal{S}$ has the form $U^j=\{(x,y)\in X\times X : (x_{i_j},y_{i_j})\in U_{i_j}\}$ for some $i_j\in I$ and $U_{i_j}\in\mathcal{B}_{i_j}$, we have that

$$
\begin{aligned}
U[x] &= \{y\in X : (x,y)\in U\} \\
&= \left\{y\in X : (x,y)\in\bigcap_{j=1}^{n}U^j\right\} \\
&= \left\{y\in X : \left(x_{i_j},y_{i_j}\right)\in U_{i_j}\ \forall\ j=1,2,\dots,n\right\} \\
&= \left\{y\in X : y_{i_j}\in U_{i_j}\left[x_{i_j}\right]\ \forall\ j=1,2,\dots,n\right\} \\
&= \prod_{i\in I\setminus\left\{i_j:j=1,2,\dots,n\right\}} X_i\times\prod_{j=1}^{n}U_{i_j}\left[x_{i_j}\right],
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
\{x\in X : A\cap U[x]\neq\emptyset\} &= \left\{x\in X : A\cap\left(\prod_{i\in I\setminus\{i_j:j=1,2,\dots,n\}} X_i\times\prod_{j=1}^{n}U_{i_j}\left[x_{i_j}\right]\right)\neq\emptyset\right\} \\
&= \left\{x\in X : \prod_{i\in I\setminus\{i_j:j=1,2,\dots,n\}}(\pi_i(A)\cap X_i)\times\prod_{j=1}^{n}\left(\pi_{i_j}(A)\cap U_{i_j}\left[x_{i_j}\right]\right)\neq\emptyset\right\} \\
&= \left\{x\in X : \prod_{j=1}^{n}\left(\pi_{i_j}(A)\cap U_{i_j}\left[x_{i_j}\right]\right)\neq\emptyset\right\} \\
&= \bigcap_{j=1}^{n}\left\{x\in X : \pi_{i_j}(A)\cap U_{i_j}\left[x_{i_j}\right]\neq\emptyset\right\} \\
&= \bigcap_{j=1}^{n}\left(\prod_{i\in I\setminus\{i_j\}} X_i\times\left\{x_{i_j}\in X_{i_j} : \pi_{i_j}(A)\cap U_{i_j}\left[x_{i_j}\right]\neq\emptyset\right\}\right).
\end{aligned}
\tag{2.6}
$$

By the definition of $L\Gamma$-spaces, we obtain that for each $j\in\{1,2,\dots,n\}$, the set $\{x_{i_j}\in X_{i_j} : \pi_{i_j}(A)\cap U_{i_j}[x_{i_j}]\neq\emptyset\}$ is Γ_{i_j}-convex. It follows from (2.6) that the set $\{x\in X : A\cap U[x]\neq\emptyset\}$ is a Γ-convex subset of X. Therefore $(X;\Gamma;\mathcal{B})$ is an $L\Gamma$-space. This completes the proof. □

Remark 2.13. Lemma 2.12 generalizes [26, Theorem 2.2] from locally FC-uniform spaces to $L\Gamma$-spaces. The proof of Lemma 2.12 is different with the proof of [26, Theorem 2.2].

3. Existence Theorems of Solutions for Systems of Generalized Quasivariational Inclusion Problems

Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, $(X_i; \Gamma_i^1; \mathcal{B}_i^1)$ be an $L\Gamma$-space, and $(Y_i; \Gamma_i^2; \mathcal{B}_i^2)$ be an $L\Gamma$-space with $1_{Y_i} \in \mathfrak{RC}(Y_i, Y_i)$. Let $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$ and $(X \times Y; \Gamma; \mathcal{B})$ be the product $L\Gamma$-space as defined in Lemma 2.12. Furthermore, we assume that $(X \times Y; \Gamma; \mathcal{B})$ is a KKM space. Throughout this paper, we use these notations unless otherwise specified, and assume that all topological spaces are Hausdorff.

The following theorem is the main result of this paper.

Theorem 3.1. *For each $i \in I$, suppose that*

(i) $A_i : X \times Y \multimap X_i$ *is a compact u.s.c. map with nonempty closed Γ_i^1-convex values,*

(ii) $T_i : X \multimap Y_i$ *is a compact continuous map with nonempty closed Γ_i^2-convex values,*

(iii) $G_i : X \times Y_i \times Y_i \multimap Z_i$ *is a closed map with nonempty values,*

(iv) *for each $(x, v_i) \in X \times Y_i$, $y_i \multimap G_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex; for each $(x, y_i) \in X \times Y_i$, $v_i \multimap G_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex-like and $0 \in G_i(x, y_i, y_i)$.*

Then, there exists $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x} = (\overline{x}_i)_{i\in I}$ and $\overline{y} = (\overline{y}_i)_{i\in I}$ such that for each $i \in I$, $\overline{x}_i \in A_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x})$ and $0 \in G_i(\overline{x}, \overline{y}_i, v_i)$ for all $v_i \in T_i(\overline{x})$.

Proof. For each $i \in I$, define $H_i : X \multimap T_i(X)$ by

$$H_i(x) = \{y_i \in T_i(x) : 0 \in G_i(x, y_i, v_i)\ \forall\ v_i \in T_i(x)\}, \quad \forall x \in X. \tag{3.1}$$

Then, $H_i(x)$ is nonempty for each $x \in X$. Indeed, fix any $i \in I$ and $x \in X$, define $Q_i^x : T_i(x) \multimap T_i(x)$ by

$$Q_i^x(v_i) = \{y_i \in T_i(x) : 0 \in G_i(x, y_i, v_i)\}, \quad \forall v_i \in T_i(x). \tag{3.2}$$

First, we show that Q_i^x is a KKM map w.r.t. $1_{T_i(x)}$. Suppose to the contrary that there exists a finite subset $\{v_i^1, v_i^2, \ldots, v_i^n\} \subset T_i(x)$ such that $\Gamma_i^2(\{v_i^1, v_i^2, \ldots, v_i^n\}) \not\subset \bigcup_{k=1}^n Q_i^x(v_i^k)$. Hence, there exists $\overline{v}_i \in \Gamma_i^2(\{v_i^1, v_i^2, \ldots, v_i^n\})$ satisfying $\overline{v}_i \notin Q_i^x(v_i^k)$ for all $k = 1, 2, \ldots, n$. Since $T_i(x)$ is Γ_i^2-convex, we have $\overline{v}_i \in \Gamma_i^2(\{v_i^1, v_i^2, \ldots, v_i^n\}) \subset T_i(x)$. By $\overline{v}_i \notin Q_i^x(v_i^k)$ for all $k = 1, 2, \ldots, n$, we know that $0 \notin G_i(x, \overline{v}_i, v_i^k)$ for all $k = 1, 2, \ldots, n$. Since $v_i \multimap G_i(x, \overline{v}_i, v_i)$ is $\{0\}$-quasiconvex-like, there exists $1 \le j \le n$ such that

$$0 \in G_i(x, \overline{v}_i, \overline{v}_i) \subset G_i\left(x, \overline{v}_i, v_i^j\right). \tag{3.3}$$

This leads to a contradiction. Therefore, Q_i^x is a KKM map w.r.t. $1_{T_i(x)}$. Next, we show that $Q_i^x(v_i)$ is closed for each $v_i \in T_i(x)$. Indeed, if $y_i \in \overline{Q_i^x(v_i)}$, then there exists a net $\{y_i^\alpha\}_{\alpha\in\Lambda}$ in $Q_i^x(v_i)$ such that $y_i^\alpha \to y_i$. For each $\alpha \in \Lambda$, we have $y_i^\alpha \in T_i(x)$ and $0 \in G_i(x, y_i^\alpha, v_i)$. By condition (ii), $T_i(x)$ is closed, and hence $y_i \in T_i(x)$. By condition (iii), G_i is closed, and hence $0 \in G_i(x, y_i, v_i)$. It follows that $y_i \in Q_i^x(v_i)$. Therefore, $Q_i^x(v_i)$ is closed. Since $1_{Y_i} \in \mathfrak{RC}(Y_i, Y_i)$ and $T_i(x)$ is Γ_i^2-convex, we have that $1_{T_i(x)} \in \mathfrak{RC}(T_i(x), T_i(x))$. Having that T_i is compact, we can deduce that $\bigcap_{v_i\in T_i(x)} Q_i^x(v_i) \neq \emptyset$. That is $H_i(x)$ is nonempty.

H_i is closed for each $i \in I$. Indeed, if $(x, y_i) \in \overline{\text{Graph}(H_i)}$, then there exists a net $\{(x^\alpha, y_i^\alpha)\}_{\alpha\in\Lambda}$ in $\text{Graph}(H_i)$ such that $(x^\alpha, y_i^\alpha) \to (x, y_i)$. One has $y_i^\alpha \in T_i(x^\alpha)$ and $0 \in G_i(x^\alpha, y_i^\alpha, v_i)$ for all $v_i \in T_i(x^\alpha)$. By condition (ii), T_i is closed, and hence $y_i \in T_i(x)$. Let $v_i \in T_i(x)$, since T_i is l.s.c., there exists a net $\{v_i^\alpha\}$ satisfying $v_i^\alpha \in T_i(x^\alpha)$ and $v_i^\alpha \to v_i$. We have $0 \in G_i(x^\alpha, y_i^\alpha, v_i^\alpha)$. Since G_i is closed, we obtain $0 \in G_i(x, y_i, v_i)$. Thus, we have shown that $(x, y_i) \in \text{Graph}(H_i)$. Hence, H_i is closed.

$H_i(x)$ is Γ_i^2-convex for each $i \in I$ and $x \in X$. Indeed, if $\{y_i^1, y_i^2, \ldots, y_i^n\} \in \langle H_i(x)\rangle$, then we have that $\{y_i^1, y_i^2, \ldots, y_i^n\} \subset T_i(x)$ and $0 \in G_i(x, y_i^k, v_i)$ for all $v_i \in T_i(x)$ and all $k = 1, 2, \ldots, n$. For any given $\overline{y}_i \in \Gamma_i^2(\{y_i^1, y_i^2, \ldots, y_i^n\})$, we have $\overline{y}_i \in T_i(x)$ because $T_i(x)$ is Γ_i^2-convex. For each $v_i \in T_i(x)$, since $y_i \multimap G_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex, there exists $1 \le j \le n$ such that

$$G_i\left(x, y_i^j, v_i\right) \subset G_i(x, \overline{y}_i, v_i). \tag{3.4}$$

Hence, $0 \in G_i(x, \overline{y}_i, v_i)$ for all $v_i \in T_i(x)$. It follows that $\overline{y}_i \in H_i(x)$ and $H_i(x)$ is Γ_i^2-convex.

Since $H_i(X) \subset \overline{T_i(X)}$ and $\overline{T_i(X)}$ is compact. It follows from Lemma 2.2(ii) that H_i is a compact u.s.c. map for each $i \in I$. Define $Q : X \times Y \multimap X \times Y$ by

$$Q(x, y) = \left[\prod_{i\in I} A_i(x, y)\right] \times \left[\prod_{i\in I} H_i(x)\right], \quad \forall (x, y) \in X \times Y. \tag{3.5}$$

It follows from the above discussions that for each $i \in I$, H_i is a compact u.s.c. map with nonempty closed Γ_i^2-convex values. Thus, Q is a compact u.s.c. map with nonempty closed Γ-convex values. By Lemma 2.10, there exists $(\overline{x}, \overline{y}) \in X \times Y$ such that $(\overline{x}, \overline{y}) \in Q(\overline{x}, \overline{y})$. That is there exists $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x} = (\overline{x}_i)_{i\in I}$ and $\overline{y} = (\overline{y}_i)_{i\in I}$ such that for each $i \in I$, $\overline{x}_i \in A_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x})$ and $0 \in G_i(\overline{x}, \overline{y}_i, v_i)$ for all $v_i \in T_i(\overline{x})$. This completes the proof. □

For the special case of Theorem 3.1, we have the following corollary which is actually an existence theorem of solutions for variational equations.

Corollary 3.2. *For each $i \in I$, suppose that conditions (i) and (ii) in Theorem 3.1 hold. Moreover,*

(iii)$_1$ *$G_i : X \times Y_i \times Y_i \to Z_i$ is a continuous mapping;*

(iv)$_1$ *for each $(x, v_i) \in X \times Y_i$, $y_i \to G_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex; for each $(x, y_i) \in X \times Y_i$, $v_i \to G_i(x, y_i, v_i)$ is also $\{0\}$-quasiconvex and $G_i(x, y_i, y_i) = 0$.*

Then, there exists $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x} = (\overline{x}_i)_{i\in I}$ and $\overline{y} = (\overline{y}_i)_{i\in I}$ such that for each $i \in I$, $\overline{x}_i \in A_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x})$ and $G_i(\overline{x}, \overline{y}_i, v_i) = 0$ for all $v_i \in T_i(\overline{x})$.

Theorem 3.3. *For each $i \in I$, suppose that conditions (i) and (ii) in Theorem 3.1 hold. Moreover,*

(iii)$_2$ *$H_i : X \multimap Z_i$ is a closed map with nonempty values and $Q_i : X \times Y_i \times Y_i \multimap Z_i$ is an u.s.c. map with nonempty compact values;*

(iv)$_2$ *for each $(x, v_i) \in X \times Y_i$, $y_i \multimap Q_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex; for each $(x, y_i) \in X \times Y_i$, $v_i \multimap Q_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex-like and $0 \in H_i(x) + Q_i(x, y_i, y_i)$.*

Then, there exists $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x} = (\overline{x}_i)_{i\in I}$ and $\overline{y} = (\overline{y}_i)_{i\in I}$ such that for each $i \in I$, $\overline{x}_i \in A_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x})$ and $0 \in H_i(\overline{x}) + Q_i(\overline{x}, \overline{y}_i, v_i)$ for all $v_i \in T_i(\overline{x})$.

Proof. For each $i \in I$, define $G_i : X \times Y_i \times Y_i \multimap Z_i$ by

$$G_i(x, y_i, v_i) = H_i(x) + Q_i(x, y_i, v_i), \quad \forall (x, y_i, v_i) \in X \times Y_i \times Y_i. \tag{3.6}$$

Obviously, G_i has nonempty values. Now, we show that G_i is closed. Indeed, if $(x, y_i, v_i, z_i) \in \overline{\text{Graph}(G_i)}$, then there exists a net $\{(x^\alpha, y_i^\alpha, v_i^\alpha, z_i^\alpha)\}_{\alpha\in\Lambda}$ in $\text{Graph}(G_i)$ such that $(x^\alpha, y_i^\alpha, v_i^\alpha, z_i^\alpha) \to (x, y_i, v_i, z_i)$. Since

$$z_i^\alpha \in G_i(x^\alpha, y_i^\alpha, v_i^\alpha) = H_i(x^\alpha) + Q_i(x^\alpha, y_i^\alpha, v_i^\alpha), \tag{3.7}$$

there exist $u_i^\alpha \in H_i(x^\alpha)$ and $w_i^\alpha \in Q_i(x^\alpha, y_i^\alpha, v_i^\alpha)$ such that $z_i^\alpha = u_i^\alpha + w_i^\alpha$. Let

$$K = \{x^\alpha : \alpha \in \Lambda\} \cup \{x\}, \quad L_i = \{y_i^\alpha : \alpha \in \Lambda\} \cup \{y_i\}, \quad M_i = \{v_i^\alpha : \alpha \in \Lambda\} \cup \{v_i\}. \tag{3.8}$$

Then K is a compact subset of X, L_i and M_i are compact subsets of Y_i. By condition $(\text{iii})_2$ and Lemma 2.2(iii), $Q_i(K \times L_i \times M_i)$ is a compact subset of Z_i. Thus, we can assume that $w_i^\alpha \to w_i$. By condition $(\text{iii})_2$, Q_i is closed, and hence $w_i \in Q_i(x, y_i, v_i)$. Since $z_i^\alpha - w_i^\alpha = u_i^\alpha \in H_i(x^\alpha)$ and H_i is closed, we have $z_i - \mathrm{w}_i \in H_i(x)$. Letting $u_i = z_i - w_i$, it follows that

$$z_i = u_i + w_i \in H_i(x) + Q_i(x, y_i, v_i) = G_i(x, y_i, v_i), \tag{3.9}$$

and so G_i is closed.

By the above discussions, we know that condition (iii) of Theorem 3.1 is satisfied. It is easy to check that condition (iv) of Theorem 3.1 is also satisfied. By Theorem 3.1, there exists $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x} = (\overline{x}_i)_{i\in I}$ and $\overline{y} = (\overline{y}_i)_{i\in I}$ such that for each $i \in I$, $\overline{x}_i \in A_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x})$ and

$$0 \in G_i(\overline{x}, \overline{y}_i, v_i) = H_i(\overline{x}) + Q_i(\overline{x}, \overline{y}_i, v_i), \tag{3.10}$$

for all $v_i \in T_i(\overline{x})$. This completes the proof. □

For the special case of Theorem 3.3, we have the following corollary which is actually an existence theorem of solutions for variational equations.

Corollary 3.4. *For each $i \in I$, suppose that conditions (i) and (ii) in Theorem 3.1 hold. Moreover,*

$(\text{iii})_3$ *$H_i : X \to Z_i$ is a continuous map and $Q_i : X \times Y_i \times Y_i \to Z_i$ is a continuous map;*

$(\text{iv})_3$ *for each $(x, v_i) \in X \times Y_i$, $y_i \to Q_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex; for each $(x, y_i) \in X \times Y_i$, $v_i \to Q_i(x, y_i, v_i)$ is also $\{0\}$-quasiconvex and $H_i(x) + Q_i(x, y_i, y_i) = 0$.*

Then, there exists $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x} = (\overline{x}_i)_{i\in I}$ and $\overline{y} = (\overline{y}_i)_{i\in I}$ such that for each $i \in I$, $\overline{x}_i \in A_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x})$ and $H_i(\overline{x}) + Q_i(\overline{x}, \overline{y}_i, v_i) = 0$ for all $v_i \in T_i(\overline{x})$.

From Theorem 3.3, we establish the following corollary which is actually an existence theorem of solutions for systems of generalized vector quasiequilibrium problems.

Corollary 3.5. *For each $i \in I$, suppose that conditions (i) and (ii) in Theorem 3.1 hold. Moreover,*

(iii)$_4$ *$C_i : X \multimap Z_i$ is a closed map with nonempty values and $Q_i : X \times Y_i \times Y_i \multimap Z_i$ is an u.s.c. map with nonempty compact values;*

(iv)$_4$ *for each $(x, v_i) \in X \times Y_i$, $y_i \multimap Q_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex; for each $(x, y_i) \in X \times Y_i$, $v_i \multimap Q_i(x, y_i, v_i)$ is $\{0\}$-quasiconvex-like and $Q_i(x, y_i, y_i) \cap C_i(x) \neq \emptyset$.*

Then, there exists $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x} = (\overline{x}_i)_{i\in I}$ and $\overline{y} = (\overline{y}_i)_{i\in I}$ such that for each $i \in I$, $\overline{x}_i \in A_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x})$, and $Q_i(\overline{x}, \overline{y}_i, v_i) \cap C_i(\overline{x}) \neq \emptyset$ for all $v_i \in T_i(\overline{x})$.

Proof. Define $H_i : X \multimap Z_i$ by $H_i(x) = -C_i(x)$ for all $x \in X$. Since C_i is a closed map with nonempty values, we have that H_i is a closed map with nonempty values. All the conditions of Theorem 3.3 are satisfied. The conclusion of Corollary 3.5 follows from Theorem 3.3. This completes the proof. □

4. Applications to Optimization Problems

Let Z be a real topological vector space, D a proper convex cone in Z. A point $\overline{y} \in A$ is called a vector minimal point of A if for any $y \in A$, $y - \overline{y} \notin -D \setminus \{0\}$. The set of vector minimal point of A is denoted by $\mathrm{Min}_D A$.

Lemma 4.1 (see [27]). *Let Z be a Hausdorff t.v.s., D be a closed convex cone in Z. If A is a nonempty compact subset of Z, then $\mathrm{Min}_D A \neq \emptyset$.*

Theorem 4.2. *For each $i \in I$, suppose that conditions (i), (ii) in Theorem 3.1 and conditions (iii)$_4$, (iv)$_4$ in Corollary 3.5 hold. Furthermore, let $h : X \times Y \multimap Z$ be an u.s.c. map with nonempty compact values, where Z is a real t.v.s. ordered by a proper closed convex cone in Z. Then, there exists a solution to:*

$$\mathrm{Min}_{(x,y)} h(x, y), \tag{4.1}$$

where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ such that for each $i \in I$, $x_i \in A_i(x, y)$, $y_i \in T_i(x)$, and $Q_i(x, y_i, v_i) \cap C_i(x) \neq \emptyset$ for all $v_i \in T_i(x)$.

Proof. By Corollary 3.5, there exists $(\overline{x}, \overline{y}) \in X \times Y$ with $\overline{x} = (\overline{x}_i)_{i\in I}$ and $\overline{y} = (\overline{y}_i)_{i\in I}$ such that for each $i \in I$, $\overline{x}_i \in A_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x})$ and $Q_i(\overline{x}, \overline{y}_i, v_i) \cap C_i(\overline{x}) \neq \emptyset$ for all $v_i \in T_i(\overline{x})$. For each $i \in I$, let

$$\begin{aligned} M_i = \{ &(x, y) \in X \times Y : x_i \in A_i(x, y), y_i \in T_i(x), \\ &Q_i(x, y_i, v_i) \cap C_i(x) \neq \emptyset \ \forall v_i \in T_i(x)\}, \end{aligned} \tag{4.2}$$

and $M = \bigcap_{i\in I} M_i$. Then $(\overline{x}, \overline{y}) \in M$ and $M \neq \emptyset$. We show that M_i is closed for each $i \in I$. Indeed, if $(x, y) \in \overline{M_i}$, then there exists a net $\{(x^\alpha, y^\alpha)\}_{\alpha\in\Lambda}$ in M_i such that $(x^\alpha, y^\alpha) \to (x, y)$. For each $\alpha \in \Lambda$, $(x^\alpha, y^\alpha) \in M_i$ implies that

$$x_i^\alpha \in A_i(x^\alpha, y^\alpha), \quad y_i^\alpha \in T_i(x^\alpha), \quad Q_i(x^\alpha, y_i^\alpha, v_i) \cap C_i(x^\alpha) \neq \emptyset \quad \forall v_i \in T_i(x^\alpha). \tag{4.3}$$

By the closedness of A_i and T_i, we have that $x_i \in A_i(x,y)$ and $y_i \in T_i(x)$. Now, we prove that $Q_i(x,y_i,v_i) \cap C_i(x) \neq \emptyset$ for all $v_i \in T_i(x)$. For any $v_i \in T_i(x)$, since T_i is l.s.c., there exists a net $\{v_i^\alpha\}_{\alpha\in\Lambda}$ satisfying $v_i^\alpha \in T_i(x^\alpha)$ and $v_i^\alpha \to v_i$. Let $u_i^\alpha \in Q_i(x^\alpha, y_i^\alpha, v_i^\alpha) \cap C_i(x^\alpha)$. Since Q_i is u.s.c. with nonempty compact values, we can assume that $u_i^\alpha \to u_i \in Z_i$. By the closedness of Q_i and C_i, we have that $u_i \in Q_i(x,y_i,v_i) \cap C_i(x)$. Thus, $Q_i(x,y_i,v_i) \cap C_i(x) \neq \emptyset$. It follows that M_i is closed. Hence, M is closed. Note that $M \subset \prod_{i\in I} A_i(X\times Y) \times \prod_{i\in I} T_i(X)$. We know that M is a nonempty compact subset of $X \times Y$. It follows from Lemma 2.2(iii) that $h(M)$ is a nonempty compact subset of Z. By Lemma 4.1, $\mathrm{Min}_D h(M) \neq \emptyset$. That is there exists a solution of the problem: $\mathrm{Min}_{(x,y)} h(x,y)$ where $(x,y) \in M$. This completes the proof. □

Theorem 4.3. *For each $i \in I$, suppose that X_i is compact and condition (ii) in Theorem 3.1 holds. Moreover,*

(iii)$_5$ *$Q_i : X \times Y_i \times Y_i \to \mathbb{R}$ is a continuous function;*

(iv)$_5$ *for each $(x,v_i) \in X \times Y_i$, $y_i \to Q_i(x,y_i,v_i)$ is $\{0\}$-quasiconvex; for each $(x,y_i) \in X \times Y_i$, $v_i \to Q_i(x,y_i,v_i)$ is also $\{0\}$-quasiconvex and $Q_i(x,y_i,y_i) \geq 0$.*

Furthermore, let $h : X \times Y \to \mathbb{R}$ is a l.s.c. function. Then there exists a solution to:

$$\min_{(x,y)} h(x,y), \tag{4.4}$$

where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ such that for each $i \in I$, $y_i \in T_i(x)$ and $Q_i(x,y_i,v_i) \geq 0$ for all $v_i \in T_i(x)$.

Proof. For each $i \in I$, define $A_i : X \times Y \multimap X_i$ and $C_i : X \multimap \mathbb{R}$ by

$$\begin{aligned} A_i(x,y) &= X_i, \quad \forall (x,y) \in X \times Y, \\ C_i(x) &= [0,+\infty), \quad \forall x \in X, \end{aligned} \tag{4.5}$$

respectively. It is easy to check that all the conditions of Corollary 3.5 are satisfied. For each $i \in I$, define

$$M_i = \{(x,y) \in X \times Y : y_i \in T_i(x), \quad Q_i(x,y_i,v_i) \geq 0 \ \forall v_i \in T_i(x)\}, \tag{4.6}$$

and $M = \bigcap_{i\in I} M_i$. Then, by Corollary 3.5, there exists $(\overline{x},\overline{y}) \in M$ and hence $M \neq \emptyset$. Arguing as Theorem 4.2, we can prove that M is a nonempty compact subset of $X \times Y$. Hence there exists a solution to the problem $\min_{(x,y)} h(x,y)$ where $(x,y) \in M$. This completes the proof. □

Remark 4.4. Theorem 4.3 generalizes [28, Corollary 3.5] from locally convex topological vector spaces to $L\Gamma$-spaces.

Theorem 4.5. *For each $i \in I$, suppose that X_i is compact and condition (ii) in Theorem 3.1 holds. Moreover,*

(iii)$_6$ *$F_i : X \times Y_i \to \mathbb{R}$ is a continuous function;*

(iv)$_6$ *for each $x \in X$, $y_i \to F_i(x,y_i)$ is $\{0\}$-quasiconvex.*

Furthermore, let $h : X \times Y \to \mathbb{R}$ *be a l.s.c. function. Then, there exists a solution to the problem:*

$$\min_{(x,y)} h(x,y), \tag{4.7}$$

where $x = (x_i)_{i\in I}$ *and* $y = (y_i)_{i\in I}$ *such that for each* $i \in I$, y_i *is the solution of the problem* $\min_{v_i\in T_i(x)} F_i(x,v_i)$.

Proof. For each $i \in I$, define $Q_i : X \times Y_i \times Y_i \to \mathbb{R}$ by

$$Q_i(x,y_i,v_i) = F_i(x,v_i) - F_i(x,y_i), \quad \forall (x,y_i,v_i) \in X \times Y_i \times Y_i. \tag{4.8}$$

It is easy to check that all the conditions of Theorem 4.3 are satisfied. Theorem 4.5 follows immediately from Theorem 4.3. This completes the proof. □

Acknowledgments

This work was supported by the Key Program of NSFC (Grant no. 70831005) and the Open Fund (PLN0904) of State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation (Southwest Petroleum University).

References

[1] S. M. Robinson, “Generalized equations and their solutions. I. Basic theory,” *Mathematical Programming Study*, vol. 10, pp. 128–141, 1979.
[2] S. S. Chang, “Set-valued variational inclusions in Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 438–454, 2000.
[3] X. P. Ding, “Perturbed proximal point algorithms for generalized quasivariational inclusions,” *Journal of Mathematical Analysis and Applications*, vol. 210, no. 1, pp. 88–101, 1997.
[4] N.-J. Huang, “A new class of generalized set-valued implicit variational inclusions in Banach spaces with an application,” *Computers & Mathematics with Applications*, vol. 41, no. 7-8, pp. 937–943, 2001.
[5] L.-J. Lin, “Systems of generalized quasivariational inclusions problems with applications to variational analysis and optimization problems,” *Journal of Global Optimization*, vol. 38, no. 1, pp. 21–39, 2007.
[6] L.-J. Lin, “Variational inclusions problems with applications to Ekeland’s variational principle, fixed point and optimization problems,” *Journal of Global Optimization*, vol. 39, no. 4, pp. 509–527, 2007.
[7] L.-J. Lin, “Systems of variational inclusion problems and differential inclusion problems with applications,” *Journal of Global Optimization*, vol. 44, no. 4, pp. 579–591, 2009.
[8] E. Tarafdar, “Fixed point theorems in locally H-convex uniform spaces,” *Nonlinear Analysis*, vol. 29, no. 9, pp. 971–978, 1997.
[9] C. J. Himmelberg, “Fixed points of compact multifunctions,” *Journal of Mathematical Analysis and Applications*, vol. 38, pp. 205–207, 1972.
[10] S. Park, “Fixed point theorems in locally G-convex spaces,” *Nonlinear Analysis*, vol. 48, no. 6, pp. 869–879, 2002.
[11] S. Park, “On generalizations of the KKM principle on abstract convex spaces,” *Nonlinear Analysis Forum*, vol. 11, no. 1, pp. 67–77, 2006.
[12] S. Park, “Fixed point theory of multimaps in abstract convex uniform spaces,” *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2468–2480, 2009.
[13] L. Deng and M. G. Yang, “Coincidence theorems with applications to minimax inequalities, section theorem, best approximation and multiobjective games in topological spaces,” *Acta Mathematica Sinica (English Series)*, vol. 22, no. 6, pp. 1809–1818, 2006.
[14] L. Deng and M. G. Yang, “Weakly R-KKM mappings-intersection theorems and minimax inequalities in topological spaces,” *Applied Mathematics and Mechanics*, vol. 28, no. 1, pp. 103–109, 2007.

[15] S. Park, "Several episodes in recent studies on the KKM theory," *Nonlinear Analysis Forum*, vol. 15, pp. 13–26, 2010.
[16] S. Park, "Generalized convex spaces, L-spaces, and FC-spaces," *Journal of Global Optimization*, vol. 45, no. 2, pp. 203–210, 2009.
[17] S. Park, "Remarks on fixed points, maximal elements, and equilibria of economies in abstract convex spaces," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1365–1383, 2008.
[18] S. Park, "Remarks on the partial KKM principle," *Nonlinear Analysis Forum*, vol. 14, pp. 51–62, 20009.
[19] S. Park, "Comments on recent studies on abstract convex spaces," *Nonlinear Analysis Forum*, vol. 13, pp. 1–17, 2008.
[20] S. Park, "The rise and decline of generalized convex spaces," *Nonlinear Analysis Forum*, vol. 15, pp. 1–12, 2010.
[21] M.-G. Yang, J.-P. Xu, N.-J. Huang, and S.-J. Yu, "Minimax theorems for vector-valued mappings in abstract convex spaces," *Taiwanese Journal of Mathematics*, vol. 14, no. 2, pp. 719–732, 2010.
[22] N. X. Tan, "Quasi-variational inequalities in topological linear locally convex Hausdorff spaces," *Mathematische Nachrichten*, vol. 122, pp. 231–224, 1985.
[23] J. P. Aubin and A. Cellina, *Differential Inclusion*, Springer, Berlin, Germany, 1994.
[24] S. Park, "Equilibrium existence theorems in KKM spaces," *Nonlinear Analysis*, vol. 69, no. 12, pp. 4352–5364, 2008.
[25] J. L. Kelley, *General Topology*, vol. 2, Springer, New York, NY, USA, 1975.
[26] X. P. Ding, "The generalized game and the system of generalized vector quasi-equilibrium problems in locally FC-uniform spaces," *Nonlinear Analysis*, vol. 68, no. 4, pp. 1028–1036, 2008.
[27] D. C. Luc, *Theory of Vector Optimization*, vol. 319 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 1989.
[28] L.-J. Lin, "Mathematical programming with system of equilibrium constraints," *Journal of Global Optimization*, vol. 37, no. 2, pp. 275–286, 2007.