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Research Article

Fixed Points for Discontinuous Monotone Operators

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We obtain some new existence theorems of the maximal and minimal fixed points for discontinuous monotone operator on an order interval in an ordered normed space. Moreover, the maximal and minimal fixed points can be achieved by monotone iterative method under some conditions. As an example of the application of our results, we show the existence of extremal solutions to a class of discontinuous initial value problems.

1. Introduction

Let X be a Banach space. A nonempty convex closed set $P \subset X$ is said to be a cone if it satisfies the following two conditions: (i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$; (ii) $x \in P$, $-x \in P$ implies $x = \theta$, where θ denotes the zero element. The cone P defines an ordering in E given by $x \leq y$ if and only if $y - x \in P$. Let $D = [u_0, v_0]$ be an ordering interval in X, and $A : D \to X$ an increasing operator such that $u_0 \leq Au_0$, $Av_0 \leq v_0$. It is a common knowledge that fixed point theorems on increasing operators are used widely in nonlinear differential equations and other fields in mathematics ([1–7]).

But in most well-known documents, it is assumed generally that increasing operators possess stronger continuity and compactness. Recently, there have been some papers that considered the existence of fixed points of discontinuous operators. For example, Krasnosel'skii and Lusnikov [8] and Chen [9] discussed the fixed point problems for discontinuous monotonically compact operator. They called an operator A to be a monotonically compact operator if $x_1 \leq \cdots \leq x_n \leq \cdots \leq w$ $(x_1 \geq \cdots \geq x_n \geq \cdots \geq w)$ implies that Ax_n converges to some $x^* \in X$ in norm and that $x^* = \sup\{Ax_n\}$ $(x = \inf\{Ax_n\})$.

A monotonically compact operator is referred to as an MMC-operator. A is said to be h-monotone if $x < y$ implies $Ax < Ay - \alpha(x,y)h$, where $h \in P$, $h \neq \theta$, and $\alpha(x,y) > 0$. They proved the following theorem.

Theorem 1.1 (see [8]). *Let $A : E \to E$ be an h-monotone MMC-operator with $u < Au \leq Av < v$. Then A has at least one fixed point $x^* \in [u,v]$ possessing the property of h-continuity.*

Motivated by the results of [3, 8, 9], in this paper we study the existence of the minimal and maximal fixed points of a discontinuous operator A, which is expressed as the form CB. We do not assume any continuity on A. It is only required that C (or B) is an MMC-operator and $B(D)$ (or $A(D)$) possesses the quasiseparability, which are satisfied naturally in some spaces. As an example for application, we applied our theorem to study first order discontinuous nonlinear differential equation to conclude our paper.

We give the following definitions.

Definition 1.2 (see [3]). Let Y be an Hausdorff topological space with an ordering structure. Y is called an ordered topological space if for any two sequences $\{x_n\}$ and $\{y_n\}$ in Y, $x_n \leq y_n$ $(n = 1,2,\ldots)$ and $x_n \to x$, $y_n \to y$ $(n \to \infty)$ imply $x \leq y$.

Definition 1.3 (see [3]). Let Y be an ordered topological space, S is said to be a quasi-separable set in Y if for any totally ordered set M in S, there exists a countable set $\{y_n\} \subset M$ such that $\{y_n\}$ is dense in M (i.e., for any $y \in M$, there exists $\{y_{n_j}\} \subset \{y_n\}$ such that $y_{n_j} \to y$ $(n \to \infty)$).

Obviously, the separability implies the quasi-separability.

Definition 1.4 (see [3]). Let X, Y be two ordered topological spaces. An operator $A : X \to Y$ is said to be a monotonically compact operator if $x_1 \leq \cdots \leq x_n \leq \cdots \leq w$ $(x_1 \geq \cdots \geq x_n \geq \cdots \geq w)$ implies that Ax_n converges to some $y^* \in Y$ in norm and that $y^* = \sup\{Ax_n\}$ $(y^* = \inf\{Ax_n\})$.

Remark 1.5. The definition of the MMC-operator is slightly different from that of [8, 9].

2. Main Results

Theorem 2.1. *Let X be an ordered topological space, and $D = [u_0, v_0]$ an order interval in X. Let $A : D \to X$ be an operator. Assume that*

(i) *there exist ordered topological space Y, increasing operator $C : D \to Y$, and increasing operator $B : [Cu_0, Cv_0] = \{y \in Y \mid Cu_0 \leq y \leq Cv_0\} \to X$ such that $A = BC$;*
(ii) *$A(D)$ is quasiseparable and C is an MMC-operator;*
(iii) *$u_0 \leq Au_0$, $Av_0 \leq v_0$.*

Then A has at least one fixed point in D.

Proof. It follows from the monotonicity of A and condition (iii) that $A : D \to D$. Set $R = \{x \in A(D) \mid x \leq Ax\}$. Since $Au_0 \in R$, R is nonempty. Suppose that M is a totally ordered set in R. We now show that M has an upper bound in R.

Since $M \subset A(D)$, by condition (ii) there exists a countable subset $\{x_i\}$ of M such that $\{x_i\}$ is dense in M. Consider the sequence

$$z_1 = x_1, \qquad z_i = \max\{z_{i-1}, x_i\}, \quad i = 1,2,\ldots. \tag{2.1}$$

Since M is a totally ordered set, z_i makes sense and

$$z_1 \le z_2 \le \cdots \le z_i \le \cdots . \tag{2.2}$$

By condition (ii), $M \subset D = [u_0, v_0]$ and Definition 1.4, there exists $y^* \in Y$ such that

$$Cz_i \longrightarrow y^* = \sup\{Cz_i\}, \quad (i \longrightarrow \infty), \tag{2.3}$$

$$Cu_0 \le y^* \le Cv_0, \tag{2.4}$$

and hence By^* make sense.

Set

$$x^* = By^*. \tag{2.5}$$

Using (2.1) and (2.2), we have

$$x_i \le Ax_i = BCx_i \le BCz_i \le By^* = x^*. \tag{2.6}$$

Since $\{x_i\}$ is dense in M, for any $x \in M$ there exists a subsequence $\{x_{i_j}\}$ of $\{x_i\}$ such that $x_{i_j} \to x$ $(j \to \infty)$. By (2.6) and Definition 1.2, we get

$$x \le x^*, \quad \forall x \in M. \tag{2.7}$$

Hence $x \le Ax \le Ax^*$, therefore Ax^* is an upper bound of M.

Now we show $Ax^* \in R$. By virtue of (2.4) and condition (iii)

$$u_0 \le Au_0 = BCu_0 \le By^* = x^* \le BCv_0 \le v_0. \tag{2.8}$$

Thus $x^* \in [u_0, v_0] = D$ and hence $Ax^* \in D$. By (2.7) and condition (ii), we get $z_i \le x^*$ and hence $Cz_i \le Cx^*$. By (2.3) and Definition 1.2, we get $y^* \le Cx^*$ and

$$x^* = By^* \le BCx^* = Ax^*. \tag{2.9}$$

Hence $Ax^* \le A(Ax^*)$, and therefore $Ax^* \in R$.

This shows that Ax^* is an upper bound of M in R. It follows from Zorn's lemma that R has maximal element $\overline{x}$. Thus $\overline{x} \le A\overline{x}$. And so $A\overline{x} \le A(A\overline{x})$, which implies that $A\overline{x} \in R$ and $\overline{x} \le A\overline{x}$. As $\overline{x}$ is a maximal element of R, $\overline{x} = A\overline{x}$; that is, $\overline{x}$ is a fixed point of A. □

Theorem 2.2. *Let X be an ordered topological space, and $D = [u_0, v_0]$ an order interval in X. Let $A : D \to X$ be an operator. Assume that*

(i) *there exist ordered topological space Y, increasing operator $C : D \to Y$, and increasing operator $B : [Cu_0, Cv_0] = \{y \in Y \mid Cu_0 \le y \le Cv_0\} \to X$ such that $A = BC$;*

(ii) $[Cu_0, Cv_0]$ *is quasiseparable and* B *is an MMC-operator;*

(iii) $u_0 \leq Au_0$, $Av_0 \leq v_0$.

Then A *has at least one fixed point in* D.

Proof. Let $y_1 = Cu_0$, $y_2 = Cv_0$. By the conditions (i) and (iii), we have

$$y_1 = Cu_0 \leq CAu_0 = CBCu_0 = CBy_1, \qquad CBy_2 = CBCv_0 = CAv_0 \leq Cv_0 = y_2. \tag{2.10}$$

Since CB is increasing, for any $y \in [y_1, y_2]$, we get

$$y_1 \leq CBy_1 \leq CBy \leq CBy_2 \leq y_2, \tag{2.11}$$

that is, $CB : [y_1, y_2] \to [y_1, y_2]$; therefore the quasiseparability of $[Cu_0, Cv_0]$ implies that $CB([y_1, y_2])$ is quasiseparable. Applying Theorem 2.1, the operator CB has at least one fixed point y^* in $[y_1, y_2]$, that is,

$$y^* = CBy^*, \qquad y^* \in [y_1, y_2]. \tag{2.12}$$

Set $x^* = By^*$. Since B is increasing, by (2.12), we have

$$\begin{gathered} u_0 \leq Au_0 = BCu_0 \leq By^* = x^* \leq Bcv_0 = Av_0 \leq v_0, \\ x^* = By^* = B(CBy^*) = BC(By^*) = Ax^*; \end{gathered} \tag{2.13}$$

that is, x^* is a fixed point of the operator A in $[u_0, v_0]$. □

Theorem 2.3. *If the conditions in Theorem 2.1 are satisfied, then* A *has the minimal fixed point* u^* *and the maximal fixed point* v^* *in* D*; that is,* u^* *and* v^* *are fixed points of* A*, and for any fixed point* x *of* A *in* D*, one has* $u^* \leq x \leq v^*$.

Proof. Set

$$\operatorname{Fix} A = \{x \in Dx \text{ is a fixed point of } A\}. \tag{2.14}$$

By Theorem 2.1, Fix $A \neq \emptyset$. Set

$$S = \{[u,v] \mid [u,v] \text{ is an order interval in } X,\ u,v \in A(D),\ u \leq Au,\ Av \leq v,\ \operatorname{Fix} A \subset [u,v]\}. \tag{2.15}$$

Since A is increasing, for any $x \in \operatorname{Fix} A$, we have

$$u_0 \leq Au_0 \leq Ax = x \leq Av_0 \leq v_0, \tag{2.16}$$

and hence

$$Au_0 \le A^2 u_0 \le Ax = x \le A^2 v_0 \le Av_0, \tag{2.17}$$

therefore $[Au_0, Av_0] \in S$, and thus $S \neq \emptyset$. An order of S is defined by the inclusion relation, that is, for any $I_1 \in S$, $I_2 \in S$, and if $I_1 \subset I_2$, then we define $I_1 \le I_2$. We show that S has a minimal element. Let $\{[u_\alpha, v_\alpha] \mid \alpha \in T\}$ be a totally subset of S and $M' = \{u_\alpha \mid \alpha \in T\}$. Obviously, M' is a totally ordered set in X. Since $A(D)$ is quasiseparable, it follows from $M' \subset A(D)$ that there exists a countable subset $\{y_i\}$ of M' such that $\{y_i\}$ is dense in M'. Let

$$w_1 = y_1, \qquad w_i = \max\{w_{i-1}, y_i\}, \quad i = 2, 3, \ldots. \tag{2.18}$$

Since M' is a totally ordered set, w_i makes sense and

$$w_1 \le w_2 \le \cdots \le w_i \le \cdots. \tag{2.19}$$

Then there exists $\overline{w} \in Y$ such that

$$Cw_i \longrightarrow \overline{w} = \sup\{Cw_i\}. \tag{2.20}$$

Using the same method as in Theorem 2.1, we can prove that $\overline{w}$ makes sense, $A\overline{u}$ (where $\overline{u} = B\overline{w}$) is an upper bound of M', and

$$A\overline{u} \le A(A\overline{u}). \tag{2.21}$$

Since $\mathrm{Fix}\, A \subset [u_\alpha, v_\alpha]$ (for all $\alpha \in T$), for any $x \in \mathrm{Fix}\, A$, we have $u_\alpha \le x$, for all $\alpha \in T$. Since $w_i \in M'$, $w_i \le x$. By (2.20), $\overline{w} \le Cx$, and hence $\overline{u} = B\overline{w} \le BCx = Ax = x$, for all $x \in \mathrm{Fix}\, A$, and therefore

$$A\overline{u} \le Ax = x, \quad \forall x \in \mathrm{Fix}\, A. \tag{2.22}$$

Consider $N = \{v_\alpha \mid \alpha \in T\}$. Similarly, we can prove that there exists $\overline{v} \in D$ such that $A\overline{v}$ is a lower bound of N and

$$A(A\overline{v}) \le A\overline{v}, \qquad A\overline{v} \ge x, \quad \forall x \in \mathrm{Fix}\, A. \tag{2.23}$$

By (2.22) and (2.23), $A\overline{u} \le A\overline{v}$. Set $\overline{I} = [A\overline{u}, A\overline{v}]$. By virtue of (2.21), (2.22), and (2.23), $\overline{I} \in S$. It is easy to see that $\overline{I}$ is a lower bound of $\{[u_\alpha, v_\alpha] \mid \alpha \in T\}$ in S. It follows from Zorn's lemma that S has a minimal element.

Let $[u^*, v^*]$ be a minimal element of S. Therefore, $u^* \le Au^*$, $Av^* \le v^*$, and $\mathrm{Fix}\, A \subset [u^*, v^*]$. Obviously, u^* is a fixed point of A. In fact, on the contrary, $u^* \neq Au^*$ and $u^* \le Au^*$. Hence

$$Au^* \le A(Au^*), \qquad Au^* \le Ax = x, \quad \forall x \in \mathrm{Fix}\, A. \tag{2.24}$$

Since A is an increasing operator, this implies that $\operatorname{Fix} A \subset [Au^*, v^*]$ and $[u^*, v^*]$ includes properly $[Au^*, v^*]$. This contradicts that $[u^*, v^*]$ is the minimal element of S. Similarly, v^* is a fixed point of A. Since $\operatorname{Fix} A \subset [u^*, v^*]$, u^* is the minimal fixed point of A and v^* is the maximal fixed point of A. □

Theorem 2.4. *If the conditions in Theorem 2.2 are satisfied, then A has the minimal fixed point u^* and the maximal fixed point v^* in D; that is, u^* and v^* are fixed points of A, and for any fixed point x of A in D, one has $u^* \leq x \leq v^*$.*

Proof. It is similar to the proof of Theorem 2.4; so we omit it. □

Theorem 2.5. *Let X be an ordered topological space, and $D = [u_0, v_0]$ an order interval in X. Let $A : D \to X$ be an operator. Assume that*

(i) *there exist ordered topological space Y, increasing operator $C : D \to Y$, and increasing operator $B : [Cu_0, Cv_0] = \{y \in Y \mid Cu_0 \leq y \leq Cv_0\} \to X$ such that $A = BC$;*

(ii) *B is an continuous operator;*

(iii) *C is a demicontinuous MMC-operator;*

(iv) *$u_0 \leq Au_0$, $Av_0 \leq v_0$.*

Then A has both the minimal fixed point u^* and the maximal fixed point v^* in $[u_0, v_0]$, and u^* and v^* can be obtained via monotone iterates:

$$u_0 \leq Au_0 \leq \cdots \leq A^n u_0 \leq \cdots \leq A^n v_0 \leq \cdots \leq Av_0 \leq v_0 \tag{2.25}$$

with $\lim_{n\to\infty} A^n u_0 = u^*$, *and* $\lim_{n\to\infty} A^n v_0 = v^*$.

Proof. We define the sequences

$$u_n = A^n u_0, \quad v_n = A^n v_0, \quad n = 1, 2, \ldots \tag{2.26}$$

and conclude from the monotonicity of operator A and the condition (iv) that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots v_n \leq \cdots \leq v_1 \leq v_0. \tag{2.27}$$

Let

$$y_n = Cu_n, \quad n = 1, 2, \ldots. \tag{2.28}$$

Since C is increasing, $y_0 \leq y_1 \leq \cdots \leq y_n \leq \cdots \leq Cv_0$ by (2.27). By the condition (iii), we get

$$y_n \longrightarrow y^* = \sup\{y_n\}, \quad n \longrightarrow \infty. \tag{2.29}$$

By (2.29) and Definition 1.2, we have

$$y^* \in [Cu_0, Cv_0], \tag{2.30}$$

and hence By^* makes sense. Set $u^* = By^*$, then $u^* \in [u_0, v_0]$. Since B is continuous,

$$u_n = Au_n = BCu_n = By_n \longrightarrow By^* = u^*. \tag{2.31}$$

By the condition (iii), $Cu_n \xrightarrow{w} Cu^*$, that is, $y_n \xrightarrow{w} Cu^*$. Note that $y_n \to y^*$; we have $y^* = Cu^*$; hence $u^* = By^* = BCu^* = Au^*$; that is, u^* is a fixed point of A. Similarly, there exists $v^* \in D$ such that $v_n \to v^*$ and v^* is a fixed point of A. By the routine standard proof, it is easy to prove that u^* is the minimal fixed point of A and v^* is the maximal fixed point of A in D. □

3. Applications

As some simple applications of Theorem 2.5, we consider the existence of extremal solutions for a class of discontinuous scalar differential equations.

In the following, R stands for the set of real numbers and $J = [0, a]$ a compact real interval. Let $C[J, R]$ be the class of continuous functions on J. $C[J, R]$ is a normed linear space with the maximum norm and partially ordered by the cone $K = \{x \in C[J, R] : x(t) \geq 0\}$. K is a normal cone in $C[J, R]$.

For any $1 \leq p < +\infty$, set

$$L^p[J, R] = \left\{ x(t) : J \to R \mid x(t) \text{ is measurable and } \int_J |x(t)|^p dt < \infty \right\}. \tag{3.1}$$

Then $L^p[J, R]$ is a Banach space by the norm $\|x\|_p = \left(\int_J |x(t)|^p dt\right)^{1/p}$.

A function $f : J \times R \to R$ is said to be a Carathéodory function if $f(x, y)$ is measurable as a function of x for each fixed y and continuous as a function of y for a.a. (almost all) $x \in J$.

We list for convenience the following assumptions.

(H1) $u_0, v_0 \in AC[J, R]$, $u_0 \leq v_0$,

$$u_0'(t) \leq f(t, u_0(t)), \qquad v_0'(t) \geq f(t, v_0(t)) \quad \text{for a.a. } t \in J. \tag{3.2}$$

(H2) $f : J \times R \to R$ is a Carathéodory function.

(H3) There exists $p > 1$ such that

$$f(t, u_0(t)) \in L^P[J, R], \qquad f(t, v_0(t)) \in L^P[J, R]. \tag{3.3}$$

(H4) There exists $M \geq 0$ such that $f(t, x) + Mx$ is nondecreasing for a.a. $t \in J$.

Consider the differential equation

$$x' = f(t, x), \quad x(0) = x_0, \tag{3.4}$$

where $f : J \times R \to R$. It is a common knowledge that the initial value problem (3.4) is equivalent to the equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds \tag{3.5}$$

if $f(t, x)$ is continuous. Therefore, when $f(t, x)$ is not continuous, we define the solution of the integral equation (3.5) as the solution of the equation (3.4).

Theorem 3.1. *Under the hypotheses (H1)–(H4), the IVP* (3.4) *has the minimal solution u^* and maximal solution v^* in $[u_0, v_0]$. Moreover, there exist monotone iteration sequences $\{u_n(t)\}, \{v_n(t)\} \subset [u_0, v_0]$ such that*

$$u_n(t) \longrightarrow u^*(t), \quad v_n(t) \longrightarrow v^*(t) \quad \text{as } n \longrightarrow \infty \text{ uniformly on } t \in J, \tag{3.6}$$

where $\{u_n(t)\}$ and $\{v_n(t)\}$ satisfy

$$\begin{gathered} u_n'(t) = f(t, u_{n-1}(t)) - M(t)(u_n(t) - u_{n-1}(t)), \quad u_n(0) = x_0, \\ v_n'(t) = f(t, v_{n-1}(t)) - M(t)(v_n(t) - v_{n-1}(t)), \quad v_n(0) = x_0, \\ u_0 \le u_1 \le \cdots \le u_n \le \cdots \le u^* \le v^* \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \end{gathered} \tag{3.7}$$

Proof. For any $h \in C[J, R]$, we consider the linear integral equation:

$$x(t) = h(t) - (Tx)(t), \tag{3.8}$$

where $(Tx)(t) \triangleq \int_0^t Mu(s)ds$. Obviously, $T : C[J, R] \to C[J, R]$ is a linear completely continuous operator. By direct computation, the operator equation $x + Tx = \theta$ has only zero solution; then by Fredholm theorem, for any $h \in C[J, R]$, the operator equation (3.8) has a unique solution in $C[J, R]$. We definition the mapping $N : C[J, R] \to C[J, R]$ by

$$Nh = u_h, \tag{3.9}$$

where u_h is the unique solution of (3.8) corresponding to h. Obviously N is a linear continuous operator; now we show that N is increasing. Suppose that $h_1, h_2 \in C[J, R]$,

$h_1 \le h_2$. Set $m(t) = (Nh_2)(t) - (Nh_1)(t)$. By the definition of the operator N we get

$$\begin{aligned} m(t) &= (Nh_2)(t) - (Nh_1)(t) \\ &= h_2(t) - M\int_0^t (Nh_2)(s)ds - \left[h_1(t) - \int_0^t (Nh_1)(s)ds\right] \\ &= h_2(t) - h_1(t) - M\int_0^t [(Nh_2)(s)ds - (Nh_1)(s)]ds \\ &\ge -M\int_0^t m(s)ds. \end{aligned} \tag{3.10}$$

This integral inequality implies $m(t) \ge 0$ (for all $t \in J$); that is, N is an increasing operator. Set

$$Qv = x_0 + \int_0^t v(s)ds. \tag{3.11}$$

Obviously, $Q : L^p[J,R] \to C[J,R]$ is an increasing continuous operator. Set

$$(Cx)(t) = f(t,x(t)) + Mx(t), \quad x \in C[J,R]. \tag{3.12}$$

By (H2), C maps element of $C[J,R]$ into measurable functions. For any $u \in [u_0, v_0]$, by (H3) and (H4) we get

$$Cu_0 \le Cu \le Cv_0. \tag{3.13}$$

This implies $Cu \in L^p[J,R]$. Hence C maps $[u_0, v_0]$ into $L^p[J,R]$ and C is an increasing operator. Set

$$C[J,R] = X, \qquad L^p[J,R] = Y, \qquad B = NQ, \qquad A = BC, \qquad D = [u_0, v_0]. \tag{3.14}$$

By above discussions we know that $C : D \to Y$ and $B : Y \to X$ are all increasing. Thus conditions (i) and (ii) in Theorem 2.5 are satisfied.

Let $h_n, h^* \in D$ such that $h_n \to h^*$ in $C[J,R]$; by (H2) we have

$$\lim_{n\to\infty} f(t,h_n(t)) + Mh_n(t) = f(t,h^*(t)) + Mh^*(t), \quad \text{for a.a. } t \in J. \tag{3.15}$$

For any $\varphi(t) \in L^q[J,R]$ $(p^{-1} + q^{-1} = 1)$, by (2.29), we have

$$\begin{aligned} 0 &\le f(t,h_n(t)) + Mh_n(t) - \left[f(t,u_0(t)) + Mu_0(t)\right] \\ &\le f(t,v_0(t)) + Mv_0(t) - \left[f(t,u_0(t)) + Mu_0(t)\right], \end{aligned} \tag{3.16}$$

and hence

$$\left|f(t,h_n(t))+Mh_n(t)\right|\le H(t), \tag{3.17}$$

where $H(t)=|f(t,v_0(t))+Mv_0(t)|+2|f(t,u_0(t))+Mu_0(t)|$. By (H3), $H(t)\in L^p[J,R]$; thus

$$\varphi(t)\left|f(t,h_n(t))+Mh_n(t)\right|\le\varphi(t)H(t), \tag{3.18}$$

where $\varphi(t)H(t)\in L^1[J,R]$. Applying the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_J\varphi(t)\left(f(t,h_n(t))+Mh_n(t)\right)dt=\int_J\varphi(t)\left(f(t,h^*(t))+Mh^*(t)\right)dt. \tag{3.19}$$

This implies that $Ch_n\xrightarrow{w}Ch^*$ in $L^p[J,R]$; that is, C is a demicontinuous operator. Since the cone in $L^p[J,R]$ is regular, it is easy to see that C is an MMC-operator. Thus condition (iii) in Theorem 2.5 is satisfied.

We now show that condition (iv) in Theorem 2.5 is fulfilled. By (H1) and (3.5), and noting the definition of operator N, we get

$$\begin{aligned}(Au_0)(t)-u_0(t)&=(NQC)u_0(t)-u_0(t)\\&=N\left(x_0+\int_0^t\left[f(s,u_0(s))+Mu_0(s)\right]ds\right)-u_0(t)\\&=x_0+\int_0^t\left[f(s,u_0(s))+Mu_0(s)\right]ds-M\int_0^t(Au_0)(s)ds-u_0(t)\\&\ge -M\int_0^t[(Au_0)(s)-u_0(s)]ds.\end{aligned} \tag{3.20}$$

This implies that $(Au_0)(t)-u_0(t)\ge 0$, for all $t\in J$, that is, $u_0\le Au_0$. Similarly we can show that $Av_0\le v_0$.

Since all conditions in Theorem 2.5 are satisfied, by Theorem 2.5, A has the maximal fixed point and the minimal fixed point in D. Observing that fixed point of A is equivalent to solutions of (3.5), and (3.5) is equivalent to (3.4), the conclusions of Theorem 3.1 hold. □

Remark 3.2. In the proof of Theorem 3.1, we obtain the uniformly convergence of the monotone sequences without the compactness condition.

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