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Research Article

Some Common Fixed Point Theorems in Menger PM Spaces

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Employing the common property (E.A), we prove some common fixed point theorems for weakly compatible mappings via an implicit relation in Menger PM spaces. Some results on similar lines satisfying quasicontraction condition as well as ψ-type contraction condition are also proved in Menger PM spaces. Our results substantially improve the corresponding theorems contained in (Branciari, (2002); Rhoades, (2003); Vijayaraju et al., (2005)) and also some others in Menger as well as metric spaces. Some related results are also derived besides furnishing illustrative examples.

1. Introduction and Preliminaries

Sometimes, it is found appropriate to assign the average of several measurements as a measure to ascertain the distance between two points. Inspired from this line of thinking, Menger [1, 2] introduced the notion of Probabilistic Metric spaces (in short PM spaces) as a generalization of metric spaces. In fact, he replaced the distance function $d : \Re \times \Re \to \Re^+$ with a distribution function $F_{p,q} : \Re \to [0,1]$ wherein for any number x, the value $F_{p,q}(x)$ describes the probability that the distance between p and q is less than x. In fact the study of such spaces received an impetus with the pioneering work of Schweizer and Sklar [3]. The theory of PM spaces is of paramount importance in Probabilistic Functional Analysis especially due to its extensive applications in random differential as well as random integral equations.

Fixed point theory is one of the most fruitful and effective tools in mathematics which has enormous applications within as well as outside mathematics. The theory of fixed points in PM spaces is a part of Probabilistic Analysis which continues to be an active area of mathematical research. By now, several authors have already established numerous fixed point and common fixed point theorems in PM spaces. For an idea of this kind of the literature, one can consult the results contained in [3–14].

In metric spaces, Jungck [15] introduced the notion of compatible mappings and utilized the same (as a tool) to improve commutativity conditions in common fixed point theorems. This concept has been frequently employed to prove existence theorems on common fixed points. However, the study of common fixed points of noncompatible mappings is also equally interesting which was initiated by Pant [16]. Recently, Aamri and Moutawakil [17] and Liu et al. [18] respectively, defined the property (E.A) and the common property (E.A) and proved some common fixed point theorems in metric spaces. Imdad et al. [19] extended the results of Aamri and Moutawakil [17] to semimetric spaces. Most recently, Kubiaczyk and Sharma [20] defined the property (E.A) in PM spaces and used it to prove results on common fixed points wherein authors claim to prove their results for strict contractions which are merely valid up to contractions.

In 2002, Branciari [21] proved a fixed point result for a mapping satisfying an integral-type inequality which is indeed an analogue of contraction mapping condition. In recent past, several authors (e.g., [22–26]) proved various fixed point theorems employing relatively more general integral type contractive conditions. In one of his interesting articles, Suzuki [27] pointed out that Meir-Keeler contractions of integral type are still Meir-Keeler contractions. In this paper, we prove the fixed point theorems for weakly compatible mappings via an implicit relation in Menger PM spaces satisfying the common property (E.A). Our results substantially improve the corresponding theorems contained in [21, 24, 26, 28] along with some other relevant results in Menger as well as metric spaces. Some related results are also derived besides furnishing illustrative examples.

In the following lines, we collect the background material to make our presentation as self-contained as possible.

Definition 1.1 (see [3]). A mapping $F : \Re \to \Re^+$ is called distribution function if it is nondecreasing and left continuous with $\inf\{F(t) : t \in \Re\} = 0$ and $\sup\{F(t) : t \in \Re\} = 1$.

Let L be the set of all distribution functions whereas H be the set of specific distribution function (also known as Heaviside function) defined by

$$H(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases} \tag{1.1}$$

Definition 1.2 (see [1]). Let X be a nonempty set. An ordered pair (X, F) is called a PM space if F is a mapping from $X \times X$ into L satisfying the following conditions:

(1) $F_{p,q}(x) = H(x)$ if and only if $p = q$,

(2) $F_{p,q}(x) = F_{q,p}(x)$,

(3) $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$, then $F_{p,r}(x + y) = 1$, for all $p, q, r \in X$ and $x, y \ge 0$.

Every metric space (X, d) can always be realized as a PM space by considering $F : X \times X \to L$ defined by $F_{p,q}(x) = H(x - d(p, q))$ for all $p, q \in X$. So PM spaces offer a wider framework (than that of the metric spaces) and are general enough to cover even wider statistical situations.

Definition 1.3 (see [3]). A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if

(1) $\Delta(a,1) = a, \Delta(0,0) = 0$,

(2) $\Delta(a,b) = \Delta(b,a)$,

(3) $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a, d \geq b$,

(4) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$ for all $a,b,c \in [0,1]$.

Example 1.4. The following are the four basic t-norms.

(i) The minimum t-norm: $\Delta_M(a,b) = \min\{a,b\}$.

(ii) The product t-norm: $\Delta_P(a,b) = a.b$.

(iii) The Lukasiewicz t-norm: $\Delta_L(a,b) = \max\{a+b-1,0\}$.

(iv) The weakest t-norm, the drastic product:

$$\Delta_D(a,b) = \begin{cases} \min\{a,b\} & \text{if } \max\{a,b\} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

In respect of above mentioned t-norms, we have the following ordering:

$$\Delta_D < \Delta_L < \Delta_P < \Delta_M. \tag{1.3}$$

Throughout this paper, Δ stands for an arbitrary continuous t-norm.

Definition 1.5 (see [1]). A Menger PM space (X, F, Δ) is a triplet where (X, F) is a PM space and Δ is a t-norm satisfying the following condition:

$$F_{p,r}(x+y) \geq \Delta(F_{p,q}(x), F_{q,r}(y)). \tag{1.4}$$

Definition 1.6 (see [6]). A sequence $\{p_n\}$ in a Menger PM space (X, F, Δ) is said to converge to a point p in X if for every $\epsilon > 0$ and $\lambda > 0$, there is an integer $M(\epsilon,\lambda)$ such that $F_{p_n,p}(\epsilon) > 1-\lambda$, for all $n \geq M(\epsilon,\lambda)$.

Definition 1.7 (see [10]). A pair (A, S) of self-mappings of a Menger PM space (X, F, Δ) is said to be compatible if $F_{ASp_n,SAp_n}(x) \to 1$ for all $x > 0$, whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \to t$, for some t in X as $n \to \infty$.

Definition 1.8 (see [23]). A pair (A, S) of self-mappings of a Menger PM space (X, F, Δ) is said to be noncompatible if and only if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \in X, \quad \text{for some } t \in X, \tag{1.5}$$

but for some $t_0 > 0, \lim_{n\to\infty} F_{ASx_n,SAx_n}(t_0)$ is either less than 1 or nonexistent.

Definition 1.9 (see [6]). A pair (A, S) of self-mappings of a Menger PM space (X, F, Δ) is said to satisfy the property (E.A) if there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \in \mathrm{X}. \tag{1.6}$$

Clearly, a pair of compatible mappings as well as noncompatible mappings satisfies the property (E.A).

Inspired by Liu et al. [18], we introduce the following.

Definition 1.10. Two pairs (A, S) and (B, T) of self-mappings of a Menger PM space (X, F, Δ) are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}, \{y_n\}$ in X and some t in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = t. \tag{1.7}$$

Example 1.11. Let (X, F, Δ) be a Menger PM space with $\mathrm{X} = [-1, 1]$ and,

$$F_{x,y}(t) = \begin{cases} e^{-|x-y|/t}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases} \tag{1.8}$$

for all $x, y \in \mathrm{X}$. Define self-mappings A, B, S and T on X as $Ax = x/3$, $Bx = -x/3$, $Sx = x$, and $Tx = -x$ for all $x \in \mathrm{X}$. Then with sequences $\{x_n\} = 1/n$ and $\{y_n\} = -1/n$ in X, one can easily verify that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 0. \tag{1.9}$$

This shows that the pairs (A, S) and (B, T) share the common property (E.A).

Definition 1.12 (see[29]). A pair (A, S) of self-mappings of a nonempty set X is said to be weakly compatible if the pair commutes on the set of coincidence points, that is, $Ap = Sp$ for some $p \in \mathrm{X}$ implies that $ASp = SAp$.

Definition 1.13 (see [8]). Two finite families of self-mappings $\{A_i\}_{i=1}^m$ and $\{B_k\}_{k=1}^n$ of a set X are said to be pairwise commuting if

(1) $A_iA_j = A_jA_i$, $i, j \in \{1, 2, \ldots, m\}$,

(2) $B_kB_l = B_lB_k$, $k, l \in \{1, 2, \ldots, n\}$,

(3) $A_iB_k = B_kA_i$, $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

2. Implicit Relation

Let F_6 be the set of all continuous functions $\Phi(t_1, t_2, t_3, t_4, t_5, t_6) : [0,1]^6 \to \mathbb{R}$ satisfying the following conditions:

(Φ_1) $\Phi(u, 1, u, 1, 1, u) < 0$, for all $u \in (0,1)$,

(Φ_2) $\Phi(u, 1, 1, u, u, 1) < 0$, for all $u \in (0,1)$,

(Φ_3) $\Phi(u, 1, u, 1, u, u) < 0$, for all $u \in (0,1)$.

Example 2.1. Define $\Phi(t_1, t_2, t_3, t_4, t_5, t_6) : [0,1]^6 \to \mathbb{R}$ as

$$\Phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\min\{t_2, t_3, t_4, t_5, t_6\}), \tag{2.1}$$

where $\psi : [0,1] \to [0,1]$ is increasing and continuous function such that $\psi(t) > t$ for all $t \in (0,1)$. Notice that

(Φ_1) $\Phi(u, 1, u, 1, 1, u) = u - \psi(u) < 0$, for all $u \in (0,1)$,

(Φ_2) $\Phi(u, 1, 1, u, u, 1) = u - \psi(u) < 0$, for all $u \in (0,1)$,

(Φ_3) $\Phi(u, 1, u, 1, u, u) = u - \psi(u) < 0$, for all $u \in (0,1)$.

Example 2.2. Define $\Phi(t_1, t_2, t_3, t_4, t_5, t_6) : [0,1]^6 \to \mathbb{R}$ as

$$\Phi(t_1, t_2, t_3, t_4, t_5, t_6) = \int_0^{t_1} \phi(t)dt - \psi\left(\int_0^{\min\{t_2,t_3,t_4,t_5,t_6\}}\right)\phi(t)dt, \tag{2.2}$$

where $\psi : [0,1] \to [0,1]$ is an increasing and continuous function such that $\psi(t) > t$ for all $t \in (0,1)$ and $\phi : \Re_+ \to \Re_+$ is a Lebesgue integrable function which is summable and satisfies

$$0 < \int_0^{\epsilon} \phi(s)ds < 1, \quad \forall 0 < \epsilon < 1, \qquad \int_0^1 \phi(s)ds = 1. \tag{2.3}$$

Observe that

(Φ_1) $\Phi(u, 1, u, 1, 1, u) = \int_0^u \phi(t)dt - \psi(\int_0^u \phi(t)dt) < 0$, for all $u \in (0,1)$,

(Φ_2) $\Phi(u, 1, 1, u, u, 1) = \int_0^u \phi(t)dt - \psi(\int_0^u \phi(t)dt) < 0$, for all $u \in (0,1)$,

(Φ_3) $\Phi(u, 1, u, 1, u, u) = \int_0^u \phi(t)dt - \psi(\int_0^u \phi(t)dt) < 0$, for all $u \in (0,1)$.

Example 2.3. Define $\Phi(t_1, t_2, t_3, t_4, t_5, t_6) : [0,1]^6 \to \mathbb{R}$ as

$$\begin{aligned}\Phi&(t_1, t_2, t_3, t_4, t_5, t_6)\\ &= \int_0^{t_1} \phi(t)dt - \psi\left(\min\left\{\int_0^{t_2} \phi(t)dt, \int_0^{t_3} \phi(t)dt, \int_0^{t_4} \phi(t)dt, \int_0^{t_5} \phi(t)dt, \int_0^{t_6} \phi(t)dt\right\}\right),\end{aligned} \tag{2.4}$$

where $\psi : [0,1] \to [0,1]$ is an increasing and continuous function such that $\psi(t) > t$ for all $t \in (0,1)$ and $\phi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is a Lebesgue integrable function which is summable and satisfies

$$0 < \int_0^{\epsilon} \phi(s)ds < 1, \quad \forall 0 < \epsilon < 1, \qquad \int_0^1 \phi(s)ds = 1. \tag{2.5}$$

Observe that

(Φ_1) $\Phi(u,1,u,1,1,u) = \int_0^u \phi(t)dt - \psi(\int_0^u \phi(t)dt) < 0$, for all $u \in (0,1)$,
(Φ_2) $\Phi(u,1,1,u,u,1) = \int_0^u \phi(t)dt - \psi(\int_0^u \phi(t)dt) < 0$, for all $u \in (0,1)$,
(Φ_3) $\Phi(u,1,u,1,u,u) = \int_0^u \phi(t)dt - \psi(\int_0^u \phi(t)dt) < 0$, for all $u \in (0,1)$.

3. Main Results

We begin with the following observation.

Lemma 3.1. *Let A, B, S and T be self-mappings of a Menger space (X, F, Δ) satisfying the following:*

(i) *the pair (A, S) (or (B, T)) satisfies the property $(E.A)$;*

(ii) *for any $p, q \in X, \Phi \in F_6$ and for all $x > 0$,*

$$\Phi(F_{Ap,Bq}(x), F_{Sp,Tq}(x), F_{Sp,Ap}(x), F_{Tq,Bq}(x), F_{Sp,Bq}(x), F_{Tq,Ap}(x)) \geq 0; \tag{3.1}$$

(iii) *$A(X) \subset T(X)$ (or $B(X) \subset S(X)$).*

Then the pairs (A, S) and (B, T) share the common property $(E.A)$.

Proof. Suppose that the pair (A, S) owns the property (E.A), then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \quad \text{for some } t \in X. \tag{3.2}$$

Since $A(X) \subset T(X)$, hence for each $\{x_n\}$ there exists $\{y_n\} \in X$ such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = t. \tag{3.3}$$

Thus in all, we have $Ax_n \to t, Sx_n \to t$ and $Ty_n \to t$. Now we assert that $By_n \to t$. Suppose that $By_n \not\to t$; then applying inequality (3.1), we obtain

$$\Phi(F_{Ax_n,By_n}(x), F_{Sx_n,Ty_n}(x), F_{Sx_n,Ax_n}(x), F_{Ty_n,By_n}(x), F_{Sx_n,By_n}(x), F_{Ty_n,Ax_n}(x)) \geq 0 \tag{3.4}$$

which on making $n \to \infty$ reduces to

$$\Phi\Big(F_{t,\lim_{n\to\infty} By_n}(x), F_{t,t}(x), F_{t,t}(x), F_{t,\lim_{n\to\infty} By_n}(x), F_{t,\lim_{n\to\infty} By_n}(x), F_{t,t}(x)\Big) \geq 0 \tag{3.5}$$

or

$$\Phi\Big(F_{t,\lim_{n\to\infty} By_n}(x), 1, 1, F_{t,\lim_{n\to\infty} By_n}(x), F_{t,\lim_{n\to\infty} By_n}(x), 1\Big) \geq 0 \tag{3.6}$$

which is a contradiction to (Φ_2), and therefore $By_n \to t$. Hence the pairs (A, S) and (B, T) share the common property (E.A). □

Remark 3.2. The converse of Lemma 3.1 is not true in general. For a counter example, one can see Example 3.17 (presented in the end).

Theorem 3.3. *Let A, B, S and T be self-mappings on a Menger PM space (X, F, Δ) satisfying inequality* (3.1). *Suppose that*

(i) *the pair (A, S) (or (B, T)) enjoys the property (E.A),*

(ii) *$A(X) \subset T(X)$ (or $B(X) \subset S(X)$),*

(iii) *$S(X)$ (or $T(X)$) is a closed subset of X.*

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Proof. In view of Lemma 3.1, the pairs (A, S) and (B, T) share the common property (E.A), that is, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \quad \text{for some } t \in X. \tag{3.7}$$

Suppose that $S(X)$ is a closed subset of X, then $t = Su$ for some $u \in X$. If $t \neq Au$, then applying inequality (3.1), we obtain

$$\Phi\big(F_{Au,By_n}(x), F_{Su,Ty_n}(x), F_{Su,Au}(x), F_{Ty_n,By_n}(x), F_{Su,By_n}(x), F_{Ty_n,Au}(x)\big) \geq 0 \tag{3.8}$$

which on making $n \to \infty$, reduces to

$$\Phi(F_{Au,t}(x), 1, F_{t,Au}(x), 1, 1, F_{t,Au}(x)) \geq 0 \tag{3.9}$$

which is a contradiction to (Φ_1). Hence $Au = Su = t$.

Since $A(X) \subset T(X)$, there exists $v \in X$ such that $t = Au = Tv$.

If $t \neq Bv$, then using inequality (3.1), we have

$$\Phi(F_{Au,Bv}(x), F_{Su,Tv}(x), F_{Su,Au}(x), F_{Tv,Bv}(x), F_{Su,Bv}(x), F_{Tv,Au}(x)) \geq 0 \tag{3.10}$$

or

$$\Phi(F_{t,Bv}(x), 1, 1, F_{t,Bv}(x), F_{t,Bv}(x), 1) \geq 0 \tag{3.11}$$

which is a contradiction to (Φ_2), and therefore $Au = Su = t = Bv = Tv$.

Since the pairs (A, S) and (B, T) are weakly compatible and $Au = Su, Bv = Tv$, therefore

$$At = ASu = SAu = St, \qquad Bt = BTv = TBv = Tt. \tag{3.12}$$

If $At \neq t$, then using inequality (3.1), we have

$$\Phi(F_{At,Bv}(x), F_{St,Tv}(x), F_{St,At}(x), F_{Tv,Bv}(x), F_{St,Bv}(x), F_{Tv,At}(x)) \geq 0 \tag{3.13}$$

or

$$\Phi\ (F_{At,t}(x), 1, F_{At,t}(x), 1, F_{At,t}(x)_0, F_{t,At}(x)_0) \geq 0 \tag{3.14}$$

which is a contradiction to (Φ_3), and therefore $At = St = t$.

Similarly, one can prove that $Bt = Tt = t$. Hence $t = Bt = Tt = At = St$, and t is a common fixed point of A, B, S and T. The uniqueness of common fixed point is an easy consequences of inequality (3.1).

By choosing A, B, S and T suitably, one can derive corollaries involving two or three mappings. As a sample, we deduce the following natural result for a pair of self-mappings by setting $B = A$ and $T = S$ (in Theorem 3.3). □

Corollary 3.4. *Let A and S be self-mappings on a Menger space (X, F, Δ). Suppose that*

(i) *the pair (A, S) enjoys the property (E.A),*

(ii) *for all $p, q \in X, \Phi \in F_6$ and for all $x > 0$,*

$$\Phi\big(F_{Ap,Aq}(x), F_{Sp,Sq}(x), F_{Sp,Ap}(x), F_{Sq,Aq}(x), F_{Sp,Aq}(x), F_{Sq,Ap}(x)\big) \geq 0, \tag{3.15}$$

(iii) *$S(X)$ is a closed subset of X.*

Then A and S have a coincidence point. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

Theorem 3.5. *Let A, B, S and T be self-mappings of a Menger PM space (X, F, Δ) satisfying the inequality* (3.1). *Suppose that*

(i) *the pairs (A, S) and (B, T) share the common property (E.A),*

(ii) *$S(X)$ and $T(X)$ are closed subsets of X.*

If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that the pairs (A, S) and (B, T) satisfy the common property (E.A), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \quad \text{for some } t \in X. \tag{3.16}$$

Since $S(X)$ and $T(X)$ are closed subsets of X, we obtain $t = Su = Tv$ for some $u, v \in X$. If $t \neq Au$, then using inequality (3.1), we have

$$\Phi\left(F_{Au,By_n}(x), F_{Su,Ty_n}(x), F_{Su,Au}(x), F_{Ty_n,By_n}(x), F_{Su,By_n}(x), F_{Ty_n,Au}(x)\right) \geq 0 \tag{3.17}$$

which on making $n \to \infty$ reduces to

$$\Phi(F_{Au,t}(x), 1, F_{t,Au}(x), 1, 1, F_{t,Au}(x)) \geq 0 \tag{3.18}$$

which is a contradiction to (Φ_1), and hence $t = Au = Tv = Su$. The rest of the proof can be completed on the lines of the proof of Theorem 3.3, hence it is omitted. □

Remark 3.6. Theorem 3.3 extends the main result of Ciric [30] to Menger PM spaces besides extending the main result of Kubiaczyk and Sharma [20] to two pairs of mappings without any condition on containment of ranges amongst involved mappings.

Theorem 3.7. *The conclusions of Theorem 3.5 remain true if condition (ii) of Theorem 3.5 is replaced by the following:*

(iii)′ $\overline{A(X)} \subset T(X)$ *and* $\overline{B(X)} \subset S(X)$.

Corollary 3.8. *The conclusions of Theorems 3.3 and 3.5 remain true if conditions (ii) (of Theorem 3.3) and (iii)′ (of Theorem 3.7) are replaced by the following:*

(iv) $A(X)$ *and* $B(X)$ *are closed subsets of* X *whereas* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$.

As an application of Theorem 3.3, we prove the following result for four finite families of self-mappings. While proving this result, we utilize Definition 1.13 which is a natural extension of commutativity condition to two finite families of mappings.

Theorem 3.9. *Let* $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_p\}$, $\{S_1, S_2, \ldots, S_n\}$ *and* $\{T_1, T_2, \ldots, T_q\}$ *be four finite families of self-mappings of a Menger PM space* (X, F, Δ) *with* $A = A_1A_2 \cdots A_m$, $B = B_1B_2 \cdots B_p$, $S = S_1S_2 \cdots S_n$ *and* $T = T_1T_2 \cdots T_q$ *satisfying condition* (3.1). *If the pairs* (A, S) *and* (B, T) *share the common property (E.A) and* $S(X)$ *as well as* $T(X)$ *are closed subsets of* X, *then*

(i) *the pair* (A, S) *as well as* (B, T) *has a coincidence point,*

(ii) A_i, B_k, S_r *and* T_t *have a unique common fixed point provided that the pair of families* $(\{A_i\}, \{S_r\})$ *and* $(\{B_k\}, \{T_t\})$ *commute pairwise, where* $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, p\}$, $r \in \{1, 2, \ldots, n\}$, *and* $t \in \{1, 2, \ldots, q\}$.

Proof. The proof follows on the lines of Theorem 4.1 according to M. Imdad and J. Ali [31] and Theorem 3.1 according to Imdad et al. [19]. □

Remark 3.10. By restricting four families as $\{A_1, A_2\}, \{B_1, B_2\}, \{S_1\}$ and $\{T_1\}$ in Theorem 3.9, we can derive improved versions of certain results according to Chugh and Rathi [4], Kutukcu and Sharma [32], Rashwan and Hedar [11], Singh and Jain [14], and some others. Theorem 3.9 also generalizes the main result of Razani and Shirdaryazdi [12] to any finite number of mappings.

By setting $A_1 = A_2 = \cdots = A_m = G, B_1 = B_2 = \cdots = B_p = H, S_1 = S_2 = \cdots S_n = I$ and $T_1 = T_2 = \cdots = T_q = J$ in Theorem 3.9, we deduce the following.

Corollary 3.11. *Let G, H, I and J be self-mappings of a Menger space (X, F, Δ) such that the pairs (G^m, I^n) and (H^p, J^q) share the common property (E.A) and also satisfy the condition*

$$\Phi\{F_{G^m x, H^p y}(z), F_{I^n x, J^q y}(z), F_{I^n x, G^m x}(z), F_{I^n x, H^p y}(z), F_{J^q y, H^p y}(z), F_{J^q y, G^m x}(z)\} \geq 0 \tag{3.19}$$

for all $x, y \in X$, for all $z > 0$ and m, n, p and q are fixed positive integers.

If $I^n(X)$ and $J^q(X)$ are closed subsets of X, then G, H, I and J have a unique common fixed point provided, $GI = IG$ and $HJ = JH$.

Remark 3.12. Corollary 3.11 is a slight but partial generalization of Theorem 3.3 as the commutativity requirements (i.e., $GI = IG$ and $HJ = JH$) in this corollary are stronger as compared to weak compatibility in Theorem 3.3. Corollary 3.11 also presents the generalized and improved form of a result according to Bryant [33] in Menger PM spaces.

Our next result involves a lower semicontinuous function $\psi : [0,1] \to [0,1]$ such that $\psi(t) > t$ for all $t \in (0,1)$ along with $\psi(0) = 0$ and $\psi(1) = 1$.

Theorem 3.13. *Let A, B, S and T be self-mappings of a Menger space (X, F, Δ) satisfying conditions (i) and (ii) of Theorem 3.5 and for all $p, q \in X, x > 0$*

$$\int_0^{F_{Ap,Bq}(x)} \phi(t)dt \geq \psi\left(\int_0^{m(p,q)} \phi(t)dt\right), \tag{3.20}$$

where $m(p,q) = \min\{F_{Sp,Tq}(x), F_{Sp,Ap}(x), F_{Tq,Bq}(x), F_{Sp,Bq}(x), F_{Tq,Ap}(x)\}$.

Then the pairs (A, S) and (B, T) have point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Proof. As both the pairs share the common property (E.A), there exist two sequences $\{x_n\}, \{y_n\} \subset X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \quad \text{for some } t \in X. \tag{3.21}$$

If $S(X)$ is a closed subset of X, then (3.21). Therefore, there exists a point $u \in X$ such that $Su = t$. Now we assert that $Au = Su$. If it is not so, then setting $p = u, q = y_n$ in (3.20), we get

$$\int_0^{F_{Au,By_n}(x)} \phi(t)dt \geq \psi\left(\int_0^{\min\{F_{Su,Ty_n}(x), F_{Su,Au}(x), F_{Ty_n,By_n}(x), F_{Su,By_n}(x), F_{Ty_n,Au}(x)\}} \phi(t)dt\right) \tag{3.22}$$

which on making $n \to \infty$, reduces to

$$\int_0^{F_{Au,t}(x)} \phi(t)dt \geq \psi\left(\int_0^{\min\{F_{t,t}(x), F_{t,Au}(x), F_{t,t}(x), F_{t,t}(x), F_{t,Au}(x)\}} \phi(t)dt\right) \tag{3.23}$$

or

$$\int_0^{F_{Au,t}(x)} \phi(t)dt \geq \psi\left(\int_0^{F_{Au,t}(x)} \phi(t)dt\right) > \int_0^{F_{Au,t}(x)} \phi(t)dt \tag{3.24}$$

a contradiction. Therefore $Au = t$, and hence $Au = Su$ which shows that the pair (A, S) has a point of coincidence.

If $T(X)$ is a closed subset of X, then (3.21). Hence, there exists a point $w \in X$ such that $Tw = t$. Now we show that $Bw = Tw$. If it is not so, then using (3.20) with $p = x_n, q = w$, we have

$$\int_0^{F_{Ax_n,Bw}(x)} \phi(t)dt \geq \psi\left(\int_0^{\min\{F_{Sx_n,Tw}(x),F_{Sx_n,Ax_n}(x),F_{Tw,Bw}(x),F_{Sx_n,Bw}(x),F_{Tw,Ax_n}(x)\}} \phi(t)dt\right) \tag{3.25}$$

which on making $n \to \infty$ reduces to

$$\int_0^{F_{t,Bw}(x)} \phi(t)dt \geq \psi\left(\int_0^{\min\{F_{t,t}(x),F_{t,t}(x),F_{t,Bw}(x),F_{t,Bw}(x),F_{t,t}(x)\}} \phi(t)dt\right) \tag{3.26}$$

or

$$\int_0^{F_{t,Bw}(x)} \phi(t)dt \geq \psi\left(\int_0^{F_{t,Bw}(x)} \phi(t)dt\right) > \int_0^{F_{t,Bw}(x)} \phi(t)dt \tag{3.27}$$

a contradiction. Therefore $Bw = t$ and hence $Tw = Bw$ which proves that the pair (B, T) has a point of coincidence.

Since the pairs (A, S) and (B, T) are weakly compatible and both the pairs have point of coincidence u and v, respectively. Following the lines of the proof of Theorem 3.3, one can easily prove the existence of unique common fixed point of mappings A, B, S and T. This concludes the proof. □

Remark 3.14. Theorem 3.13 generalizes the main result of Kohli and Vashistha [9] to two pairs of self-mappings as Theorem 3.13 never requires any condition on the containment of ranges amongst involved mappings besides weakening the completeness requirement of the space to closedness of suitable subspaces along with suitable commutativity requirements of the involved mappings. Here one may also notice that the function ψ is lower semicontinuous whereas all the involved mappings may be discontinuous at the same time.

Remark 3.15. Notice that results similar to Theorems 3.5 –3.9 and Corollaries 3.4–3.11 can also be outlined in respect of Theorem 3.13, but we omit the details with a view to avoid any repetition.

We conclude this paper with two illustrative examples which demonstrate the validity of the hypotheses of Theorem 3.3 and Theorem 3.13.

Example 3.16. Let (X, F, Δ) be a Menger space, where $X = [0,2)$ with a t-norm defined by $\Delta(a,b) = \min\{a,b\}$ for all $a,b \in [0,2)$, $\psi(s) = \sqrt{(s)}$ for all $s \in [0,1]$ and

$$F_{p,q}(t) = \frac{t}{t + |p-q|}, \quad \forall p,q \in X,\ t > 0. \tag{3.28}$$

Define A, B, S and T by: $Ax = Bx = 1$,

$$S(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ \dfrac{2}{3} & \text{if } x \text{ is irrational}, \end{cases} \qquad T(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ \dfrac{1}{3} & \text{if } x \text{ is irrational}. \end{cases} \tag{3.29}$$

Also define $\Phi(t_1, t_2, t_3, t_4, t_5, t_6) : [0,1]^6 \to \mathbb{R}$ as

$$\begin{aligned} &\Phi(t_1, t_2, t_3, t_4, t_5, t_6) \\ &\quad = \int_0^{t_1} \phi(t)dt - \psi\left(\min\left\{\int_0^{t_2} \phi(t)dt, \int_0^{t_3} \phi(t)dt, \int_0^{t_4} \phi(t)dt, \int_0^{t_5} \phi(t)dt, \int_0^{t_6} \phi(t)dt\right\}\right). \end{aligned} \tag{3.30}$$

It is easy to see that for all $x, y \in X$ and $t > 0$

$$\begin{aligned} 1 &= \int_0^{F_{Ap,Bq}(x)} \phi(t)dt \\ &\geq \psi\left(\min\left\{\int_0^{F_{Sp,Tq}(x)} \phi(t)dt, \int_0^{F_{Sp,Ap}(x)} \phi(t)dt, \int_0^{F_{Tq,Bq}(x)} \phi(t)dt, \int_0^{F_{Sp,Bq}(x)} \phi(t)dt, \int_0^{F_{Tq,Ap}(x)} \phi(t)dt\right\}\right). \end{aligned} \tag{3.31}$$

Also $A(X) = \{1\} \subset \{1, 2/3\} = S(X)$. Thus all the conditions of Theorem 3.3 are satisfied, and 1 is the unique common fixed point of A, B, S and T.

Example 3.17. Let (X, F, Δ) and F be the same as in Example 3.16. Define A, B, S and T by

$$\begin{aligned} A(x) &= \begin{cases} 1 & \text{if } x \text{ is rational}, \\ \dfrac{3}{4} & \text{if } x \text{ is irrational}, \end{cases} \qquad B(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ \dfrac{1}{2} & \text{if } x \text{ is irrational}, \end{cases} \\ S(x) &= \begin{cases} 1 & \text{if } x \text{ is rational}, \\ \dfrac{2}{3} & \text{if } x \text{ is irrational}, \end{cases} \qquad T(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ \dfrac{1}{3} & \text{if } x \text{ is irrational}. \end{cases} \end{aligned} \tag{3.32}$$

By a routine calculation, one can verify that for all $x, y \in X$ and $t > 0$

$$\begin{aligned}\int_0^{F_{Ap,Bq}(x)} \phi(t)dt &\geq \psi\left(\int_0^{m(p,q)} \phi(t)dt\right) \\ &\geq \psi\left(\int_0^{\min(F_{Sp,Tq}(x),F_{Sp,Ap}(x),F_{Tq,Bq}(x),F_{Sp,Bq}(x),F_{Tq,Ap}(x))} \phi(t)dt\right).\end{aligned} \tag{3.33}$$

Also $A(X) = \{1, 3/4\} \not\subset \{1, 2/3\} = S(X)$, $B(X) = \{1, 1/2\} \not\subset \{1, 1/3\} = T(X)$, $\psi(s) > s$. Thus all conditions of Theorem 3.13 are satisfied, and 1 is the unique common fixed point of A, B, S and T.

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